Beyond the Nine-Point Conic

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Abstract. The nine-point conic n contains the three diagonal points and the midpoints of the six sides of a complete quadrangle. We show that for any quadrilateral $\mathcal{Q} = P_1P_2P_3P_4$ and an arbitrarily chosen point P there exists a conic l passing through ten points: P, the three diagonal points of $\mathcal{Q}$, and the six inverses of the poles of the lines $[P_i, P_j]$ with respect to any circumconic k of $\mathcal{Q}$ and the inversion center P. The circumconic k can be any conic from the pencil circumscribed to $\mathcal{Q}$. The nine-point conic shows up as a special case of the conic l. The projective nature of the definition of the conic l has implications on a certain normal problem of asymptotic quadrilaterals in the hyperbolic plane.

Keywords: Nine-point conic · quadrangle · projective inversion · pencil of conics · hyperbolic plane · asymptotic quadrangle

1 Introduction

Let four points P_1, P_2, P_3, P_4 in the Euclidean plane form a quadrilateral $\mathcal{Q}$, *i.e.*, no three points are collinear. Assume further that $D_1 = [P_1, P_2] \cap [P_3, P_4]$, $D_2 = [P_1, P_3] \cap [P_2, P_4]$, $D_3 = [P_1, P_4] \cap [P_2, P_3]$ are the diagonal points and M_{12}, M_{13}, $\ldots$, M_{34} are midpoints of the edges P_1P_2, P_1P_3, $\ldots$, P_3P_4. Now, it is well-known that the diagonal points and the six midpoints of $\mathcal{Q}$'s edges lie on a single conic n which is frequently referred to as the *nine-point* conic. This result is usually ascribed to the American mathematician M. Bôcher (cf. [3]). However, there is evidence that Bôcher later recognized that W.K. Clifford and J.J. Sylvester may have been aware of the existence of such a conic a little bit earlier (in 1864).

Whitworth's monograph [15] on trilinear coordinates mentions the nine-point conic in two exercises without explicitly calling it a nine-point conic. Perhaps, in [2], this particular conic was called nine-point conic for the first time. Later on, various attempts by means of analytic and synthetic geometry towards the nine-point conic were made in [1, 4, 9, 12, 13] and some spatial analog was described in [7]. It is clear that the midpoints of the quadrilateral's sides are the harmonic conjugates of their ideal points with respect to the pair of incident vertices (as illustrated in Fig. 1). Therefore, the nine-point conic can even be defined in the more general setting of projective geometry as this turned out to be the case for many geometric objects associated to triangles (cf. [10]). Clearly, real projective geometry is not the only framework with a nine-point conic within nine-point conics can be studied. The nine-point conic is well-defined even in Rational Trigonometry, see [8].

