CURVE FLOWS ON RULED SURFACES

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ABSTRACT: We examine special curve flows on ruled surfaces and their properties. Therefore we recall some basic facts on curves and introduce curvature flows and curvature radius flows in two- and three-dimensional Euclidean space. After that we focus on implementation techniques. As we want to study flows on ruled surfaces, we discuss differential geometric properties of ruled surfaces. With these preparations, we are able to study Gaussian curvature and geodesic curvature flows on ruled surfaces.

Keywords: curve flow, curvature radius, geodesic flow, ruled surface, striction curve, Gaussian curvature

1. INTRODUCTION

During the last few years a great interest in curve and surface flows arose. For a survey on the curve shortening flow see [1]. We focus on curve flows on ruled surfaces.

The paper is organized as follows. In section two, we recall some basic differential geometric properties of curves. Some equivalent expressions for the curvature of a curve are provided. These formulas will be the basis for discrete curvature approximation methods. After that, we introduce the flow equations, which are derived by applying the heat equation to curves [7]. This results in flows defined by the partial differential equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial s^2} = \kappa e_2.$$

Section three deals with the implementation of these curve flows. We present a not FEM-based, and therefore more geometric method for the implementation. The curve is considered as a discrete set of points. We use discrete analogues to differential geometric invariants. The development in time is approximated by a Runge-Kutta method of sufficient order. Due to numerical instabilities while calculating discrete curvature, we present a comparison of different methods for curvature approximation, cf. [5, 8]. Furthermore, we also present an implementation of the Crank-Nicolson method. In addition, we show examples for planar and spatial curve flows. As we preserve the length of the curves under the curvature flow and also under the curvature radius flow in Euclidean space, interesting topological results can be observed.

Curvature flows on ruled surfaces in the direction of the rulings are presented in section four. Therefore some basic properties of ruled surfaces are described. Different equations for flows are implemented. On one hand we consider flows that depend on the Gaussian curvature and its partial derivatives. On the other hand we investigate geodesic curvature flows. Additionally, we examine flows depending on intrinsic geometric properties of the ruled surface. It turns out that all points of a given curve converge to points on the striction curve of the ruled surface. Compared to this, the geodesic flow is also studied. In the latter case the development of the curve converges to a geodesic curve on the ruled surface.

2. CURVATURE AND CURVATURE RADIUS FLOWS FOR CURVES

Before we start the examination of flows on curves and on curves on ruled surfaces respectively, we recall some basic facts from differential geometry.

2.1 Differential geometry of curves

Definition 1: A differentiable curve

$$c(s),\, c : I \subset \mathbb{R} \mapsto \mathbb{R}^3,$$

which is parametrized by its arc length, is called a *Frenet curve* if the vectors c', c'' are linearly independent and $c'' \neq 0$ for all parameters $s \in I$. The *Frenet frame* is uniquely determined by

$$\begin{aligned} e_1 &= c' && \text{(tangent vector)},\\ e_2 &= \frac{c''}{\|c''\|} && \text{(principal normal vector)},\\ e_3 &= e_1 \times e_2 && \text{(binormal vector)}. \end{aligned}$$

Remark 1: The $c' = \frac{dc}{ds}$ denotes the derivative with respect to the arc length. In contrast to this $\dot{c} = \frac{dc}{dt}$ denotes the derivative with respect to an arbitrary curve parameter.

We formulate the Frenet equations for the three-dimensional case. The Frenet formulas for two-dimensional curves are included. There we have $\tau = 0$. For $n = 3$ the *Frenet equations* exhibit the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \qquad (1)$$

The function κ is called the curvature of the curve and τ is called the torsion of the curve. Differential geometric properties, especially the curvature, of the curve have to be approximated, thus we provide an equation to compute the curvature of a given curve, cf. [2]. The function $\kappa = \|c''\|$ is defined as the curvature of the curve c. It is also possible to calculate the curvature via

$$\kappa(t) = \frac{\|\dot{c} \times \ddot{c}\|}{\|\dot{c}\|^3}. \qquad (2)$$

Note that the curvature also can be expressed as the derivative of the tangent rotation angle with respect to the arc length φ

$$\kappa = \frac{d\varphi(s)}{ds}. \qquad (3)$$

Furthermore, we give an expression for the curvature that will be helpful for an implicit discretization of the curvature flow (5).

$$\kappa e_2 = \frac{\langle \dot{c}, \dot{c} \rangle \ddot{c} - \langle \dot{c}, \ddot{c} \rangle \dot{c}}{\langle \dot{c}, \dot{c} \rangle^2}. \qquad (4)$$

