Some triangle centers associated with the circles tangent to the excircles

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Abstract

We study those tritangent circles of the excirlces of a triangle which enclose exactly one excircle and touch the two others from the outside. It turns out that these three circles share exactly the Spieker point. Moreover we show that these circles give rise to some triangles which are in perspective with the base triangle. The respective perspectors turn out to be new polynomial triangle centers.

1 Introduction

Let $\Delta := \{A, B, C\}$ be a triangle in the Euclidean plane. Let Γ_i be the excircles with centers I_i and radii ρ_i with $i \in \{1, 2, 3\}$. We assume that Γ_1 lies opposite to A.

There are eight circles tangent to all three excircles: the side lines of Δ (considered as circles with infinitely large radius), the Feuerbach circle (cf. [2, 4]), the so-called Apollonius circle (enclosing all the three circles Γ_i, see for example [3, 6, 9]) and three remaining circles which will in the following be denoted by K_i. Any circle K_i encloses a circle Γ_i and touches the remaining circles from the outside. Until now only their radii are known, see [1].

In the following we focus on the circles K_i and show that these circles share the Spieker point X_{10}. Further the triangle of contact points $\{K_{1,1}, K_{2,2}, K_{3,3}\}$ of Γ_i and K_i is perspective to Δ. Surprisingly the triangle built by the centers M_i of circles K_i is also perspective to Δ.

2 Main results

The problem of constructing the tritangent circles to three given circles is well studied. Applying the ideas of M. Gergonne [5] to the three excircles Γ_i we see that the construction of the tritangent circles K_j is linear. At first we find the contact points $K_{i,j}$ of Γ_i with K_j (see Fig. 1): We project the contact points $I_{i,j}$ of the i-th excircle with the j-th side[1] of Δ from the radical center of the excircles to the i-th excircle. The radical center of Γ_i is the Spieker point, cf. [8]. Knowing the contact points of $K_{i,j}$ it is easy to find the centers M_j of K_j.

[1]The lines $[B, C]$, $[C, A]$, $[A, B]$ are the first, second, and third side of Δ.

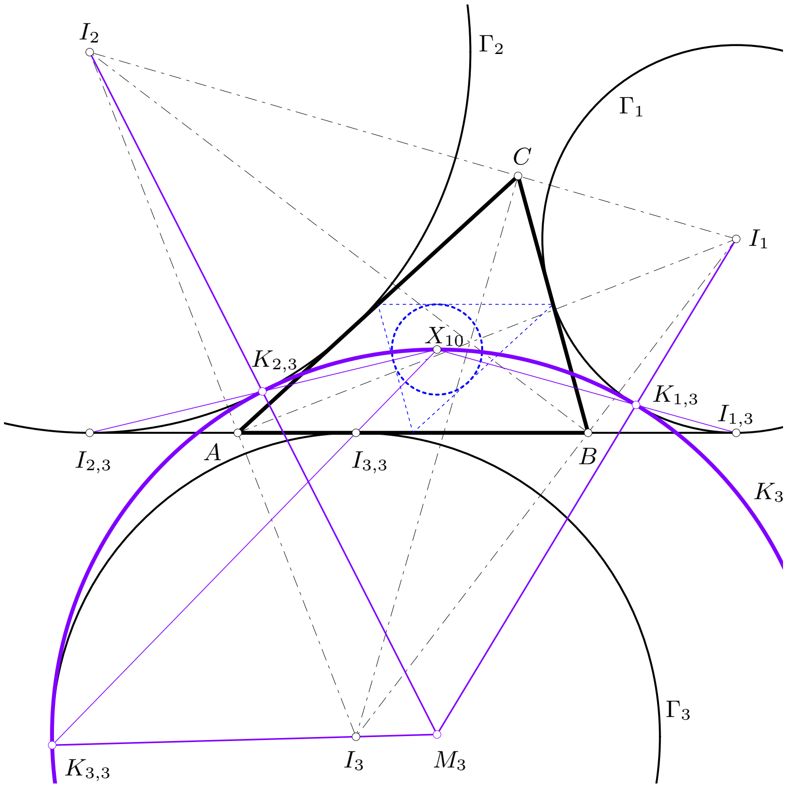

Figure 1: Linear construction of the tritangent circle enclosing Γ_3.