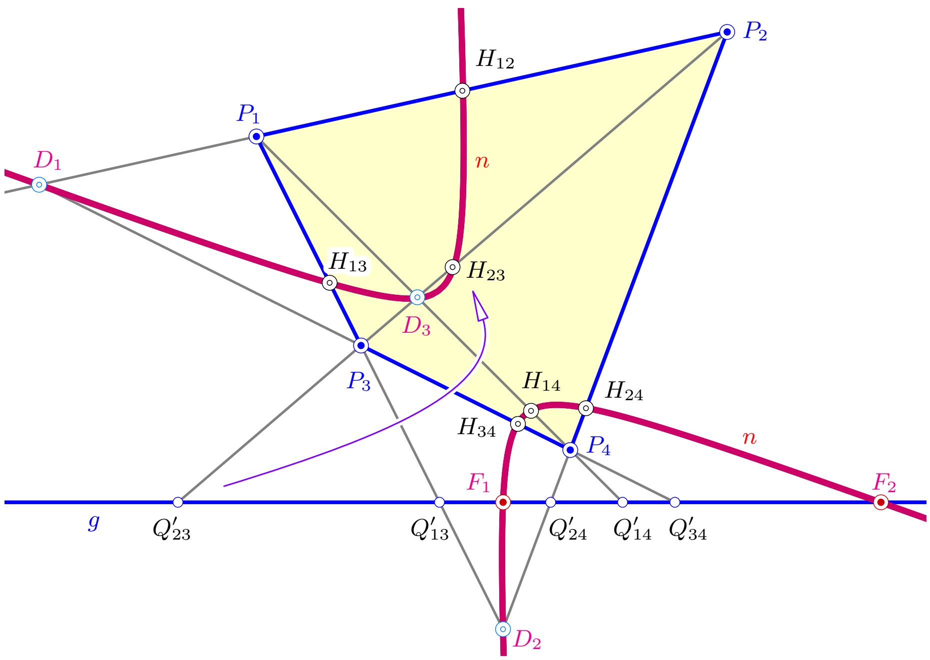

Fig. 1. The nine-point conic n associated with a complete quadrangle $\mathcal{Q} = P_1P_2P_3P_4$ and a straight line g contains the harmonic conjugates H_{12}, H_{13}, ..., H_{34} of the six points $Q'_{ij} = [P_i, P_j] \cap g$, the 3 diagonal points D_1, D_2, D_3, and (if they exist) the fixed points F_1, F_2 of the Desargues involution induced by $\mathcal{Q}$ on g.

In Section 2, we shall present an apparently new result on a complete quadrangle whose vertices lie on a conic. Unfortunately, this result can only be shown by means of computation.

Indeed, any quadrilateral $\mathcal{Q}$ lies on some conic k and any quadrilateral determines a pencil of circumconics. We will see that the pencil of conics on $\mathcal{Q}$ defines a pencil of ten-point conics. In comparison to the definition and construction of the nine-point conic, the arbitrarily chosen line g is replaced by the point P and the conic k. Nevertheless, a certain line instead of g will show up. Section 3 briefly describes the ten-point conic of cyclic quadrilaterals. It will turn out that the nine-point conic is in fact a special case of a ten-point conic, at least for cyclic quadrilaterals with P being the center of the circumcircle. Section 4 is devoted to the pencil of conics defined by the quadrilateral and the associated pencil of ten-point conics. In Section 5, we leave the purely projective setting and even the Euclidean plane and show that the existence of the ten-point conic has implications on totally asymptotic quadrangles in the hyperbolic plane.

2 Ten points on one conic

Let k be a conic in the projective plane $\mathbb{P}^2$ and assume that $P_1, \ldots, P_4$ are four (pairwise different) points on k. (It is thereby guaranteed that these points form a quadrilateral, *i.e.*, no three points are collinear.) Further, let P be an *arbitrarily chosen* point in $\mathbb{P}^2$ that does not lie on the tangents T_i of k at P_i. The four given points determine six chords $p_{ij} = [P_i, P_j]$ ($i \neq j$ and $i, j \in \{1, 2, 3, 4\}$) of k with their respective poles P_{ij} with regard to k. The projections of the poles P_{ij} onto the chords p_{ij} shall be denoted by Q_{ij} (cf. Fig. 2). Now, we have the rather surprising result:

Theorem 1. *The six points Q_{ij}, the three diagonal points of $\mathcal{Q}$, and the point P lie on a single conic l.*

This result seems to be unknown and does not appear in the classical literature, neither does it in the newer.

Proof. Unfortunately, we can only give an analytic proof. For that purpose, we impose a projective frame (cf. [6]) such that the given points have the homogeneous coordinates

$$P_1 = 1 : 0 : 0,\ P_2 = 0 : 0 : 1,\ P_3 = 1 : 1 : 1,\ P_4 = 1 : t : t^2, \tag{1}$$

where $t \in \mathbb{R} \setminus \{0, 1\}$ (hence, $P_4 \neq P_1, P_2, P_3$) and

$$P = p_0 : p_1 : p_2.$$

In this case, the conic k circumscribed to P_i is the *standard conic*

$$k:\ x_0x_2 - x_1^2 = 0. \tag{2}$$

(Any other four points can be mapped to these and any arbitrary point will be mapped to P, see [6]).

