A one-parameter family of triangle cubics

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Abstract. Points in the plane of a given triangle whose trilinear distances form a constant product gather on a planar cubic curve. All these cubics constitute a pencil of cubics in which the three-fold ideal line of the triangle plane and the three side lines of the base triangle are the only two degenerate cubics in the pencil. Among the non-degenerate cubics, there is only one rational curve with an isolated node at the centroid of the triangle. Independent of the chosen distance (product), the inflection points of the cubics are the ideal points of the triangle sides. It turns out that the harmonic polars of the inflection points are the medians of the base triangle. We shall study especially those cubics that are defined by triangle centers. Each triangle center defines its own distance product cubic and, in contrast to all other known triangle cubics, only a rather small number of centers share their cubic.

Keywords: triangle, cubic, triangle center, trilinear distance, constant product, triangle center

1 Introduction

In triangle geometry, cubic curves appear frequently as loci of points satisfying certain constraints. These may be the so-called *locus properties* as is, for example, the case with the Neuberg cubic $\mathcal{K}_{001}$ (see Fig. 1). This particular cubic appears as the locus of all points P in the plane of a triangle $\Delta = ABC$ such that the triangle $\Delta' = P_aP_bP_c$ is perspective to Δ, where P_a, P_b, and P_c are the reflections of P in $\Delta's$ side lines. This characterization of $\mathcal{K}_{001}$ and fifteen more can be found on Bernard Gibert's page [2]. However, there are other possible ways to determine cubics in a triangle plane. The loci of points P which lie collinear with their isogonal conjugates or their isotomic conjugates P' and some point $Z \neq P, P'$ (for example, Z may be equal to some triangle center) are called *pivotal isogonal* or *pivotal isotomic* cubics with *pivot* Z. The Neuberg cubic is the self-isogonal cubic with the Euler infinity point X_{30} as its pivot. Clearly, such cubics are invariant with respect to these elementary planar quadratic Cremona transformations. For details on quadratic Cremona transformations see [3].

We shall go a different way and assume that P is a point in the plane of Δ which does neither lie on any of Δ's side lines nor on the line at infinity (ideal line). Therefore, P has well-defined and proper distances to each side lines.

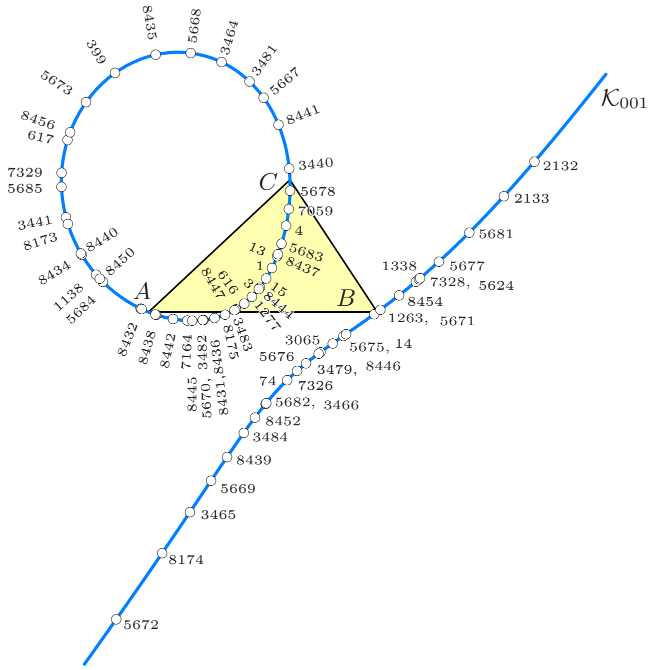

Fig. 1. The Neuberg cubic $\mathcal{K}_{001}$ and some centers on it.

Consequently, the product of these distances is well-defined and we can ask for all points in Δ's plane that form the same product of its distances to the side lines of Δ. It is clear that this results in cubic curves. This can also be confirmed by the elementary approach of multiplying the equations of the three side lines in Hessian normal form and setting this product equal to some constant.

