Distances and Central Projections

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Abstract

Given a point P in Euclidean space $\mathbb{R}^3$ we look for all points Q such that the length $\overline{PQ}$ of the line segments PQ from P to Q equals the length of the central image of the segment. It turns out that for any fixed point P the set of all points Q is a quartic surface Φ. The quartic Φ carries a one-parameter family of circles, has two conical nodes, and intersects the image plane π along a proper line and the three-fold ideal line p_2 of π if we perform the projective closure of the Euclidean three-space. In the following we shall describe and analyze the surface Φ.

Keywords: central projection, distance, principal line, distortion, circular section, quartic surface, conical node

MSC 2010: 51N20, 14H99, 70B99

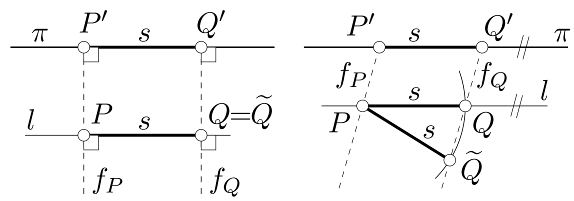

Figure 1: Principal lines: orthogonal projection (left), oblique parallel projection (right).

1 Introduction

It is well-known that segments on lines which are parallel to the image plane π or, equivalently, orthogonal to the fibres of an *orthogonal projection* have images of the same length, *i.e.*, they appear undistorted, see [1, 4, 5, 7]. The lines orthogonal to the fibres of an orthogonal projection are usually called *principal lines* and they are the only lines with undistorted images under this kind of projection.

In case of an *oblique parallel projection*, *i.e.*, the fibres of the projection are not orthogonal (and, of course, not parallel) to the image plane, the principal lines are still parallel to the image plane π. Nevertheless, there is a further class of principal lines in the case of a parallel projection $\iota: \mathbb{R}^3 \to \mathbb{R}^2$. As illustrated in Figure 1, we can see that in between the parallel fibres f_P and f_Q of two arbitrary points P and Q on a principal line $l \parallel \pi$ we can find a second segment emanating from P and ending at $\widetilde{Q}$ with $\overline{PQ} = \overline{P\widetilde{Q}} = \overline{P'Q'}$. (Here and in the following we write P' for the image point of P instead of $\iota(P)$.) In case of an orthogonal projection, we have $Q = \widetilde{Q}$, cf. Figure 1.

In both cases, the orthogonal projection and the oblique parallel projection, the principal lines are mapped *congruent* onto their images.

What about the central projection? Let $\kappa : \mathbb{R}^3 \setminus \{O\} \to \pi$ be the a central projection with center (eyepoint) O and image plane π. For the sake of simplicity, we shall write P' instead of $\kappa(P)$. Again the lines parallel to π serve as principal lines. Of course, the restriction $\kappa|_l$ of κ to a line $l \parallel \pi$ is a similarity mapping. The mapping $\kappa|_l$ is a congruent transformation if, and only if, $l \subset \pi$ because it is the identity in this case.

From Figure 2 we can easily guess that even in the case of central projections there are more line segments than those in the image plane π having central images of the same length. Once we have chosen a point P on the fibre f_P through P' we can find up to two points Q, $\widetilde{Q}$ on the fibre f_Q through Q' such that $\overline{P'Q'} = \overline{PQ} = \overline{P\widetilde{Q}}$ holds as long as $\overline{Pf_Q} < \overline{P'Q'}$. The points Q and $\widetilde{Q}$ coincide exactly if $\overline{Pf_Q} = \overline{P'Q'}$. Finally, there are no points Q and $\widetilde{Q}$ if $\overline{Pf_Q} > \overline{P'Q'}$.

In the case of a central projection κ, only the lines in the image plane are mapped *congruent* onto their images. All the other lines which carry segments whose images are of the same length are *not mapped congruent* onto their images. Just one segment on all these lines has a κ-image of the same length.

Note that if either Q or P equals O the line $[P, Q]$ is mapped to a point. Thus $s = \overline{PQ} \neq \overline{P'Q'}$ since the latter quantity is undefiend for either Q' or P' does not exist.

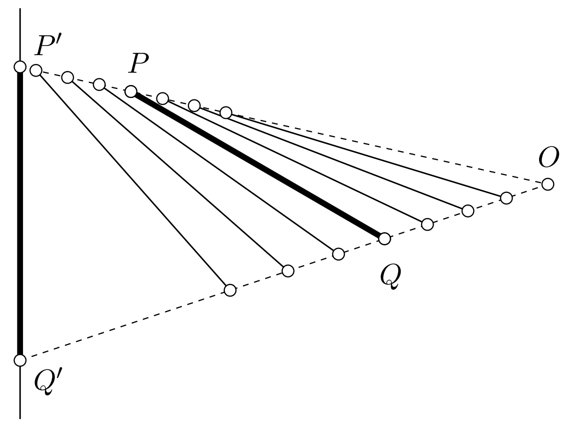

Figure 2: Some of infinitely many segments of length s with the same image $P'Q'$ and, therefore, also of length s.

Assume further that $P \neq O$ is an arbitrary point in Euclidean three-space. Now we can ask for the set of all points Q at fixed distance, say $s \in \mathbb{R} \setminus \{0\}$, such that

$$s = \overline{PQ} = \overline{P'Q'} \tag{1}$$

where $P' := \kappa(P)$ and $Q' = \kappa(Q)$ and $s \in \mathbb{R} \setminus \{0\}$. The left-hand equation of (1) can also be skipped. Then, we are looking for all points Q being the endpoints of line segments emanating from P whose central image has the same length. It is clear that the set of all Q is an algebraic surface. In Section 2 we shall describe and analyze this surface in more detail. Section 3 is devoted to the study of algebraic properties of this surface. Surprisingly, this type of quartic surface does appear among the huge number of quartic surfaces in [3].