By simple and straight foreward computation following the construction of the contact points $K_{i,j}$ of either circle K_i with the excircle Γ_j we can show:

Theorem 2.1
The triangle $\{K_{1,1}, K_{2,2}, K_{3,3}\}$ built by the contact points of K_i with the i-th excircle Γ_i is in perspective with Δ, see Fig. 2. The perspector P_K is has homogeneous barycentrics

$$P_K = \left(\frac{s-a}{a^2} : \frac{s-b}{b^2} : \frac{s-c}{c^2} \right). \tag{1}$$

Proof: We denote the contact points of Γ_i with the j-th side of Δ by $I_{i,j}$. Their homogeneous barycentrics are

$$\begin{array}{llll} I_{1,1} = (0 : s-b : s-c), & I_{1,2} = (b-s : 0 : s), \\ I_{1,3} = (c-s : s : 0), & I_{2,1} = (0 : a-s : s), \\ I_{2,2} = (s-a : 0 : s-c), & I_{2,3} = (s : c-s : 0), \\ I_{3,1} = (0 : c : a-s), & I_{3,2} = (s : 0 : b-s), \\ I_{3,3} = (s-a : s-b : 0), & \end{array} \tag{2}$$

where $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$ are the side lengths of Δ and $s = (a+b+c)/2$ is the half of the perimeter. Note that $K_{i,j} = [I_{i,j}, X_{10}] \cap \Gamma_j \setminus I_{i,j}$ and the Spieker point is given by

$$X_{10} = (b+c : c+a : a+b),$$

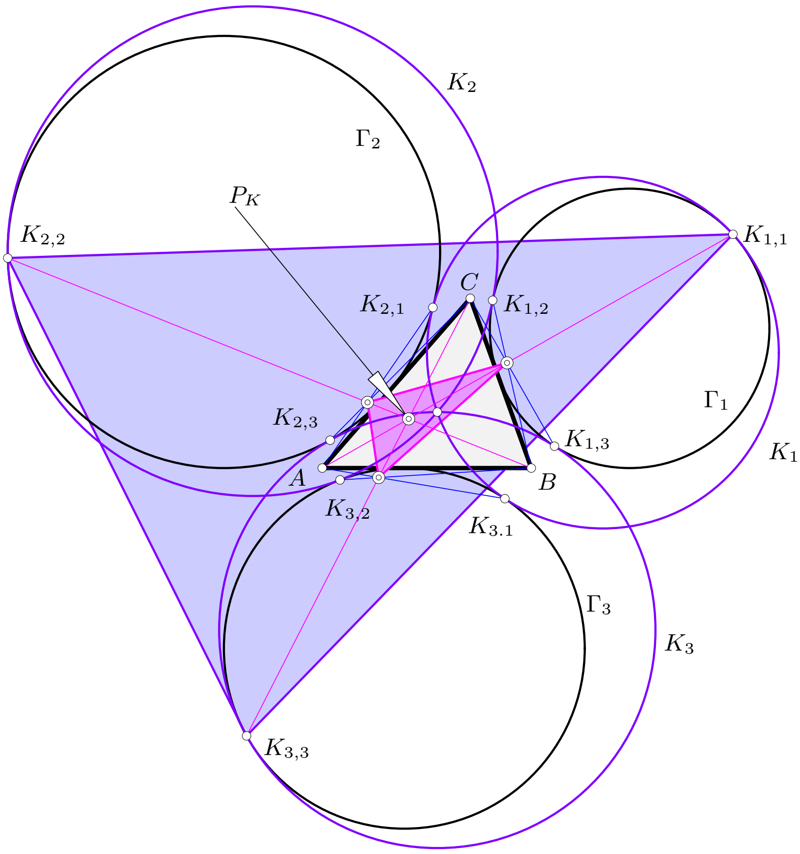

Figure 2: Two triangles in perspective with Δ: the triangle of contact points $K_{i,i}$, the center P_K from Th. 2.1, and the triangle mentioned in Th. 2.2.

cf. [7, 8]. In the following we use the abbreviations $\widehat{a} := b + c$, $\widehat{b} = a + c$, and $\widehat{c} = a + b$.