There are some natural restrictions on the position of P. They can be written in algebraic form as follows.
(i) P may not lie on any of the six lines p_{ij}:

$$p_1(p_1 - p_2)(p_2 - tp_1)(p_1 - p_0)(p_1 - tp_0)(p_0t - (1 + t)p_1 + p_2) \neq 0.$$

(ii) P may not lie on any of the four tangents of k at P_i:

$$p_0p_2(p_0 - 2p_1 + p_2)(t^2p_0 - 2tp_1 + p_2) \neq 0.$$

(iii) P may not lie on the sides of the diagonal triangle of $\mathcal{Q}$:

$$(tp_0 - 2p_1 + p_2)(tp_0 - 2tp_1 + p_2)(tp_0 - p_2) \neq 0.$$

Therefore, none of the latter 13 factors can be zero, and we are allowed to cancel any of them whenever they occur.

Any five of the six points $Q_{ij} = p_{ij} \cap [P, P_{ij}]$ determine a unique conic l whose equation can be derived from a 6×6 determinant, see [6, p. 241, eq. (6.13)]. Independent of the choice of the five points P_{ij}, the equation of l reads

$$\begin{aligned} l:\ & tp_2x_0^2 + 2(tp_1 - tp_2 - p_2)x_0x_1 + (p_2 - tp_0)x_0x_2 + \\ & +2(p_2 - tp_0)x_1^2 + 2(tp_0 + p_0 - p_1)x_1x_2 - p_0x_2^2 = 0. \end{aligned} \tag{3}$$

The coordinates of any remaining sixth point annihilate (3). This holds also true for the three diagonal points and the point P itself. □

Fig. 2 shows the ten-point conic for a cyclic quadrangle. The cyclicity of the points P_i is not a projective property. However, the contents of Thm. 1 are invariant under arbitrary projective transformations, in particular, that transformation that maps the *standard conic* (2) to a circle k.

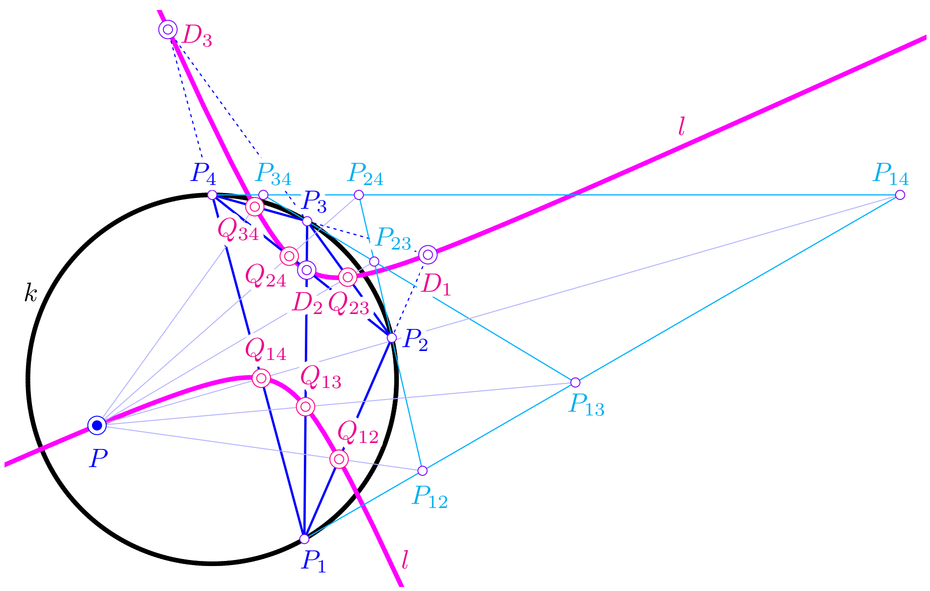

Fig. 2. The ten-point conic l of a cyclic quadrilateral $P_1P_2P_3P_4$.