In the following $\mathbf{x} = (x, y, z)^{\mathrm{T}} \in \mathbb{R}^3$ are Cartesian coordinates. For any two vectors $\mathbf{u}$ and $\mathbf{v}$ from $\mathbb{R}^3$ we denote the canonical scalarproduct by

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_x v_x + u_y v_y + u_z v_z.$$

Based on the canonical scalarproduct, we can compute the length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

2 The set of all end-points

In the following we assume that there is the central projection $\kappa : \mathbb{R}^{3*} \to \pi \cong \mathbb{R}^2$ with the *image plane* π where $\mathbb{R}^{3*} := \mathbb{R}^3 \setminus \{O\}$ and $O \notin \pi$ shall be the center of the projection, *i.e.*, the *eyepoint*. The *principal (vanishing) point* $H \in \pi$ is π's closest point to the eyepoint O and $d := \overline{OH} = \overline{O\pi}$ is called the *distance* of κ. Therefore, H is the pedalpoint of the normal from the eyepoint O to the image plane π.

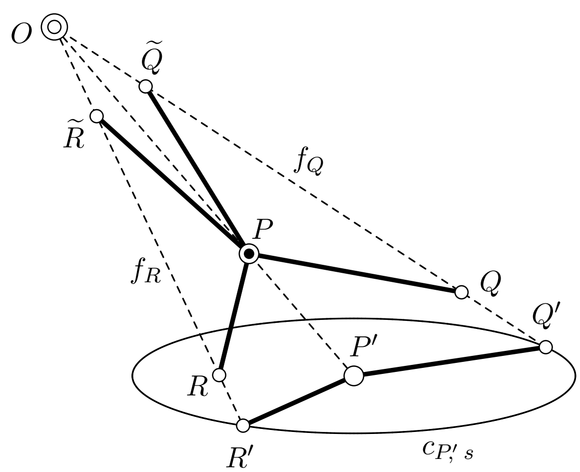

Figure 3: Line segments in π and their equally long preimages.

Let us assume that $P \in \mathbb{R}^{3*} \setminus \{\pi\}$ is a point in Euclidean three-space (neither coincident with O nor in π). With $P' = [O, P] \cap \pi$ we denote the κ-image of P. The set of all points $Q' \in \pi$ with a certain fixed distance $s \in \mathbb{R} \setminus \{0\}$ from P' is a circle $c_{P',s}$ in the image plane π centered at P' with radius s, see Figure 3.

We find all possible preimages of Q' on the quadratic cone $\Gamma_{P',s} = c_{P',s} \vee O$ of κ-fibres through all points on $c_{P',s}$. The preimages shall satisfy

$$s = \overline{P'Q'} = \overline{PQ}$$

and, therefore, they are located on a Euclidean sphere $\Sigma_{P,s}$ centered at P with radius s. Consequently, we can say:

Theorem 1. *The set of all points $Q \in \mathbb{R}^3$ with $\overline{PQ} = \overline{P'Q'} = s \in \mathbb{R} \setminus \{0\}$ for some point $P \in \mathbb{R}^{3*} \setminus \{\pi\}$ is a quartic space curve q being the intersection of a sphere $\Sigma_{P,s}$ (centered at P with radius s) with a quadratic cone $\Gamma_{P',s}$ whose vertex is the eyepoint O and the circle $c_{P',s}$ (lying in π, centered at P's κ-image P', and with radius s) is a directrix.*

The quartic curve q mentioned in Theorem 1 has always two branches, since the two points on each generator f_Q of $\Gamma_{P',s}$ are the points of intersection of the generator f_Q with the sphere $\Sigma_{P,s}$. Therefore, q is in general not rational. An example of such a quartic is displayed in Figure 4 where the sphere $\Sigma_{P,s}$ and the cone $\Gamma_{P',s}$ are also shown.

Not even in the cases $[O, P] \perp \pi$ and $P \in \pi$ an exeption occurs: q happens to be the union of two circles (rational curves). However, the union of rational curves is (in general) not rational. In the first case $\Gamma_{P',s}$ is a cone of revolution and $\Sigma_{P,s}$ is centered on

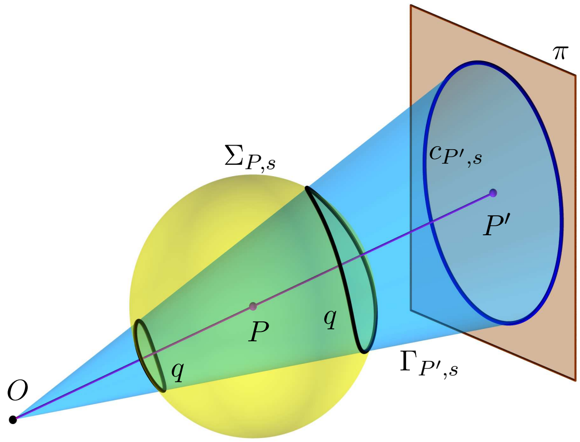

Figure 4: The quartic curve q of possible endpoints of line segments starting at P with length s and equally long image segments. The curve q is the intersection of the quadratic cone $\Gamma_{P',s}$ and the sphere $\Sigma_{P,s}$.

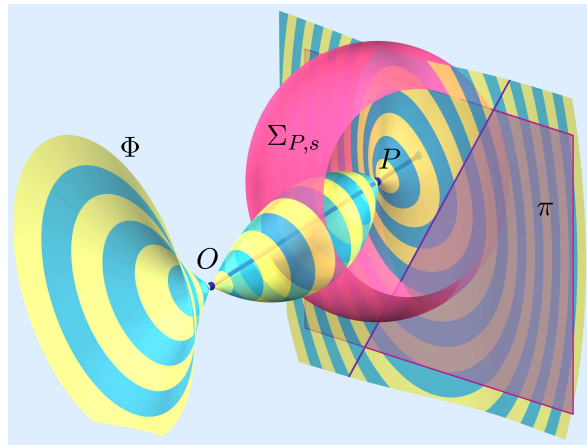

Figure 5: The linear one-parameter family of spherical quartic curves covers a quartic surface.

the cone's axis. Consequently, q degenerates and becomes a pair of parallel circles on both surfaces. In the second case the quartic q is also the union of two circles, namley $c_{P',s}$ and a further circle on $\Sigma_{P,s}$ and $\Gamma_{P',s}$.