2.2 Flow Equation

The flow equations we determine are derived through the application of the heat equation to curves. In general the heat equation for one-dimensional problems is a partial differential equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} \right),$$

where α is called the *thermal diffusivity*, t is the parameter for the time and x denotes the place. Applying it to curves results in the equation:

$$\frac{\partial c(s)}{\partial t} = \frac{\partial^2 c(s)}{\partial s^2} = \kappa e_2. \qquad (5)$$

We remark that the Frenet equations (1) were used to obtain formula (5). The parameter t stands for the time and s for the arc length. This flow equation is also called the curve shortening flow, see [1]. As lengths are preserved during the flow, we are dealing with, we obtain curves with constant curvature at every curve point. In contrast to the curve shortening flow or the curvature flow we additionally examine the curvature radius flow, which is given by

$$\frac{\partial c(s)}{\partial t} = \frac{1}{\kappa} e_2. \qquad (6)$$

It is obvious that both flows move the points of the curve in the direction of the principal normal vector. The curvature flow ensures that the curvature of the resulting curve is constant along the curve. The curvature radius flow, however, ensures that the radius of curvature is constant all over the curve. In the planar case this always leads to a circle that is traced once or more times.

3. IMPLEMENTATION

This section gives a detailed description of the implementation of the algorithms. The approximation in place is done by finite differences. For the time derivative a Runge-Kutta method is used. Due to numerical instabilities it is necessary to use a method with a sufficient high order. The given curve is approximated by a list of points. As we restrict ourselves to closed curves, the last entry in the list is equal to the first one. For the approximation of the derivatives, finite differences are used. Therefore finite differences of order two

$$\dot{c} \approx \frac{c_{i-1} - c_{i+1}}{\Delta}, \quad \ddot{c} \approx \frac{c_{i-1} - 2c_i + c_{i+1}}{\Delta^2}$$

and finite differences of order six are computed by

$$\begin{aligned} \dot{c} &\approx \frac{1}{60\Delta}\big(-c_{i-3} + 9c_{i-2} - 45c_{i-1} \\ &\qquad + 45c_{i+1} - 9c_{i+2} + c_{i+2}\big), \\ \ddot{c} &\approx \frac{1}{180\Delta^2}\big(2c_{i-3} - 27c_{i-2} + 270c_{i-1} - 490c_i \\ &\qquad + 270c_{i+1} - 27c_{i+2} + 2c_{i+3}\big), \end{aligned}$$

where Δ is the constant increment in the parameter domain of the curve. Under this assumption the curve is not parametrized by arc length. In a first attempt we just use explicit iterative Runge-Kutta methods to approximate the time derivative, for details see [4]. The advantage of these methods is that no system of linear equation occurs in each step of the Runge-Kutta method.

Here we give a short overview on the implemented Runge-Kutta methods. In general a Runge-Kutta method is given by a Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b, \end{array}, \quad \sum_{j=1}^{r} a_{ij} = c_i \text{ for } i = 1, \ldots, r.$$

The Runge-Kutta method is then given by

$$y_{n+1} = y_n + \sum_{i=1}^{r} b_i k_i,$$

Table 1: Runge-Kutta methods

<table>
<tr><th></th><th>Euler</th><th>Heun</th><th>Runge-Kutta</th></tr>
<tr>
<td>$\begin{array}{c|c} c & \mathrm{A} \\ \hline & b^{\mathrm{T}} \end{array}$</td>
<td>$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$</td>
<td>$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$</td>
<td>$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & & \frac{1}{2} & & \\ 1 & & & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$</td>
</tr>
</table>

where

$$k_i = hf(t_n + c_i h, y_n + \sum_{j=1}^{r} a_{ij} k_j).$$

In these formulas h defines the step size in time direction and r the dimension of the square matrix A. The fact that we restrict ourselves to explicit methods implies that the square matrix A is a lower triangular matrix. An explicit Euler method (order 1) as the simplest possible method, the method of Heun (order 2), the classic fourth-order method, see Table 1, a Runge-Kutta-Verner method (order 5) and a Runge-Kutta-Verner method (order 6) are implemented.

The corresponding Butcher tableaus can be found in [4]. It is important to note that the flow equations under consideration are parabolic partial differential equations and therefore a condition for the place and time increment is needed, if explicit methods are used. This condition is called the Courant-Friedrichs-Lewy condition (CFL condition), cf. [3]

$$\frac{h}{\Delta^2} \leq C, \tag{7}$$

where C is constant and depends mainly on the partial differential equation. C is related to the minimum respectively maximum of the curvature.