In this paper, we shall first give the equations of the distance product cubics in Section 2 and work out some basic properties which are common to all the cubics in this one-parameter family in Section 3. Then, we focus on those cubics which are defined by triangle centers in Section 4. We shall see that only a small number of known triangle centers share their distance product cubic with others. Section 5 describes how to find further triangle centers on such a cubic once it is defined by a certain center. Finally, in Section 6, we briefly describe some ideas for future work which is related to the group structure on elliptic cubics.

2 Prerequsites

Assume $\Delta = ABC$ is a triangle in the Euclidean plane with the side lengths $\overline{AB} = c$, $\overline{BC} = a$, and $\overline{CA} = b$. We prefer the representation of points in terms of homogeneous trilinear coordinates. Therefore, the vertices of Δ are given by

$$A = (1 : 0 : 0), \quad B = (0 : 1 : 0), \quad C = (0 : 0 : 1)$$

and the unit point of the projective frame equals the incenter X_1 of Δ which is given by the unit vector, *i.e.*, $X_1 = (1 : 1 : 1)$.

The subscript 1 of X_1 refers to the number of the particular triangle center in Clark Kimberling's *Encyclopedia of Triangle Centers*, cf. [4, 5].

Since we are interested in distances, we need the notion of actual trilinear coordinates or trilinear distances of points in the plane of Δ. The trilinear distances (actual trilinear coordinates) (ξ^a, η^a, ζ^a) of a point $X = (\xi : \eta : \zeta)$ are related to its homogeneous trilinear coordinates by

$$(\xi^a, \eta^a, \zeta^a) = \frac{2F}{a\xi + b\eta + c\zeta}(\xi, \eta, \zeta), \tag{1}$$

where F equals the area of Δ. The linear form $a\xi + b\eta + c\zeta$ in the denominator vanishes for points on the ideal line. Clearly, these points are excluded from our considerations.

The normalization (1) of the homogeneous trilinear coordinates is based on the following observation: Let $(\xi : \eta : \zeta)$ be the homogeneous trilinear coordinates of a point X in the triangle plane. The point X determines the three subtriangles BCX, CAX, and ABX of Δ and $\xi^a = \overline{X\,[BC]}$, $\eta^a = \overline{X\,[CA]}$, and $\zeta^a = \overline{X\,[CA]}$ are their altitudes. Thus, their areas sum up to twice the area of Δ, *i.e.*,

$$a\xi^a + b\eta^a + c\zeta^a = 2F.$$

For each point X in the plane of Δ, we can convert its homogeneous trilinear coordinates $(\xi : \eta : \zeta)$ into actual trilinear coordinates. This allows us to ask for all points whose actual trilinear coordinates, *i.e.*, distances to the sides of Δ, form a constant product. This yields

$$\frac{8F^3\xi^a\eta^a\zeta^a}{(a\xi^a + b\eta^a + c\zeta^a)^3} = \delta,$$

where $\delta \in \mathbb{R} \setminus \{0\}$ is an arbitrary constant.

As a first result, we can state:

Theorem 1. *The locus $k(\delta)$ of all points X in the plane of a triangle $\Delta = ABC$ whose distances to the sides of Δ form a constant product $\delta \in \mathbb{R} \setminus \{0\}$ is a planar cubic curve with the equation*

$$k(\delta):\ 8F^3\xi\eta\zeta - \delta(a\xi + b\eta + c\zeta)^3 = 0, \tag{2}$$

where ξ, η, ζ are actual or homogeneous trilinear coordinates.