Now, we can close the gap between the nine-point conic and the ten-point conic:

Theorem 2. *The ten-point conic l described in Thm. 1 is also the nine-point conic of the quadrilateral $\mathcal{Q}$ with respect to the polar line p of P with respect to the conic k.*

Proof. The arbitrarily chosen point P has a polar line p with regard to k. The points Q_{ij} on the polars p_{ij} are the harmonic conjugates of $Q'_{ij} = p \cap [P_i, P_j]$ with respect to the pairs $[P_i, P_j]$, cf. [6, Ch. 7.1]. □

Thm. 2 shows that the results in [4] can be seen from a superordinate standpoint.

As an immediate consequence of Thm. 2, we can formulate:

Theorem 3. *The nine-point conic n of a quadrilateral $\mathcal{Q}$ on a conic k equals the ten-point conic l if P is chosen as the center of k.*

Proof. If P is the center of k, then its polar line with regard to k is the ideal line (line at infinity). The harmonic conjugates of the ideal point of the each of the six lines p_{ij} with respect to the pair (P_i, P_j) (with $i \neq j$) are the midpoints of the segments P_iP_j. □

Moreover, the points Q_{ij} are the images of the poles P_{ij} under the projective inversion $\iota: \mathbb{P}^{2\star} \to \mathbb{P}^{2\star}$ (acting on the projective plane $\mathbb{P}^2$ sliced along a triangle) in k with center P (cf. [6, p. 343]).

3 Cyclic quadrilaterals

We find a very special situation if the initial conic k is chosen as the Euclidean unit circle and P as its center. The polar line of P (the center of k) is the ideal line ω (line at infinity) of the projectively closed Euclidean plane. Thus, the points Q'_{ij} are the ideal points of the lines p_{ij} and their harmonic conjugates Q_{ij} (inverses of the poles P_{ij}) are the midpoints of the segments P_iP_j. Then, the projective inversion becomes the "ordinary" inversion in a Euclidean circle. In terms of Cartesian coordinates, the unit circle k has the equation

$$x^2 + y^2 = 1 \tag{4}$$

which can be written in terms of complex coordinates as

$$z\overline{z} = 1,$$

where $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$ is the complex conjugate of z. With the help of complex coordinates, the inversion is described by $\iota: \ z \mapsto \frac{1}{\overline{z}}$.

If the four given points $P_1, \dots, P_4$ are now described by four complex numbers a, b, c, d of norm 1, then the equation of the ten-point conic l can be given in the form

$$l: \ 2z^2 - 2abcd\,\overline{z}^2 - (a+b+c+d)z + (abc + abd + acd + bcd)\overline{z} = 0. \tag{5}$$

The points Q_{ij} are then given by $\frac{1}{2}(a+b)$, …, $\frac{1}{2}(c+d)$ and it is a matter of simple computations to show that the center of l equals

$$\frac{1}{4}(a+b+c+d),$$

i.e., that is the *centroid* of the quadrilateral $\mathcal{Q} = abcd$. In this very special case, the conic l coincides with the ordinary nine-point conic. In any case, l from (5) is an equilateral hyperbola since its ideal points are

$$F_{1,2} = 0 : abcd - 1 : \mathrm{i}(1 \pm \sqrt{abcd})^2$$

which are conjugate in the elliptic involution on the ideal line (that joins ideal points of pairs of orthogonal directions).

We can summarize (cf. Thm. 3):

Theorem 4. *For a cyclic quadrilateral with P being the center of the circumcircle, the ten-point l conic equals the ordinary nine-point conic n.*

The conic l is not a notion of inversive geometry since its inverse in the circle k equals the cubic curve

$$\iota(l):\ 2z^2 - 2abcd\,\overline{z}^2 - (a+b+c+d)z^2\overline{z} + (abc+abd+acd+bcd)z\overline{z}^2 = 0$$

which is a *strophoid* (circular cubic with orthogonal tangents at the double point, cf. [14]). Its node lies in the center of k, *i.e.*, the center of inversion (see Fig. 3).