Figure 4 shows an example of such a quartic curve (in the non-rational or generic case) carrying the preimages of possible endpoints Q.

As the length s of PQ as well as of $P'Q'$ can vary freely, there is a linear family of quartic curves depending on s. Thus, from Theorem 1 we can deduce the following:

Theorem 2. *The set of all points Q being the endpoints of line segments PQ starting at an arbitrary point $P \in \mathbb{R}^{3*} \setminus \{\pi\}$ with $\overline{PQ} = \overline{P'Q'}$ is a quartic surface Φ.*

Proof. There exists a $(1,1)$-correspondence between the pencil of quadratic cones $\Gamma_{P',s}$ and the pencil of spheres $\Sigma_{P,s}$. Consequently, the manifold of common points, *i.e.*, the set of points common to any pair of assigned surfaces is a quartic variety, cf. [6]. $\square$

Figure 5 shows the one-parameter family of quartic curves mentioned in Theorem 2.

Figures 5 and 6 show the quartic surface Φ mentioned in Theorem 1.

3 The quartic surface

In order to describe and investigate the quartic surface Φ, we introduce a Cartesian coordinate system: It shall be centered at H, the x-axis points towards O, and π shall serve as the $[yz]$-plane. Thus, $O = (d, 0, 0)^{\mathrm{T}}$ and the image plane π is given by the equation $x = 0$.

For any point $P \in \mathbb{R}^{3*}$ with coordinate vec-

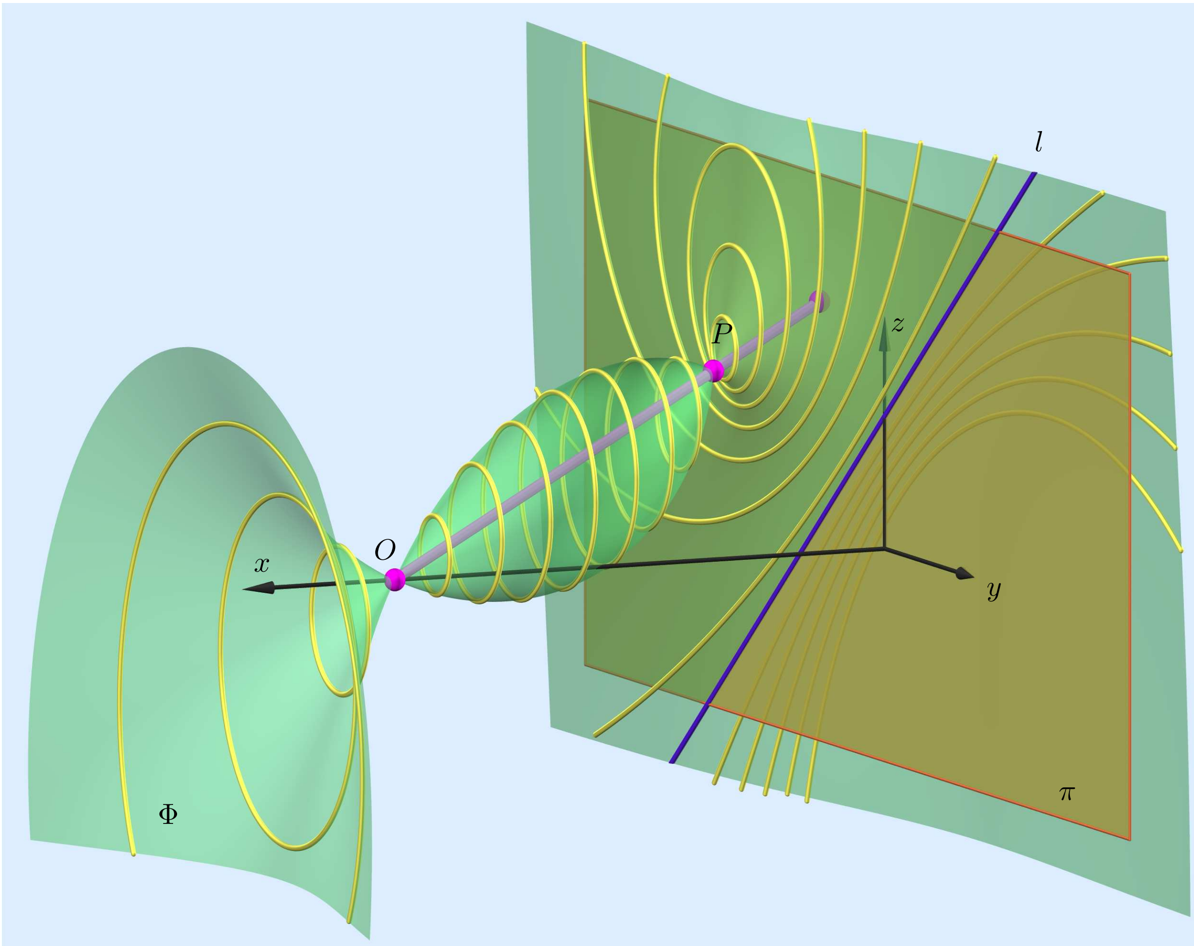

Figure 6: The quartic surface Φ with its circles in planes parallel to π has a singularity at O and P. Φ intersects π in the line l and the ideal line p_2 of π, the latter with multiplicity three.

tor $\mathbf{p} = (\xi, \eta, \zeta)^{\mathrm{T}}$ with $\xi \neq d$ the central image $P' := \kappa(P) = [O, P] \cap \pi$ is given by

$$\mathbf{p}' = \left(0, \frac{d\eta}{d-\xi}, \frac{d\zeta}{d-\xi}\right)^{\mathrm{T}}. \qquad (2)$$

Obviously, $P' = P$ if $P \in \pi$, *i.e.*, $\xi = 0$. The points in the plane

$$\pi_v : \; x = d \qquad (3)$$

have no image in the affine part of the plane π_v. Therefore, the plane π_v is called *vanishing plane*. The plane π_v contains the center O and is parallel to π at distance d. Performing the projective closure of $\mathbb{R}^3$ the images of all points of $\pi_v \setminus \{O\}$ are the ideal points of π gathering on π's ideal line p_2 .