Proof. The obvious difference between (2) and the equation given above lies in the superscripts pointing to actual trilinear coordinates. The equations of the cubics (2) remain unchanged if we replace the homogeneous trilinear coordinates by actual trilinear coordinates, as follows from a simple computation. $\square$

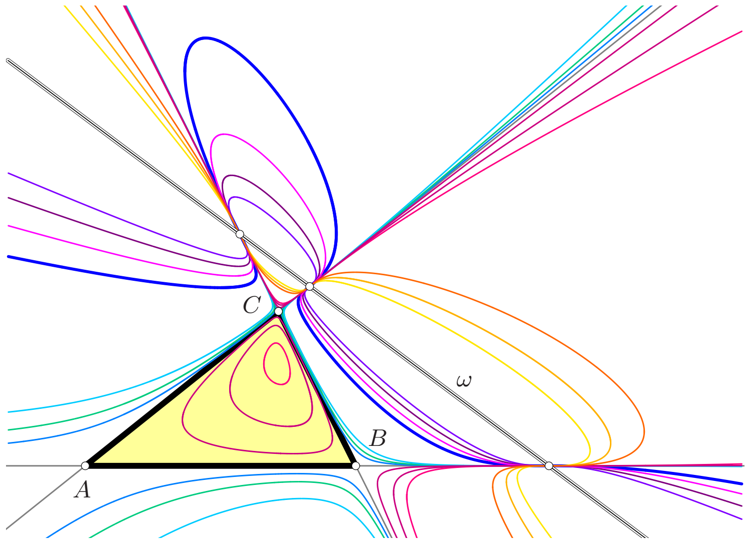

Fig. 2. Projective view onto the pencil of distance product cubics.

The cubics defined by (2) form a linear one-parameter family, *i.e.*, a pencil of cubics, since the equations depend linearly on the parameter δ. It is clear that $\delta = 0$ corresponds to a degenerate cubic $k(0): \ \xi\eta\zeta = 0$ that consists of the three side lines of the triangle, see Fig. 2. Extending the range of δ by letting $\delta \in \mathbb{R} \cup \{\infty\}$, we find that the line ω at infinity considered as a line with multiplicity three is also a degenerate cubic in the pencil.

3 Some properties of the cubics

We shall exclude isosceles, equilateral, and right triangles from our considerations. Clearly, the distance product cubics of equilateral or isosceles triangles share the symmetries with the base triangle. Later, when we deal with distance product cubics that are defined by triangle centers, we will make use of the fact that triangle centers of a generic triangle usually do not coincide with a side line of the triangle. For example, the orthocenter of a right triangle equals the vertex with the right angle and the circumcenter is the midpoint of the hypothenuse. Obviously, these two points have zero distance to at least one triangle side, and thus, the corresponding distance product cubics degenerate into the triplet of Δ's side lines. In an isosceles triangle, not as many triangle centers coincide as is the case with equilateral triangles. However, the symmetry with respect to a certain median of Δ remains, as can be seen in Fig. 3.

We can describe the cubics in the pencil (2) with regard to their singularities:

Theorem 2. *The family* (2) *of cubic curves contains one singular (rational) non-degenerate curve which correspond to* $\delta_2 = \frac{8F^3}{27abc}$.

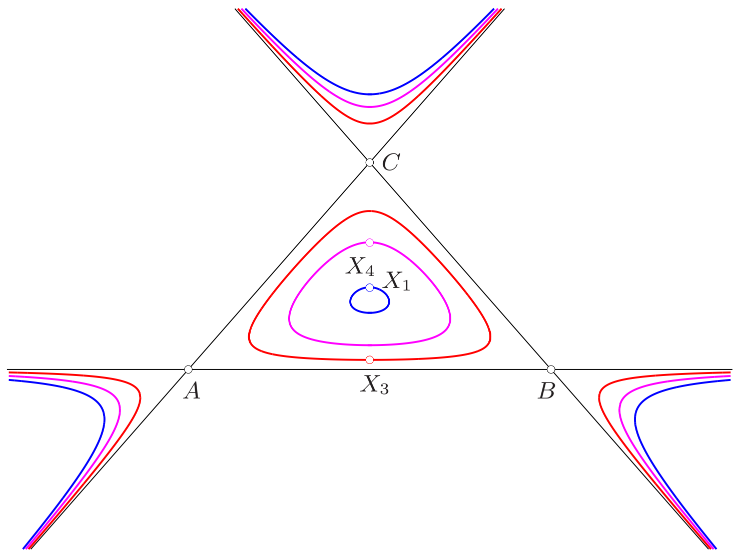

Fig. 3. Distance product cubics of an isosceles triangle share its symmetries with the triangle.