3.1 Equidistant distribution of curve points

Near singularities the distance between two neighboring points can become very small. Therefore, we consider a second flow, a flow that distributes the points on the curve newly. In so doing, it is guaranteed that the distance between two neighboring points does not become too small. This would lead to a very small time step because of the CFL condition (7). To avoid this effect, we use the partial differential equation

$$\frac{\partial c}{\partial t} = \frac{\langle \dot{c}, \ddot{c} \rangle}{\|\dot{c}\|^3} e_1. \tag{8}$$

Figure 1 shows the effect of this flow on the curve.

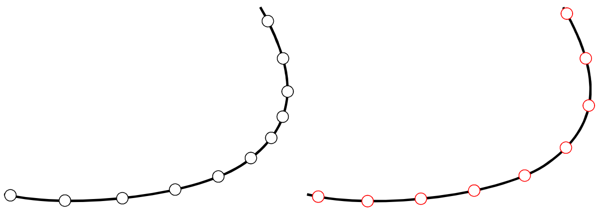

Figure 1: Old distribution (left), new distribution (right)

3.2 Approximation methods for the curvature

In this section we discuss methods for the approximation of the curvature that were implemented. A total of four different methods are implemented.

(i) First we use finite differences for $\dot{c}$ and $\ddot{c}$ and use these discrete analogues to calculate a discrete curvature with equation (2).

(ii) We follow the procedure introduced in [8] and use finite differences for the angle between tangents. Let t_{i-1}, t_i, t_{i+1} be three consecutive tangents and let $\varphi^- = \sphericalangle(t_{i-1}, t_i)$, $\varphi^+ = \sphericalangle(t_i, t_{i+1})$, then equation (3) is approximated by

$$\kappa = \frac{1}{2d_i} \left(\varphi^+ - \varphi^- \right),$$

see Figure 2. The distance d_i is measured between the points m_{i-1} and m_{i+1}. These points are the midpoints of the intersections of the tangents, see also Figure 2. Note that the tangents are approximated by finite differences. Tangent lines can be skew in the spatial case. In this case, we take the intersection points of the tangents with its common normal.

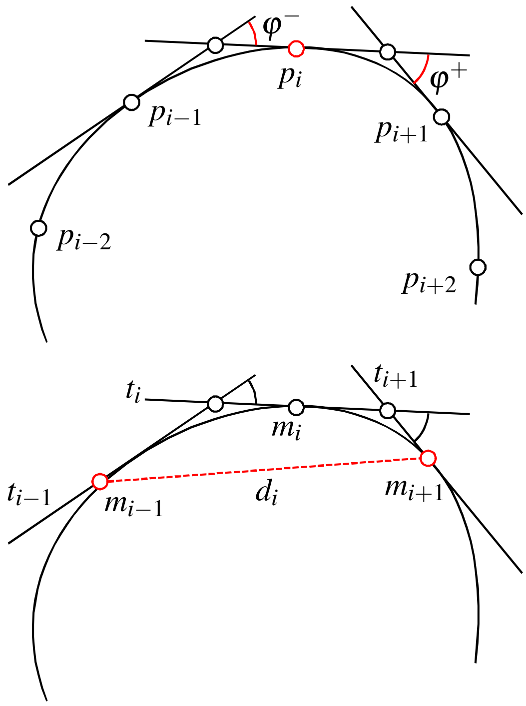

Figure 2: Finite differences for the tangent angle

(iii) The next method follows [5]. The curvature at point p_i is approximated by the curvature of the circle through p_i and its neighbors p_{i-1}, p_{i+1}, see Figure 3.

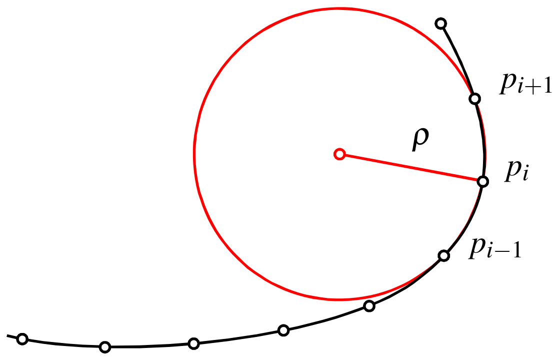

Figure 3: Curvature approximation with circle

(iv) In the fourth method we approximate the curvature by the curvature of a best fitting parabola. This also follows [5], see Figure 4.