Let now Q be the variable endpoint of a segment starting at P. The point Q shall be given by its coordinate vector $\mathbf{x} = (x, y, z)^{\mathrm{T}}$. Then, an implicit equation of Φ is given by

$$\Phi : \; \overline{PQ}^2 - \overline{P'Q'}^2 = 0. \qquad (4)$$

Using Eq. (2) we can write Eq. (4) in terms of coordinates as

$$\begin{aligned}\Phi:\ & d^2((\eta(d-x)-y\delta)^2+\\ &+(\zeta(d-x)-z\delta)^2)=\\ =&((x-\xi)^2+(y-\eta)^2+(z-\zeta)^2)\cdot\\ &\cdot\delta^2(d-x)^2\end{aligned}\tag{5}$$

where $\delta := d-\xi$.

4 Properties of Φ

A closer look at the equation of Φ as given by Eq. (5) allows us to formulate the following theorem which holds in projectively extended Euclidean space $\mathbb{R}^3$:

Theorem 3. *Let $\kappa : \mathbb{R}^{3*} \to \pi$ be a central projection from a point $O \in \mathbb{R}^3$ to a plane $\pi \not\ni O$ and let further $P \in \mathbb{R}^{3\star}$ be a point in Euclidean three-space. The set of all points Q satisfying*

$$\overline{PQ} = \overline{P'Q'}$$

(where $P' = \kappa(P)$ and $Q' = \kappa(Q)$) is a uni-circular algebraic surface Φ of degree four. The ideal line p_2 of π is a double line of Φ.

Proof. The algebraic degree Φ can be easily read off from Eq. (5).

In order to show the circularity of Φ, we perform the projective closure of $\mathbb{R}^3$ and write Φ's equation (5) in terms of homogeneous coordinates: We substitute

$$x = X_1X_0^{-1},\quad y = X_2X_0^{-1},\quad z = X_3X_0^{-1}$$

and multiply by X_0^4. The intersection of the (projectively) extended surface Φ with the ideal plane $\omega :\ X_0 = 0$ is given by inserting $X_0 = 0$ into the homogeneous equation of Φ which yields the equations of a quartic cycle

$$\phi:\ X_1^2(X_1^2+X_2^2+X_3^2) = X_0 = 0. \tag{6}$$

The first factor of the latter equation tells us that the ideal line p_2 of the image plane $\pi :\ X_1 = 0$ is a part of $\phi = \omega \cap \Phi$ and has multiplicity two. In order to be sure that p_2 is a double line on Φ, we compute the Hessian $\mathrm{H}(\Phi)$ of the homogeneous equation of Φ and evaluate at

$$p_2 = (0:0:X_2:X_3)$$

(with $X_2 : X_3 \neq 0 : 0$ or equivalently $X_2^2 + X_3^2 \neq 0$). This yields

$$\mathrm{H}(\Phi)=2\delta^2(X_2^2+X_3^2)\begin{pmatrix} 0 & -d & 0 & 0\\ -d & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{7}$$

which shows that all but two partial derivatives of Φ's homogeneous equation do not vanish along p_2. Therefore, p_2 is a double line on Φ.

The second factor of the left-hand side of (6) is the equation of the *absolute conic* of Euclidean geometry with multiplicity one. Thus, Φ is uni-circular. □

A part of the double line p_2 is shown in Figure 7 which shows a perspective image of the surface Φ and the circles and lines on Φ.

Corollary 1. *In the case $P \in \pi$,* i.e., *$\xi = 0$, the surface Φ is the union of the image plane π (a surface of degree one) and a cubic surface.*