Proof. This result is easily verified by computing the gradient grad k of the cubics' equations (with variable δ) an eliminating, *e.g.*, ζ and η from all three coordinate functions using (2) which results in

$$2^{24}3^2a^3bc^{19}F^{24}\delta^9(27abc\delta + 64F^3)(8F^3 - 27abc\delta)\xi^4 = 0.$$

(The results of the elimination of the other pairings of variables does not differ from this substantially.) For the solution $\xi = 0$, the gradient of k does not vanish. The same holds true for $\xi = 0$ and $\eta = 0$ which show up as solutions in the other elimination processes.

Since $a, b, c \neq 0$, and thus, $F \neq 0$, and also $\delta \neq 0$, we can infer that either $27abc\delta + 64F^3 = 0$ or $8F^3 - 27abc\delta = 0$. The first equation yields $\delta_0 = -\frac{64F^3}{27abc}$ which defines an elliptic cubic without any singularity. The second equation yields the value $\delta_2 = \frac{8F^3}{3abc}$ and defines a rational cubic with an isolated node at $X_2 = (bc : ca : ab)$. $\square$

It is worth noting that $\delta_2 = \frac{8F^3}{27abc}$ can be obtained as the product of the trilinear distances of X_2 (centroid): The homogeneous trilinear coordinates of X_2 are $(a^{-1} : b^{-1} : c^{-1}) = (bc : ca : ab)$. Hence, the actual trilinear coordinates are $\frac{2F}{3abc}(bc, ca, ab)$. Thus, the cubic that collects all points that share the distance product with the centroid has the equation

$$k_2 : \; 27abc\xi\eta\zeta - (a\xi + b\eta + c\zeta)^3 = 0. \tag{3}$$

This curve has an isolated node at X_2 with the complex conjugate pair of tangents given by

$$a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 - ab\eta\xi - ac\xi\zeta - bc\eta\zeta = 0.$$

The cubic curve defined by $\delta = \delta_0$ has the equation

$$k_0 : \; 27abc\xi\eta\zeta + 8(a\xi + b\eta + c\zeta)^3 = 0,$$

has no singularity and does not contain a single triangle center X_i for all $i \leq 12000$ (of the ones listed in [5]).

If a planar cubic has three real points of inflection, these are known to be collinear. In our special case, we have:

Theorem 3. *All cubics* (2) *in the pencil share the three real points of inflection, which are located on the line at infinity.*

Proof. In this case it is not necessary to compute the Hessian curve of $k(\delta)$ and intersect it with each curve in the family.

We recall that $a\xi + b\eta + c\zeta = 0$ is the equation of the ideal line ω in the triangle's plane. Therefore, the intersection of $k(\delta)$ and ω is described by $\xi\eta\zeta = 0$. Consequently, either $\xi = 0$, $\eta = 0$, or $\zeta = 0$ determines the intersections of k and ω. All three points are of multiplicity three and have the homogeneous trilinear coordinates

$$W_1 = (0 : c : -b), \quad W_2 = (-c : 0 : a), \quad W_3 = (b : -a : 0). \tag{4}$$

The side lines of Δ are the inflection tangents of k. Obviously, δ does not appear in the coordinates of any point of inflection. □

Fig. 4 shows a collinear image of a distance product cubic. The three real points of inflection gather on the line ω at infinity.

As a consequence of Theorem 3, any two cubics in the pencil (2) have no other points than the inflection points in common

The harmonic polar of a point P on a planar cubic curve k is defined as follows: Each line l (except the tangent) in the pencil about P meets k in two further points, say Q and R. Then, there exists a unique point S on l such that $\mathrm{H}(Q, R; P, S)$, *i.e.*, S is the harmonic conjugate of P with respect to Q and R. If we do this for all lines in the pencil, we obtain the *harmonic polar* of P as the locus of all points S with respect to k. In our very special case, we have the following result:

Theorem 4. *The harmonic polars of the points of inflection of the cubics* (2) *are the medians of the base triangle.*

Proof. It is well-known that the harmonic polar of an inflection point on a planar cubic is a straight line, cf. [1].