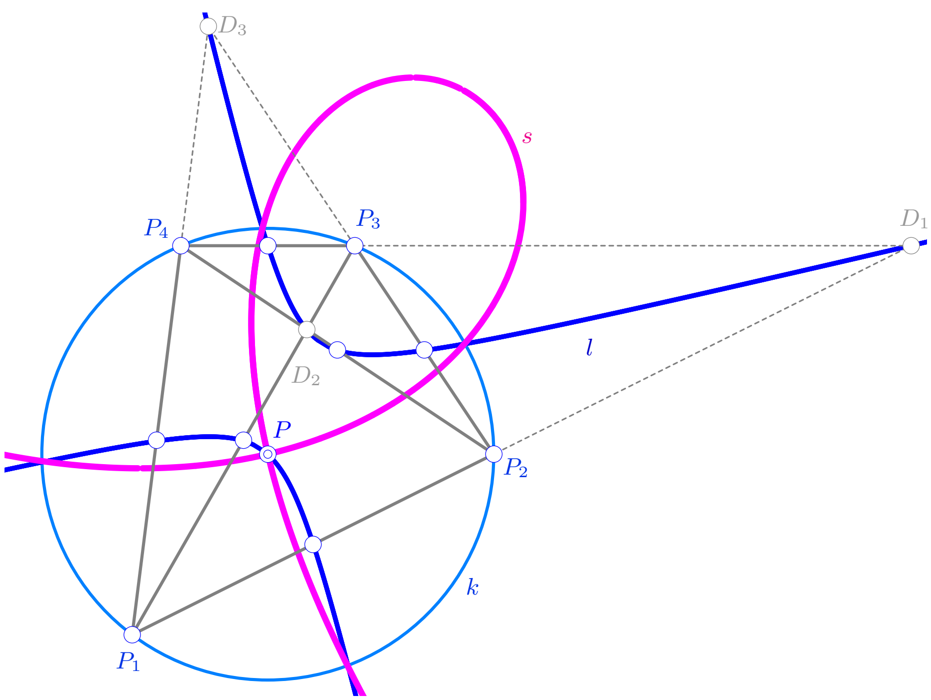

Fig. 3. The ten-point conic l of a cyclic quadrilateral $P_1P_2P_3P_4$ is an equilateral hyperbola. Its inverse in k (center of inversion = center of k) is a strophoid s.

4 A property of pencils

The four points we have chosen prior to Thm. 1 can be considered as the base points of a pencil of conics of the first kind, *i.e.*, the one-parameter family of all conics passing through these four points (cf. [6]). The constructions done above are invariant under projective transformations: Tangents, polars, joins of points, and intersections of lines are not altered under collineations and correlations. Consequently, we can state the following:

Theorem 5. *Let $P_1, \dots, P_4$ form a quadrilateral in a projective plane, let c be any conic from the pencil of the first kind spanned by the quadrilateral, and let further P be a point not contained in any line $[P_i, P_j]$ ($i \neq j, i, j \in \{1,2,3,4\}$). Then, the projections of the six poles P_{ij} of $[P_i, P_j]$ with regard to c from P onto $[P_i, P_j]$ lie on a single conic n which also houses P and the diagonal points of the quadrilateral.*

If we use the assumptions made on the coordinatization in the proof of Thm. 1, we can span the pencil of conics through $P_1, \dots, P_4$ by k and a singular conic in the pencil, *e.g.*, by the union of the lines $[P_1, P_3]$ and $[P_2, P_4]$. This yields the equations of the conics as

$$B(\lambda,\mu):\ \lambda\underbrace{(x_2-x_1)(x_1-tx_0)}_{[P_1,P_3]\cup[P_2,P_4]}+\mu\underbrace{(x_0x_2-x_1^2)}_{k}=0,$$