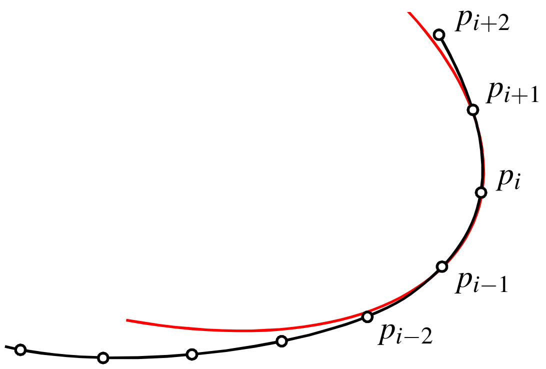

Figure 4: Curvature approximation with best fitting parabola

In practice, it has turned out, that the tangent angle method for curvature approximation delivers the best results.

3.3 Crank-Nicolson method

The flow equations we are working with, are stiff differential equations, cf. [4]. Naturally, we want to formulate our discretization without a restriction to the increment of time. For this reason an implicit Crank-Nicolson method is implemented to approximate equation (4). This method uses the central difference quotient in time and space

$$\partial_t c_i^{n+\frac{1}{2}} = \frac{c_i^{n+1} - c_i^n}{\Delta_t}.$$

Furthermore, the unknown points $c_i^{n+\frac{1}{2}}$ and their derivatives are determined by the linear approximation

$$c_i^{n+\frac{1}{2}} = \frac{1}{2}\left(c_i^n + c_i^{n+1}\right).$$

Note, that the flow equation is a non-linear partial differential equation and therefore the Crank-Nicolson scheme is also non-linear. To avoid terms of higher order we have modified this method by using c_i^n instead of $c_i^{n+1/2}$ for all occurring inner products

$$\partial_t c_i^{n+\frac{1}{2}} = \frac{\langle \dot{c}_i^n, \dot{c}_i^n \rangle \ddot{c}_i^{n+\frac{1}{2}} - \langle \dot{c}_i^n, \ddot{c}_i^n \rangle \dot{c}_i^{n+\frac{1}{2}}}{\langle \dot{c}_i^n, \dot{c}_i^n \rangle^2}.$$

This modification enables us to formulate and solve systems of linear equations for each coordinate

$$\left[N^2 + \mu M\right]^n c_x^{n+1} = \left[N^2 - \mu M\right]^n c_x^n,$$

with the abbreviations

$$\begin{aligned} N &= D(\langle \dot{c}, \dot{c} \rangle), \\ M &= N \cdot D_2^h - h \cdot D(\langle \dot{c}, \ddot{c} \rangle) \cdot D_1^h. \end{aligned}$$

The matrices D_1^h and D_2^h are representations of the finite differences of first respectively second order. Furthermore, the operator D maps the inner products of all points to a diagonal matrix. Compared to explicit Runge-Kutta schemes, this method is unconditionally stable, especially in the neighborhood of singularities. The cost of solving three systems of linear equations in each time step is lower than ensuring the CFL condition and redistributing the points.

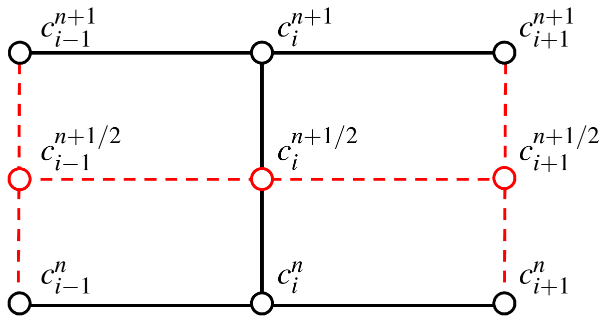

Figure 5: Crank-Nicolson method

3.4 Flows in two- and three-dimensional space

In this section we show some examples for the curvature flow and also for the curvature radius flow. The first example shows Pascal's limaçon, that is parametrized by

$$c(t) = \left(2\cos^2 t + \cos t, \sin 2t + \sin t\right).$$

In Figure 7 the resulting evolution of c is illustrated. We remark that nice topological properties can be observed. For planar and also for spatial curves the curvature flow yields a circle, which is covered once. In contrast to this, the curvature radius flow results in a circle that is covered twice. So this flow does not influence topological properties like the winding number.