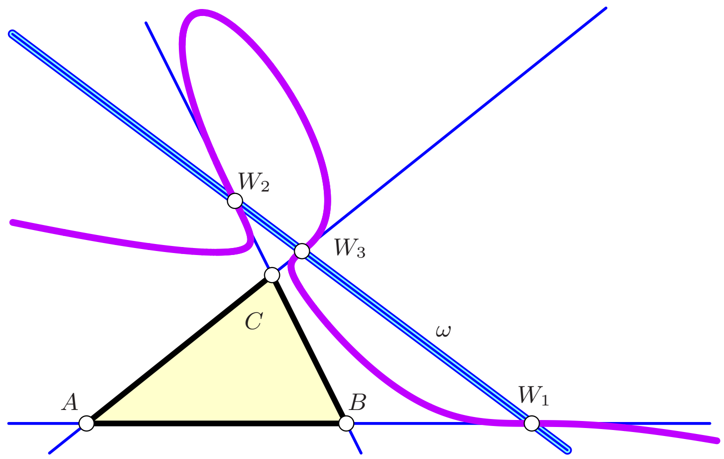

Fig. 4. A distance product cubic k in the projectively extended Euclidean plane has three points W_1, W_2, W_3 of inflection on the line ω at infinity. The tangents to k at the inflection points equal the side lines of the base triangle Δ.

In order to show that the harmonic polars are precisely the medians of the triangle, we compute the harmonic polar for the point $W_1 = (0 : c : -b)$. The remaining polars can be computed in the same way. Assume that the lines in the pencil about W_1 are spanned by W_1 and a further cubic point $P = (\xi' : \eta' : \zeta')$. The lines l in the pencil admit the parametrization $l(\lambda, \mu) = \lambda W_1 + \mu P$ with $\lambda : \mu \neq 0 : 0$. Intersecting the lines with the cubic means inserting the latter parametrization into (2). Since $P \in k$ and $W_1 \in k$, their coordinates satisfy (2), and thus, the two linear factors λ and μ split off from the homogeneous cubic equation. The remaining linear factor yields

$$\lambda : \mu = -b\eta' + c\zeta' : bc,$$

which are the homogeneous coordinates (on l) of the third intersection Q of l and k. The fourth harmonic point (or harmonic conjugate, cf. [3]) is then uniquely defined by

$$\lambda : \mu = -b\eta' + c\zeta' : 2bc.$$

Therefore, $R = (2bc\xi'; bc\eta' + c^2\zeta' : b^2\eta' + bc\zeta')$ which is a homogeneous parametrization of the line $b\eta - c\zeta = 0$.

The remaining harmonic polars are $c\zeta - a\xi = 0$ and $a\xi - b\eta = 0$. These three lines pass through X_2 and the vertices of Δ, which makes them the medians. $\square$

4 Cubics determined by triangle centers

The triangle centers listed in Kimberling's *Encyclopedia of Triangle Centers* [4, 5] determine cubic curves as loci of points with the equal product of trilinear distances. Finding triangle centers located on the same cubic curve is equivalent to finding triangle centers with the same product of trilinear distances. This would be another classification of groups of triangle centers.

Surprisingly, among the many known, listed, and in principle arbitrarily numbered triangle centers, there is only a small number of triangle centers that gather on the same cubic.