where $\lambda : \mu \neq 0 : 0$ is a homogeneous parameter. Proceeding in the same way as done in the proof of Thm. 1, we arrive at the equations of the ten-point conics associated with the conics of the pencil

$$\begin{aligned} l(\lambda,\mu):\ & \lambda\big(tx_0-2tx_1+x_2\big)\big(t(p_1-p_2)x_0+(p_2-tp_0)x_1+(tp_0-p_1)x_2\big)+\\ & \mu\big(tp_2x_0^2+2(tp_1-tp_2-p_2)x_0x_1+(p_2-tp_0)x_0x_2+2(p_2-tp_0)x_1^2+\\ & \quad +2(tp_0+p_0-p_1)x_1x_2-p_0x_2^2\big)=0 \end{aligned}$$

Obviously, the conics $l(\lambda, \mu)$ form a pencil. Since the determinant of the coefficient matrix equals (up to non-vanishing factors)

$$4\mu(t\lambda-\mu)(\lambda(t-1)-\mu),$$

the singular conics in this pencil correspond to

$$\mu_1=0,\quad \mu_2=\lambda t,\quad \mu_3=\lambda(t-1).$$

These are the pairs of lines

$$\begin{aligned} s_{1,2}:\ & (tx_0-2tx_1+x_2)(t(p_1-p_2)x_0+(p_2-tp_0)x_1+(tp_0-p_1)=0,\\ s_{3,4}:\ & (tx_0-x_2)(tx_0p_1-tx_1p_0-x_1p_2+x_2p_1)=0,\\ s_{5,6}:\ & (tx_0-2x_1+x_2)(tx_0p_1-tx_1p_0-x_0p_2+x_1p_2+x_2p_0-x_2p_1)=0 \end{aligned}$$

forming a complete quadrilateral with the diagonal points

$$\begin{aligned} D_1 &= s_1\cap s_2 = 2t^2p_0-t(p_0+2p_1)+p_2 : t^2p_0+t(p_2-2p_1) : t^2(p_0-2p_1+2p_2)-tp_2,\\ D_2 &= s_3\cap s_4 = tp_0+p_2 : 2tp_1 : t(tp_0+p_2),\\ D_3 &= s_5\cap s_6 = tp_0-2p_0+2p_1-p_2 : t(2p_1-p_0)-p_2 : -t^2p_0+t(2p_1+p_2)-2p_2 \end{aligned}$$

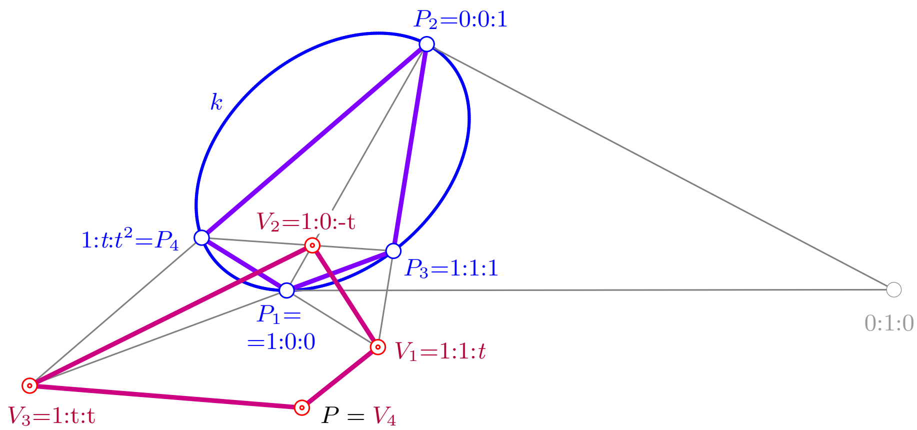

Fig. 4. The standard frame attached to k and the base points of the pencil of ten-point conics as described in Thm. 6

and the vertices

$$\begin{aligned}
V_1 &= s_1 \cap s_3 \cap s_6 = 1 : 1 : t,\\
V_2 &= s_1 \cap s_4 \cap s_5 = 1 : 0 : -t,\\
V_3 &= s_2 \cap s_3 \cap s_5 = 1 : t : t,\\
V_4 &= s_2 \cap s_4 \cap s_6 = p_0 : p_1 : p_2.
\end{aligned}$$