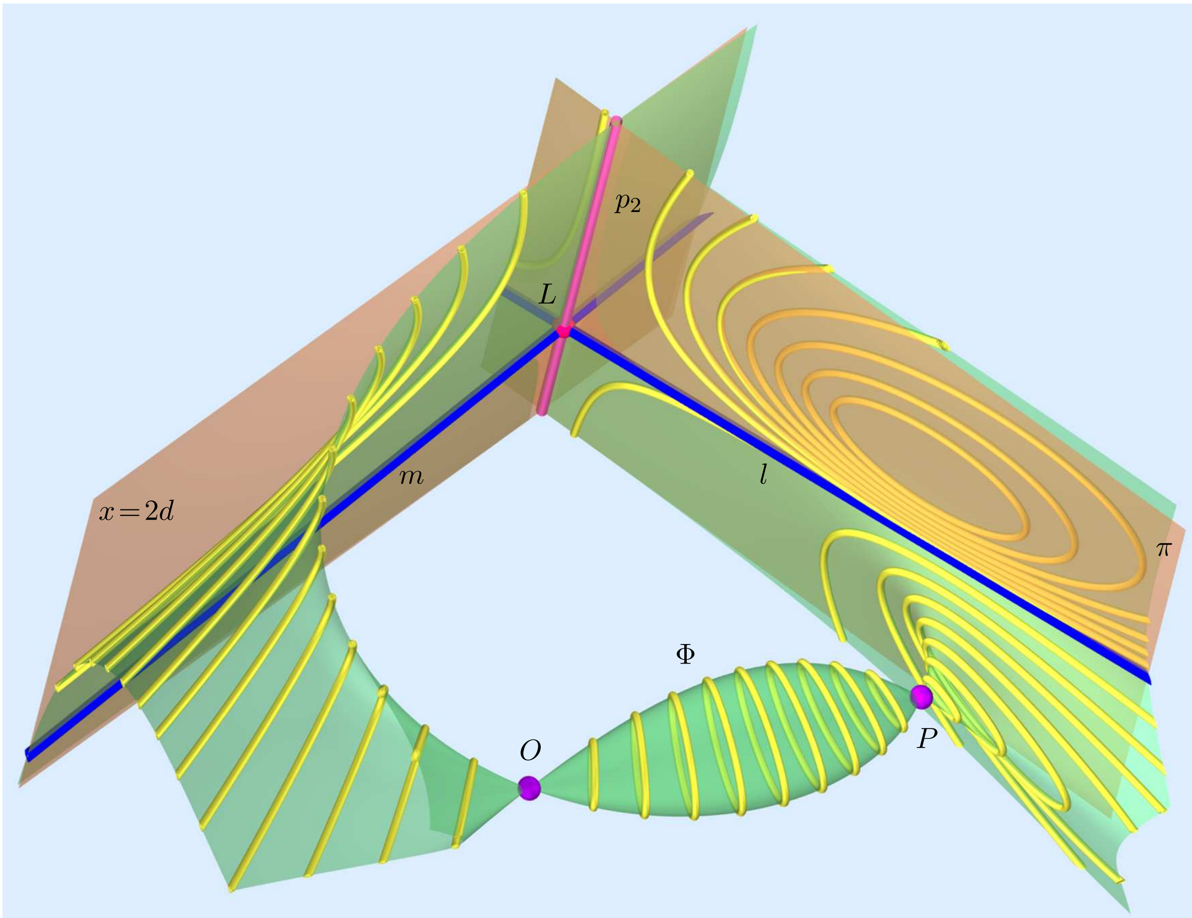

Figure 7: A perspective image of the situation in space: The ideal line p_2 of the image plane π of κ is a part of the double curve of Φ. The two parallel lines l and m meet in the common ideal point $L \in p_2$. The two planes π and $x = 2d$ serve as tangent planes of Φ along p_2 and meet Φ along p_2 with multiplicity three and l and m appear as the remaining linear part.

Proof. If $P \in \pi$, we have $\xi = 0$. Inserting $\xi = 0$ into Eq. (5) we find

$$x(\|\mathbf{x}\|^2(x-2d)-2(x-d)(\eta y+\zeta z)+d^2x)=0.$$

Obviously, Φ is the union of the plane π (with the equation $x = 0$) and a cubic surface. $\square$

The spheres of the one-parameter family of concentric spheres centered at P carrying the one-parameter family of quartic curves $q \subset \Phi$ intersect Φ along the quartics q and the absolute circle of Euclidean geometry. At the latter the spheres are in concact with each other and with the quartic surface Φ. This can easily be shown by computing the resultants of Φ's and the spheres' homogeneous equations with respect to X_0. From this resultant the factor $X_1^2+X_2^2+X_3^2$ splits off with multiplicity 2. In other words: Φ

and all spheres about P share an isotropic tangent cone with vertex at P.

The shape of the curve $\omega \cap \Phi$ together with $\eta^2+\zeta^2 \neq 0$, *i.e.*, $P \notin [O,H]$, tells us:

Theorem 4. *A plane $x = k$ ($k \in \mathbb{R}$) parallel to the image plane π intersects Φ along*

1. *the union of a circle whose center lies on a rational planar cubic curve γ and the two-fold ideal line p_2 if $k \neq 0, d, 2d, \xi$,*
2. *the union of a line l and the three-fold line p_2 if $k = 0$,*
3. *the union of a line $m \parallel l$ and the three-fold line p_2 if $k = 2d$, and*
4. *the union of a pair of isotropic lines and the two-fold line p_2 if $k = d, \xi$.*

Proof. Each planar section of the affine part of Φ is an algebraic curve whose degree is at most 4. As we have seen in the proof of Theorem 3, the ideal line p_2 of the image plane π is a two-fold line in Φ. Thus, the intersection of (the projectively extended) surface Φ with any plane parallel to π also contains this repeated line. The remaining part r of these planar intersetions is at most of degree 2.

The planes parallel to π meet the absolute conic of Euclidean geometry at their *absolute points* which induce Euclidean geometry in these planes. Since the absolute conic is known to be a part of ϕ, the curves r are Euclidean circles (including pairs of isotropic lines and the join p_2 of the two absolute points as limiting cases). The equations of the intersections of Φ with planes parallel to π can be found by rearranging Φ's equation (5) considering y and z as variables in these planes. The coefficients are univariate functions in x and we find

$$\begin{gathered} x(x-2d)\delta^2(\underline{y^2}+\underline{z^2})+ \\ +2\delta(d-x)(\delta x+d\xi)(\eta\underline{y}+\zeta\underline{z})+ \\ +(d-x)^2\delta^2(\langle \mathbf{p},\mathbf{p}\rangle + x(x-2\xi)) \\ -d^2(\eta^2+\zeta^2)=0. \end{gathered} \tag{8}$$

The essential monomials y^2, z^2, y, and z are underlined in order to emphasize them. Note that the monomial yz does not show up. Since $\operatorname{coeff}(x^2) = \operatorname{coeff}(y^2)$ the curves in Eq. (8) are Euclidean circles.

1. We only have to show that the centers of the circles given in Eq. (8) on Φ in planes $x = k$ (with $k \neq 0, d, 2d, \xi$) are located on a rational planar cubic curve. For that purpose we consider Φ's inhomogeneous equation (5) as an equation of conics in the $[y,z]$ plane. By completing the squares in Eq. (8), we find the center of these conics. Keeping in mind that x varies freely in $\mathbb{R} \setminus \{0, d, 2d, \xi\}$ we can parametrize the centers by

$$\gamma(x) = \begin{pmatrix} x \\ \dfrac{\eta(d-x)(d\xi+dx-x\xi)}{\delta x(2d-x)} \\ \dfrac{\zeta(d-x)(d\xi+dx-x\xi)}{\delta x(2d-x)} \end{pmatrix} \tag{9}$$

which is the parametrization of a rational cubic curve. The cubic passes through O and P which can be verified by inserting either $x = d$ or $x = \xi$. In order to show that m is planar, we show that any four points on γ are coplanar. We insert $t_i \neq 0, d, 2d, \xi$ with $i \in \{1,2,3,4\}$ into (9) and show that the inhomogeneous coordinate vectors of the four points $\gamma(t_i)$ are linearly dependent for any choice of mutually distinct t_i.