Until now, we know that only the following groups of triangle centers are located on the same cubic:

Theorem 5. *The triangle centers with the following groups of triangle centers have the same product of trilinear distances,* i.e.*, they are located on the same distance product cubic:*

$$(1,764),\ (2,3081,6545,8027,8028,8029,8030,8031,8032),\ (4,5489),\ (6,22260),$$
$$(8,21132),\ (25,394),\ (42,321,8034),\ (57,200),\ (75,21143),\ (76,23099),$$
$$(86,21131),\ (99,14444),\ (145,23764),(324,418),\ (459,3079),\ (649,693),$$
$$(669,850,32320),\ (671,14443),\ (756,8042),\ (875,4375,4444),\ (903,14442),$$
$$(1022,3251),\ (1026,3675),\ (1648,5468),\ (1649,5466),\ (1650,4240),$$
$$(2501,3265),\ (3051,8024),\ (3227,14441),\ (3233,12079),\ (3239,3676),$$
$$(3733,4036),\ (4024,7192),\ (4358,8661),\ (4500,4507),\ (6384,8026),$$
$$(6544,6548),\ (8013,8025),\ (8023,8039).$$

Proof. In order to verify the results given in the above theorem, just insert the trilinear representations of the respective centers into to the equations of the curves.

Right before the deadline for this article, we have tested the incidence of triangle centers on cubics up to the Kimberling number 12000. It is not at all surprising that triangle centers with higher Kimberling number occur in the list given in Theorem 5 as we shall see in the next section.

5 How to find triangle centers on such curves?

If trilinear representations of centers are available, then one can simply check by inserting coordinates into cubic forms. But that is not an efficient search. Moreover, it requires the trilinear respresentations of all triangle centers. Many of the triangle centers in [5] miss a coordinate representation. In some cases the trilinear representation involves cubic roots, some may even be not even algebraic. Therefore, in some, cases computer algebra systems fail to evaluate whether a point is on a certain cubic or not.

It is well-known that the tangent T_i at X_i of the triangle cubic k_i (defined by the center X_i) intersects the cubic in a further triangle center, say $R_{1,i}$. Thus, in a first step, we compute the points $R_{1,i}$. The numerical value of the first trilinear

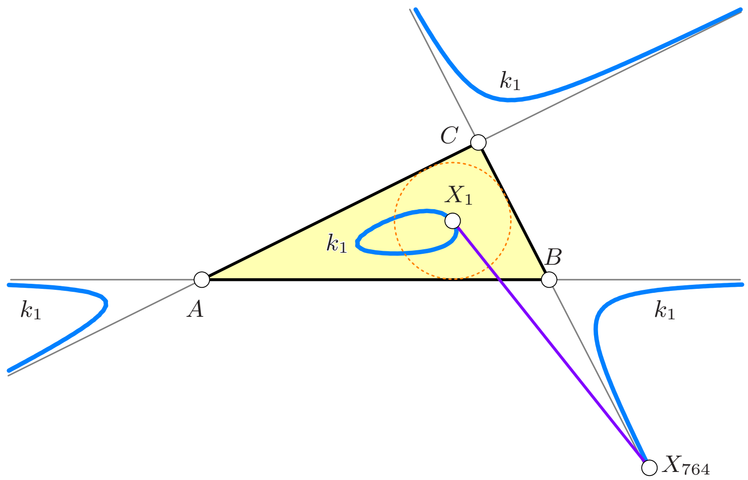

Fig. 5. The distance product cubic k_1 defined by the incenter X_1. The center $X_{764} \in k_1$ is the intersection of k_1's tangent at X_1.

coordinate of $R_{1,i}$ is sufficient and can be compared with the values given on the search page at [5] in order to find out whether $R_{1,i}$ is among the known centers or not.

For example, the point $R_{1,1}$ defined by $X_1 \in k_1$ equals the triangle center X_{764}, see Fig. 5. Table 1 shows the points $R_{1,i}$ for distance product cubics that contain at least two centers. The question marks indicate that these points $R_{1,i}$ are centers which are not yet listed in Kimberlings encyclopedia. In this way, the triangle centers with the high Kimberling numbers enter the scene. For example: $R_{1,6} = X_{22260}$ and $R_{1,145} = X_{23764}$.

X_i	$R_{1,i}$	X_i	$R_{1,i}$	X_i	$R_{1,i}$
1	764	99	14444	3051, 8024	?
4	5489	145	23764	3227	14441
6	22260	324, 418	?	4240	1650
8	21132	459,	3079	4358	8661
25, 294	?	671	14443	5466	1649
42, 321	8034	756, 8042	?	5468	1648
57, 200	?	903	14442	6384, 8026	?
75	21143	1022	3251	6548	6544
76	23099	1026	3675	8013, 8025	?
86	21131	1641	14423	8023, 8039	?