We can summarize this in:

Theorem 6. *Let $P_1, \dots, P_4$ form a quadrilateral $\mathcal{Q}$ in a projective plane with the diagonal points D_1, D_2, D_3. The ten-point conics associated with the pencil of conics (of the first kind) defined by $\mathcal{Q}$ form themselves a pencil of conics (of the first kind) with the pivot P and the diagonal points D_1, D_2, D_3 of $\mathcal{Q}$ for its base points (provided that P does not lie on the sides of the diagonal triangle).*

Fig. 4 shows the construction of the base points of the associated pencil of ten-point conics from the base points of the initial pencil of conics on $\mathcal{Q}$. In Fig. 5, besides the quadrilateral $\mathcal{Q}$ and a conic k on $\mathcal{Q}$, the thus determined ten-point conic l and its projective inverse $s \cap p$ in k (with center of inversion P) is shown. The curve s is a cubic with its node at P and the line p is the polar of P with respect to k.

5 Implications on non-Euclidean quadrilaterals

The projective model of the hyperbolic plane $\mathbb{H}^2$ is the interior of the absolute conic Ω. The points in the interior of Ω are the points of the hyperbolic plane, the hyperbolic lines are the chords of Ω. (A point is an *interior point* of a conic if it does not send (real) tangents to the conic.) The points on Ω are called absolute or improper points of $\mathbb{H}^2$.

For the sake of simplicity, we choose the Euclidean unit circle (4) as the absolute conic Ω which delivers the well-known Cayley-Klein model of the hyperbolic

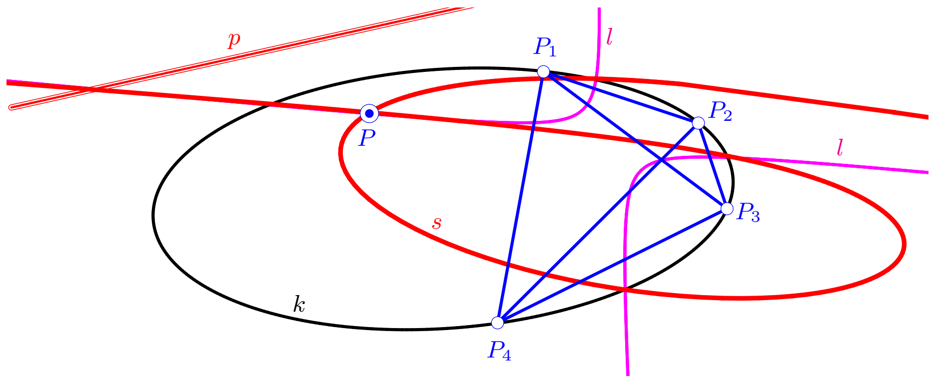

Fig. 5. The cubic curve s is a part of the inverse of l in k (with center P) and can be viewed as a projective version of the strophoid mentioned after Thm. 4.

plane (cf. [5]). A quadrilateral in $\mathbb{H}^2$ is called *asymptotic* or *ideal* if all its vertices lie on Ω.

Now, we can use Thm. 1 to show:

Theorem 7. *Let $\mathcal{Q} = P_1P_2P_3P_4$ be an asymptotic quadrilateral in the hyperbolic plane and let P be an arbitrary point in $\mathbb{H}^2$. Then, the six pedal points of the hyperbolic normals from P to the six sides of the complete quadrilateral on $\mathcal{Q}$ are located on a single conic l independent of the choice of P. P is also located on l.*