The equations of the incircles in terms of homogeneous barycentric coordinates can be written in the form $x^{\mathrm{T}} B_i x = 0$, where $x^{\mathrm{T}} = (x_0, x_1, x_2)$ is the vector collecting the homogeneous barycentric coordinates of a point X and B_i is a (regular) symmetric 3×3-matrix. The three coefficient matrices are

$$
\begin{aligned}
B_1 &= \begin{bmatrix} s^2 & s(s-c) & s(s-b) \\ s(s-c) & (s-c)^2 & -(s-b)(s-c) \\ s(s-b) & -(s-b)(s-c) & (s-b)^2 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} (s-c)^2 & s(s-c) & -(s-a)(s-c) \\ s(s-c) & s^2 & s(s-a) \\ -(s-a)(s-c) & s(s-a) & (s-a)^2 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} (s-b)^2 & -(s-a)(s-b) & s(s-b) \\ -(s-a)(s-b) & (s-a)^2 & s(s-a) \\ s(s-b) & s(s-a) & s^2 \end{bmatrix}.
\end{aligned} \tag{3}
$$

It is elementary to verify that the homogeneous barycentrics of the contact points $K_{i,j}$ are given by:

$$\begin{array}{lllll}
K_{1,1}=(& -\widehat{a}^2(s-b)(s-c) & : & c^2s(s-b) & : & b^2s(s-c)), \\
K_{1,2}=(& -c^2s(s-b) & : & \widehat{b}^2(s-b)(s-c) & : & (as+bc)^2), \\
K_{1,3}=(& -b^2s(s-c) & : & (as+bc)^2 & : & \widehat{c}^2(s-b)(s-c)), \\
K_{2,1}=(& \widehat{a}^2(s-a)(s-c) & : & -c^2s(s-a) & : & (bs+ac)^2), \\
K_{2,2}=(& c^2s(s-a) & : & -\widehat{b}^2(s-a)(s-c) & : & a^2s(s-c)), \\
K_{2,3}=(& (bs+ac)^2 & : & -a^2s(s-c) & : & \widehat{c}^2(s-a)(s-c)), \\
K_{3,1}=(& \widehat{a}^2(s-a)(s-b) & : & (cs+ab)^2 & : & -b^2s(s-a)), \\
K_{3,2}=(& (cs+ab)^2 & : & \widehat{b}^2(s-a)(s-c) & : & -a^2s(s-b)), \\
K_{3,3}=(& b^2s(s-a) & : & a^2s(s-b) & : & -\widehat{c}^2(s-a)(s-b)).
\end{array} \tag{4}$$

Now it is easily verified that the three vectors $A\times K_{1,1}$, $B\times K_{2,2,}$, and $C\times K_{3,3}$ are linearly dependent und thus the respective lines meet in the point given in (1). Obviously the point P_K is a triangle center which is not listed in [7]. □

Moreover with the computations from the previous proof we can show:

Theorem 2.2
The lines $[K_{1,1},A]$, $[K_{1,2},B]$, $[K_{1,3},C]$ are concurrent. The lines $[K_{2,1},A]$, $[K_{2,2},B]$, $[K_{2,3},C]$ are concurrent. The lines $[K_{3,1},A]$, $[K_{3,2},B]$, $[K_{3,3},C]$ are concurrent.

Proof: Use Eq. (4) compute the barycentric coordinates of the lines and show the linear dependency. □

Remark: The loci of concurrency of the three triplets of lines mentioned in Th. 2.2 are three points building a triangle which is itself in perspective with Δ. The pespector is the point P_K, cf. Fig. 2.