From

$$\det\begin{pmatrix} 1 & \gamma(t_1)^{\mathrm{T}} \\ 1 & \gamma(t_2)^{\mathrm{T}} \\ 1 & \gamma(t_3)^{\mathrm{T}} \\ 1 & x\ y\ z \end{pmatrix} = 0$$

we obtain the equation

$$\eta y - \eta z = 0$$

of the plane that carries γ.

Figure 8 shows the cubic curve γ with its three asymptotes.

2. The image plane $\pi: \ x = 0$ of the underlying central projection κ touches (the projective extended surface) Φ along the ideal line p_2 of π. This can be concluded from the following: We write down the quadratic form

$$\mathbf{X}^{\mathrm{T}}\mathrm{H}(\Phi)\mathbf{X} = X_1(X_1 - 2dX_0) = 0$$

with $\mathrm{H}(\Phi)$ being the Hessian from (7) and $\mathbf{X} = (X_0, X_1, X_2, X_3)^{\mathrm{T}}$ being homogeneous coordinates. (Nonvanishing factors are cancelled out.) This form gives the equations of the two planes through p_2 that intersect Φ along p_2 with higher multiplicity than two, *i.e.*, in this case with multiplicity three. Thus, the multiplicity of the line p_2 considered as the intersection of π and Φ is of multiplicity three and a single line l of multiplicity one remains. This line is given by

$$l\colon (2d-\xi)\langle \mathbf{p}, \mathbf{p}\rangle - d^2\xi = 2\delta(\eta y + \zeta z)$$

where y and z are used as Cartesian coordinates in the image plane π.

3. In a similar manner we find the line m which is the only proper intersection of Φ with the plane $x = 2d$:

$$m\colon d(2d^2 - 5d\xi + 4\xi^2) - \xi\langle \mathbf{p}, \mathbf{p}\rangle = \\ = 2\delta(\eta y + \zeta z)$$

The plane of the cubic curve γ is orthogonal to the lines l and m.

4. In case of $x = \xi$, the plane runs through P. Again, the ideal line p_2 splits off with multiplicity two. The remaining part r is the pair of isotropic lines through P with the equation

$$x = \xi, \quad (y-\eta)^2 + (z-\zeta)^2 = 0.$$

The same situation occurs at O, *i.e.*, $x = d$ where the isotropic lines have the equation

$$x = d, \quad y^2 + z^2 = 0. \qquad \square$$

The circles as well as the line l on the quartic surface Φ can be seen in Figures 6, 9, and 8. In Figure 8, a small piece of the line m shows up.

Remark 1. *In the case of $P \in [O, H]$, or equivalently, $\eta^2 + \zeta^2 = 0$ the lines l and m coincide with the ideal line of π and, thus, $\pi \cap \Phi$ is the ideal line of π with multiplicity four. The same holds true for the plane $x = 2d$ if $P \in [O, H]$.*