Table 1. The tangent of the triangle center X_i meets the cubic k_i at a further triangle center given in the column $R_{1,i}$. If this remainder is a known triangle center, then its number is given.

Some of the groups of centers with equal distance product mentioned in Theorem 5 do not show up in the Table 1. For example, the centers with indices $i \in (669, 850, 32320)$ define three mutually different points $R_{1,i}$. The latter triple of points forms a triangle, while the centers with the numbers $(875, 4375, 4444)$ are three collinear centers on k_{875} (see Fig. 6). Therefore, we can also search for triangle centers on distance product cubics once we know a chord of a cubic. If $L_{i,j}$ is the line spanned by the two triangle centers X_i and X_j, it meets k_i in a further triangle center $S_{i,j}$, provided that $X_j \in k_i$ ($X_i \in k_i$ holds by the very definition of k_i).

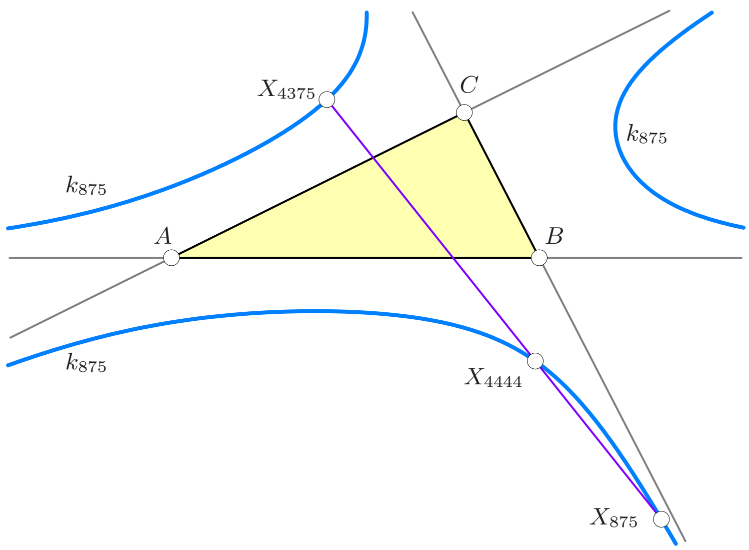

Fig. 6. A triple of three collinear centers on k_{875}: X_{875}, X_{4375}, and X_{4444}.

In some rare cases, two triangle centers on a particular cubic share the point $R_{1,i}$. For example, the tangents to k_{42} at X_{42} and at X_{321} meet in the triangle center X_{8034}, *i.e.*, $R_{1,42} = R_{1,321} = X_{8034}$, see Fig. 7.

The search of further centers on a cubic k_i be means of the computation of $R_{1,i}$ proved useful only in the very beginning, but failed for higher indices (Kimberling numbers). It is obvious that the computation of tangent intersections can be iterated which yields a sequence of points $R_{2,i}$, $R_{3,i}$, ... The algebraic representations of the points $R_{j,i}$ for increasing j are getting more complicated the larger the value j. We did not find any known triangle center as a point $R_{2,i}$.

Note that the cubic k_2 defined by the centroid X_2 contains a group of nine centers X_i with $i \in \{2, 3081, 6545, 8027, 8028, 8029, 8030, 8031, 8032\}$ (see Fig. 8). In this case, any line joining X_2 with any other already detected center on the cubic meets k_2 in X_2 two times, for X_2 is a double point on k_2.