Theorem 2.3
The triangle $\{M_1,M_2,M_3\}$ built by the centers of the circle K_1, K_2, and K_3 is in perspective with Δ, see Fig. 3. The perspector is the point P_M with homogeneous barycentrics $(\alpha:\beta:\gamma)$, where

$$\alpha=(a^5+a^4\widehat{a}-a^3(b-c)^2-a^2\widehat{a}(b^2+c^2)-2abc(\widehat{a}^2-bc)-2\widehat{a}b^2c^2)^{-1}. \tag{5}$$

Proof: The center M_3 of the tritangent circle K_3 (enclosing Γ_3) is found as the meet of two straight lines

$$M_3=[I_1,K_{1,3}]\cap[I_2,K_{2,3}].$$

Analogously we find the remaining centers. Their barycentric coordinates are thus

$$\begin{aligned}
M_1 &= (-2\widehat{a}a^4-(3\widehat{a}^2-2bc)a^3+\widehat{a}(b^2+c^2)a^2+\widehat{a}^2(b-c)^2a-\widehat{a}^3(b-c)^2 : \\
&\quad : (\widehat{a}^2+b^2)a^3+\widehat{a}(2b^2+c^2)a^2+\widehat{a}c(b-c)(2b+c)a+\widehat{a}^2c^2(b-c) : \\
&\quad : (\widehat{a}^2+c^2)a^3+\widehat{a}(b^2+2c^2)a^2-\widehat{a}b(b-c)(b+2c)a-\widehat{a}^2b^2(b-c)), \\
M_2 &= ((\widehat{b}^2+a^2)b^3+\widehat{b}(2a^2+c^2)b^2+\widehat{b}c(a-c)(2a+c)b+\widehat{b}^2c^2(a-c) : \\
&\quad : -2\widehat{b}b^4-(3\widehat{b}^2-2ac)b^3+\widehat{b}(a^2+c^2)b^2+\widehat{b}^2(a-c)^2b-\widehat{b}^3(a-c)^2 : \\
&\quad : (\widehat{b}^2+c^2)b^3+\widehat{b}(a^2+2c^2)b^2-\widehat{b}a(a-c)(a+2c)b-\widehat{b}^2a^2(a-c)), \\
M_3 &= ((\widehat{c}^2+a^2)c^3+\widehat{c}(2a^2+b^2)c^2+\widehat{c}b(a-b)(2a+b)c+\widehat{c}^2b^2(a-b) : \\
&\quad : (\widehat{c}^2+b^2)c^3+\widehat{c}(a^2+2b^2)c^2-\widehat{c}a(a-b)(a+2b)c-\widehat{c}^2a^2(a-b) : \\
&\quad : -2\widehat{c}c^4-(3\widehat{c}^2-2ab)c^3+\widehat{c}(a^2+b^2)c^2+\widehat{c}^2(a-b)^2c-\widehat{c}^3(a-b)^2).
\end{aligned} \tag{6}$$

Now the concurrency of the lines $[M_1,A]$, $[M_2,B]$, and $[M_3,C]$ is easily shown. The point P_M is a center and it is not listed in [7]. □