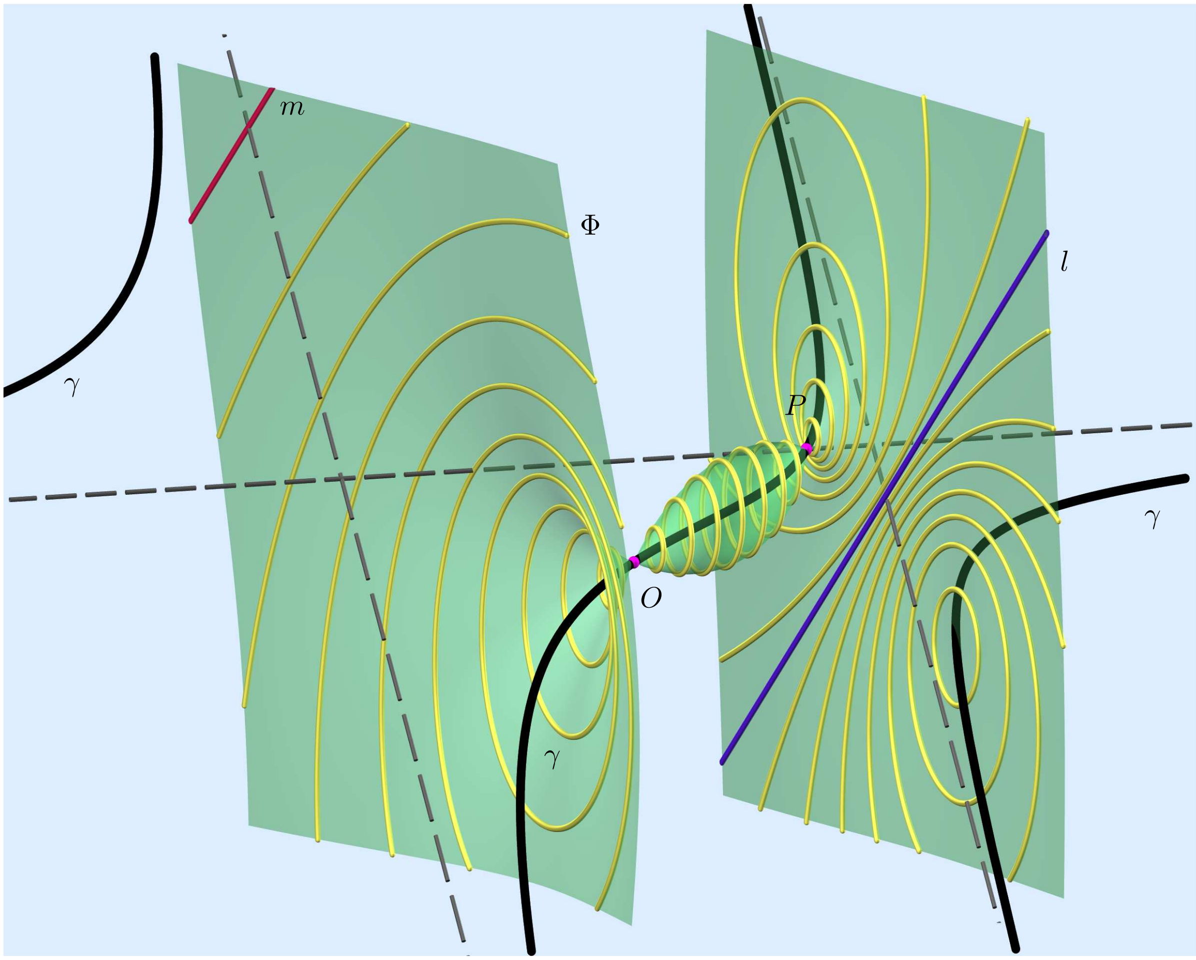

Figure 8: The cubic curve γ carries the centers of all circles on Φ. Its ideal doublepoint $(0 : 0 : \eta : \zeta)$ is the ideal point of the lines orthogonal to $l \parallel m$. The tangent of c at the third ideal point $(0 : 1 : 0 : 0)$ passes through P. The three dashed lines are γ's asymptotes.

Remark 2. *The planes π and $x = 2d$ behave like the* tangents *of a planar algebraic curve c at an ordinary double point D because these tangents intersect c at D with multiplicity three. This cannot just be seen from Figure 7.*

The lines l and m from the proof of Theorem 4 are parallel to each other but skew and orthogonal to the line $[O, P]$ as long as $\xi(\xi - 2d) \neq 0$. If $\xi = 0$ or $\xi = 2d$, we have the case mentioned in Remark 1 and l and m are ideal lines. They are still skew to $[O, P]$ but orthogonality is not defined in that case.

The set of singular surface points on Φ contains only points of multiplicity two. A more detailed description of the set of singular surfaces points is given by:

Theorem 5. *The set of singular surface points on Φ is the union of eyepoint O, the*

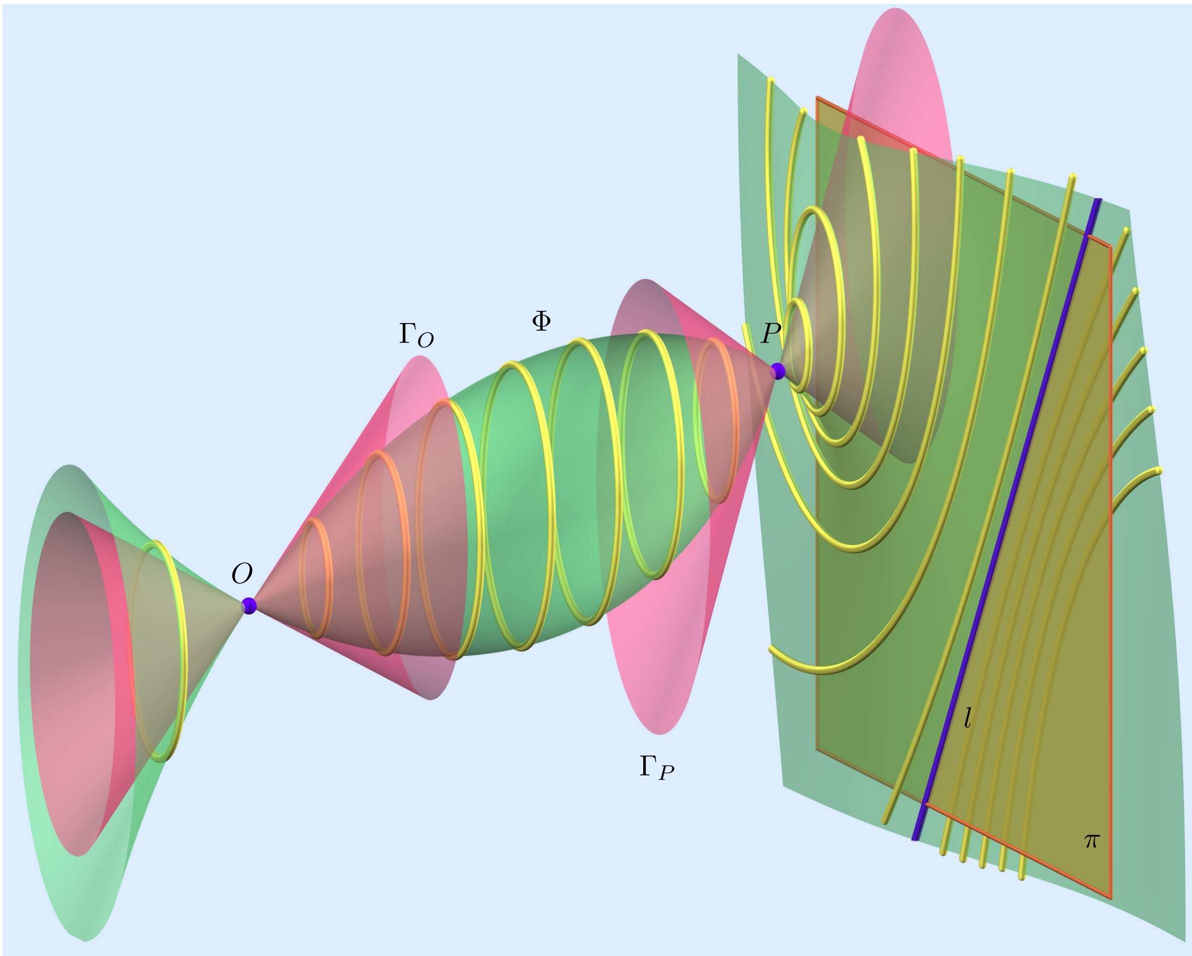

Figure 9: The two singular points O and P are conical nodes, *i.e.*, the terms of degree two of Φ's equation when translated to O or P are the equations of quadratic cones. The circular sections of Φ lie in planes that meet the quadratic cones Γ_O and Γ_P along circles.

object point P, and the ideal line p_2 of the image plane π. The eyepoint O and the object point P are conical nodes on Φ.