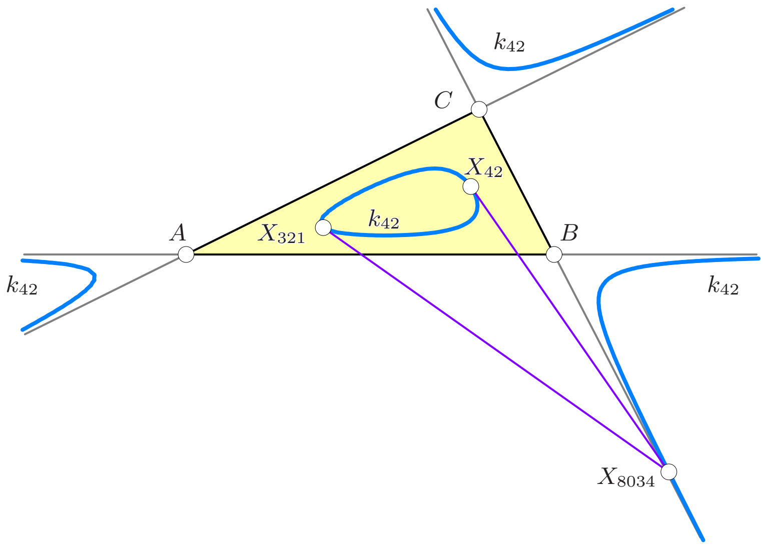

Fig. 7. The tangents to k_{42} at X_{42} and X_{321} meet in $X_{8034} \in k_{42}$.

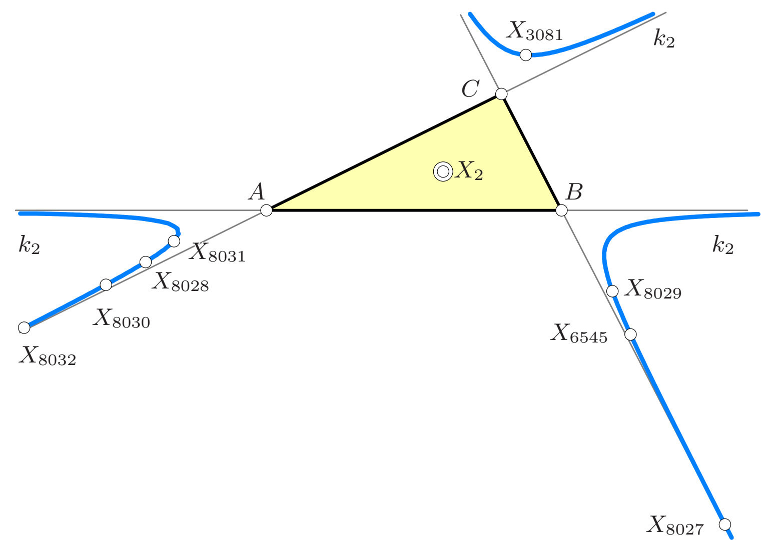

Fig. 8. The cubic k_2 defined by the centroid with nine triangle centers on it: Although X_2 seems not to lie on the curve, it does and equals the isolated node.

6 Future work

It is well-known that elliptic (cubic) curves carry a group structure (cf. [7]). This allows us to add points: Let U be an arbitrary point on an elliptic curve k. For any two points P and Q on k, there is a unique third point $R \in k$ which is collinear with P and Q. We shall briefly write $R = P \circ Q := [P, Q] \cap k \setminus \{P, Q\}$. Then, the sum of P and Q equals the point $P + Q := U \circ R$. (In the case $P = Q$, we define $R := t_P \cap k \setminus \{P\}$.) The point $U \in$ can be chosen arbitrarily.

A point $P \in k$ is said to be of *finite order* n if

$$nP := \underbrace{P + \ldots + P}_{n} = U$$

(with $n \in \mathbb{N} \setminus \{0\}$). In the theory of elliptic curves, points of *finite order* play an important role. The points of order 2 and 3 are well-known (cf. [7]) and can be easily characterized.

It would be interesting to see triangle centers of finite order on triangle cubics. Until now, on neither of the cubics on Gibert's page [2] triangle centers of finite order are known. Only some of the distance product cubics of triangle centers allow for a computation of multiples of centers within reasonable time. Clearly, the trilinear (and also the barycentric) representations of multiples of centers become more and more complicated in each step. Maybe, at least on k_1 such a center of finite order can be found.

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