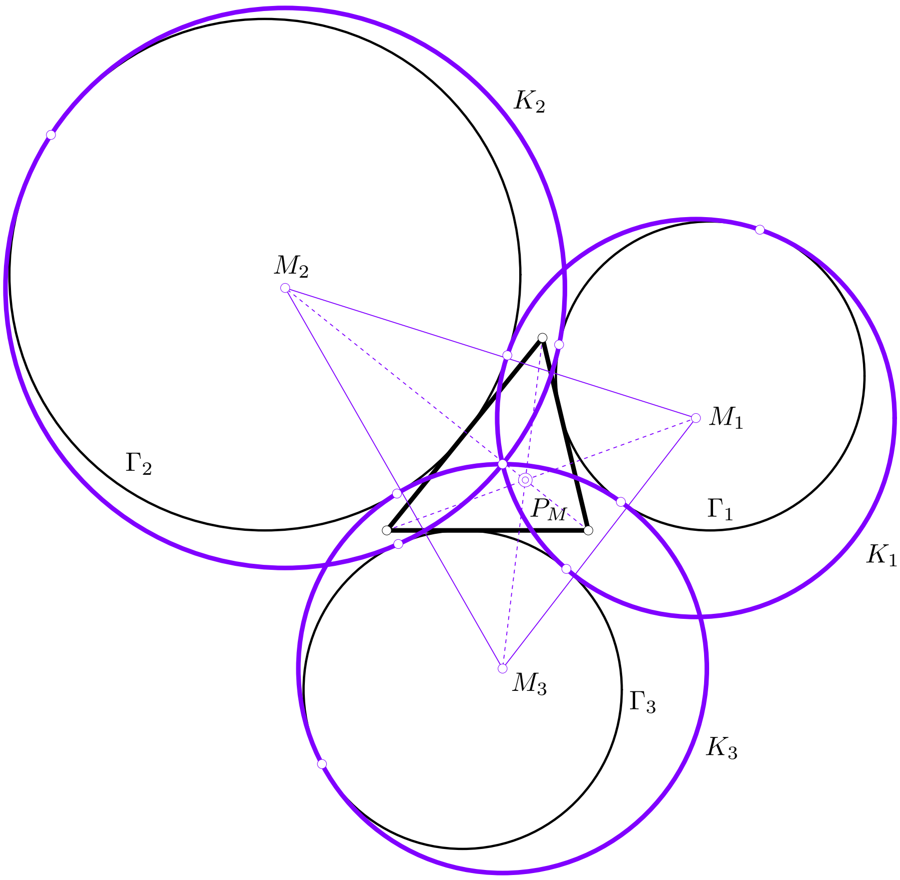

Figure 3: The three tritangent circles, the triangle of the respective centers, and the perspector P_M.

Theorem 2.4
The three tritangent circles K_i enclosing the i-th excircle of Δ and touching the others from the exterior share exactly the Spieker *point X_{10} of Δ.*

Proof: We switch to trilinear coordinates of points and use the formula for the disctance between two points given in terms of their respective actual trilinear coordinates (cf. [8, p. 31] and find the radii R_i of the tritangent circles K_i

$$\widehat{a}R_1 = \widehat{b}R_2 = \widehat{c}R_3 = s(ab+bc+ca)/4F - R, \tag{7}$$

where R is the circumradius of Δ and F is the area of Δ. Eq. (7) is a relation between the radii which is equivalent to those given in [1].
The equations of the circles K_i in terms of homogeneous barycentrics are determined uniquely (up to a non-vanishing scalar) according to [8, p. 223] and can thus be written in the form $x^{\mathrm{T}}C_ix = 0$ with the following (regular) symmetric

3×3-matrices

$$
\begin{aligned}
C_1 &= \begin{bmatrix} -2s(b^2+c^2+a\widehat{a}) & \widehat{a}(c^2-a^2)-ab^2 & \widehat{a}(b^2-a^2)-ac^2 \\ \widehat{a}(c^2-a^2)-ab^2 & 4\widehat{a}(s-a)(s-c) & 2a\widehat{a}s \\ \widehat{a}(b^2-a^2)-ac^2 & 2a\widehat{a}s & 4\widehat{a}(s-a)(s-b) \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 4\widehat{b}(s-b)(s-c) & \widehat{b}(c^2-b^2)-a^2b & 2b\widehat{b}s \\ \widehat{b}(c^2-b^2)-a^2b & -2s(a^2+c^2+b\widehat{b}) & \widehat{b}(a^2-b^2)-bc^2 \\ 2b\widehat{b}s & \widehat{b}(a^2-b^2)-bc^2 & 4\widehat{b}(s-a)(s-b) \end{bmatrix}, \qquad (8) \\
C_3 &= \begin{bmatrix} 4\widehat{c}(s-b)(s-c) & 2c\widehat{c}s & \widehat{c}(b^2-c^2)-a^2c \\ 2c\widehat{c}s & 4\widehat{c}(s-a)(s-c) & \widehat{c}(a^2-c^2)-b^2c \\ \widehat{c}(b^2-c^2)-a^2c & \widehat{c}(a^2-c^2)-b^2c & -2s(a^2+b^2+c\widehat{c}) \end{bmatrix}.
\end{aligned}
$$

By substituting the barycentrics of X_{10} into the equations of K_i we prove that the SPIEKER point is a common point of the three circles K_i. $\square$

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