Proof. The ideal line of π is a line with multiplicity two on Φ. The planes $\pi: \ x = 0$ and $x = 2d$ intersect Φ along this ideal line with multiplicity three as shown in the proof of Theorem 4. Therefore, the points on π's ideal line are singular points considered as points on Φ.

The points O and P are singular surface points on Φ since the gradients of Φ vanish at both points:

$$\operatorname{grad}(\Phi)(\mathrm{d}, 0, 0) = (0, 0, 0)^{\mathrm{T}}$$

and

$$\operatorname{grad}(\Phi)(\xi, \eta, \zeta) = (0, 0, 0)^{\mathrm{T}}$$

Now we apply the translation $\tau_1: \ O \mapsto (0, 0, 0)^{\mathrm{T}}$ to Φ, *i.e.*, the singular point O moves to the origin of the new coordinate system. The equation of Φ does not alter its degree. However, the monomials in the equation of Φ are at least of degree two in the variables x, y, z. If we remove the monomials of degree three and four, we obtain the equation of a quadratic cone Γ_O centered at O. Its equation (in the new coordinate system, but still labelled x, y, z) reads

$$\Gamma_O: \; d^2\delta^2\langle \mathbf{x},\mathbf{x}\rangle + 2d^2\delta x(\eta y+\zeta z) = \\ = (\delta^4 + \xi(2d+\xi)\langle \mathbf{p},\mathbf{p}\rangle + \xi^3(d+\delta))x^2.$$

Γ_O is the second order approximation of Φ at O. Since Γ_O is a quadratic cone the singular point O is a conical node, see [2].

In order to show that P is also a conical node of Φ we apply the translation $\tau_2: \; P \mapsto (0,0,0)^{\mathrm{T}}$. Again we use x, y, z as the new coordinates and the quadratic term of the transformed equation of Φ given by

$$\Gamma_P: \; \xi(\delta+d)\delta^2\langle \mathbf{x},\mathbf{x}\rangle + 2d^2\delta x(\eta y+\zeta z) + \\ + d^2(\langle \mathbf{p},\mathbf{p}\rangle - \delta^2 - 2\xi^2)x^2 = 0.$$

is the equation of a quadratic cone Γ_P centered at P. Consequently, P is also a conical node (cf. [2]). $\square$

Remark 3. *The homogeneous equations of the quadratic cones Γ_O and Γ_P are the quadratic forms whose coefficient matrices are (non-zero) scalar multiples of the Hessian matrix of Φ's homogeneous equation evaluated at O and P.*

Figure 9 illustrates the two quadratic cones Γ_O and Γ_P. The planes parallel to π (except $x = k$ with $k \in \{d, \xi\}$) intersect both quadratic cones Γ_O and Γ_P along circles.

If $P = P'$ but $[0,P] \not\angle \pi$, *i.e.*, $P \in \pi$ and $P \neq H$, then Φ is the union of the image plane π and a cubic surface $\overline{\Phi}$ with the equation

$$(x-2d)\langle \mathbf{x},\mathbf{x}\rangle = 2(x-d)(\eta y+\zeta z) - d^2x. \quad (10)$$

The cubic surface $\overline{\Phi}$ has only one singularity at O which is a conical node.

If $P \in [O,H]$ (but $P \neq O, H$), then $\overline{\Phi}$ is a surface of revolution with the equation

$$x(x-2d)\langle \mathbf{x},\mathbf{x}\rangle + \xi(\xi-2x)(x-d)^2 - d^2x^2 = 0 \quad (11)$$

where $\eta^2 + \zeta^2 \neq 0$ in contrast to earlier assumptions.

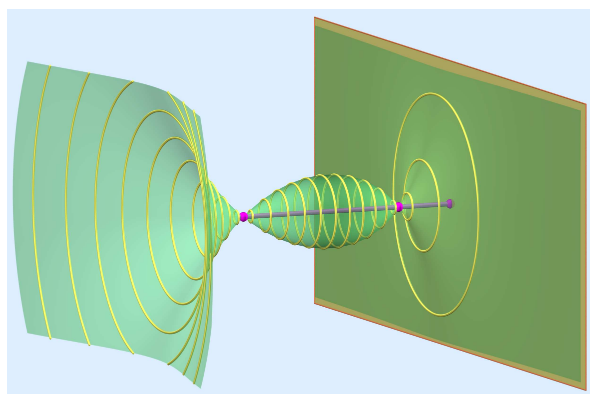

Figure 10: The set Φ of all points Q is a quartic surface of revolution if $P \in [O,H]$ and $P \neq O, H$.

Figures 10 and 11 show the two distinct cases where Φ is a surface of revolution.

References

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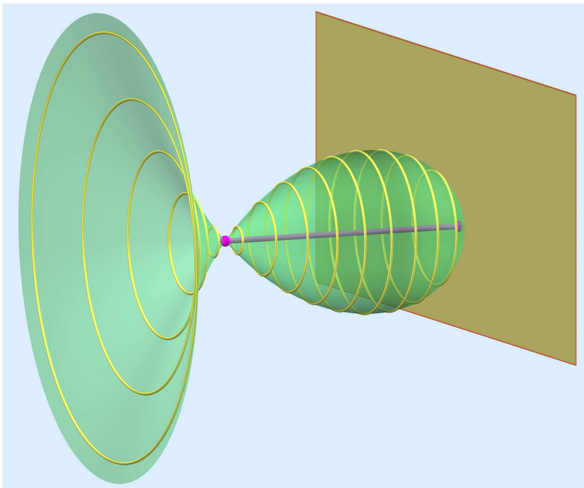

Figure 11: Φ is the union of π and a cubic surface of revolution touching π at H if $P = H$.

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