Proof. The hyperbolic normal n_{12} through a point P of a hyperbolic line $[P_1, P_2]$ passes through the absolute pole P_{AB} of $[P_1, P_2]$, *i.e.*, the pole of $[P_1, P_2]$ with regard to Ω. Since all vertices of $\mathcal{Q}$ are located on a conic (here it is Ω), the hyperbolic pedals on the lines of the complete quadrangle are the projections of the respective absolute poles onto these lines. Therefore, the hyperbolic pedal points meet the requirements of Thm. 1 and line up on a single conic. □

The behaviour of the locus $\mathcal{C}$ of all points $P \in \mathbb{H}^2$ with conconic pedal conics shows a completely different behaviour than that in the Euclidean plane as shown in [11]. However, in the Euclidean plane this locus is an algebraic curve of degree 7 (in general) or 6 (in special cases) and it can be shown that the degree does not drop below 6 (cf. [11]). In the hyperbolic plane, $\mathcal{C}$ is of degree 12 and with each vertex of the initial quadrilateral that happens to lie on Ω, the degree drops about 3. Hence, the degree of $\mathcal{C}$ assigned to a completely asymptotic quadrilateral is of degree 0, *i.e.*, $\mathcal{C}$ equals the entire hyperbolic plane.

Fig. 6 illustrates the contents of Thm. 7: In fact, the point P can be chosen freely and the six hyperbolic pedal points on the sides of a completely asymptotic quadrilateral are conconic anyhow. The pedal conic l also contains P and the only hyperbolic diagonal point D. Therefore, l is an eight-point conic.

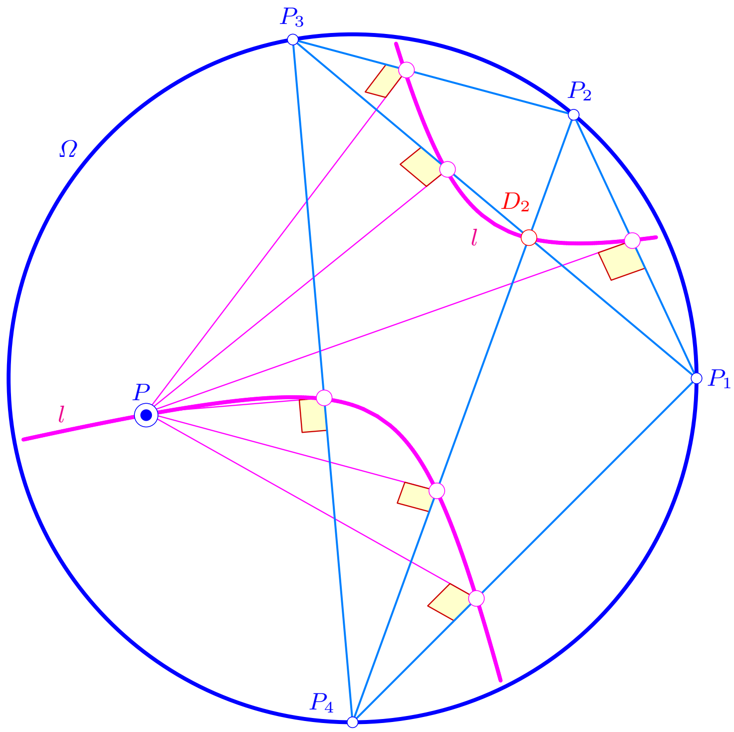

Fig. 6. The six hyperbolic pedal points of P on the sides of a completely asymptotic quadrilateral lie on a single conic l independent of the choice of P. Further, the conic l passes through P and the only (hyperbolic) diagonal point D_2.

6 Final remarks

In Section 4, we have discovered the pencil of ten-point conics associated to a pencil of conics. All the pencils we have met so far are pencils of the first kind, *i.e.*, the one-parameter family of conics through four points (forming a quadrilateral). It would be interesting to see whether we can assign pencils of ten-point conics to pencils of the other (four projectively distinguished) types of pencils. The base quadrilateral will then have 3 or 2 vertices, or even 1 vertex. Maybe it is possible to study these cases by means of limit procedures.

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