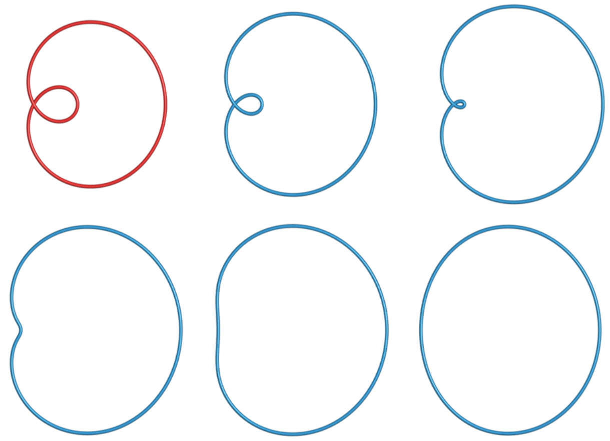

Figure 6: Planar curvature flow with Crank-Nicolson method, initial curve (red), timesteps 1,7,11,12,51,101.

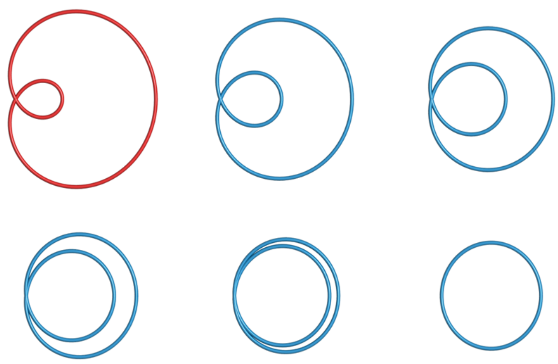

Figure 7: Planar curvature radius flow with Runge-Kutta method, initial curve (red), timesteps 1,5,9,15,21,61.

Now we present the curvature flow applied to a spatial curve, which is located on a torus

$$c = \begin{pmatrix} (R + r\cos 2u)\cos u \\ (R + r\cos 2u)\sin u \\ r\sin 2u \end{pmatrix}.$$

We choose the values $R = 2, r = 1$. In Figure 8 the evolution of this curve under the curvature flow (5) is illustrated for several time steps.

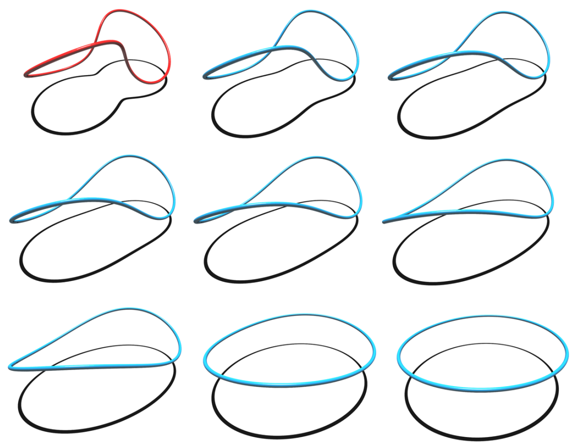

Figure 8: Curvature flow (spatial curve).

4. CURVATURE AND CURVATURE RADIUS FLOWS ON RULED SURFACES

Like in the curve case, we also need some notions of ruled surfaces. Therefore, we introduce differential geometric properties of surfaces that we need for curve flows on ruled surfaces.

4.1 Differential geometry of ruled surfaces

First of all, we have to clarify the notations we shall use.

Definition 2: A surface is called a *ruled surface*, if it has a parametrization of the form

$$f(u,v) = c(u) + v \cdot e(u), \tag{9}$$

where $c : I \to \mathbb{R}^3$ is a directrix and $e : I \to S^2$ is a spherical curve, which corresponds to the direction of the rulings. The partial derivatives of the ruled surfaces are denoted by f_u respectively f_v.

The striction curve of a ruled surface is the curve of points on the surface with extremal Gaussian curvature, see [6]. It can be parametrized by

$$s(u) = c - \frac{\langle \dot{c}, \dot{e} \rangle}{\langle \dot{e}, \dot{e} \rangle} e. \tag{10}$$

We call a point of the ruled surface *singular* if the partial derivatives f_u and f_v are linearly dependent.

Definition 3: The *parameter of distribution* δ for a ruled surface describes the speed of the tangentplanes winding about the ruling, cf. [6]. In the striction point it can be determined by

$$\delta := \frac{\det(\dot{c}, e, \dot{e})}{\|\dot{e}\|^2},$$

which only depends on u.

Therefore, the Gaussian curvature is defined by Lamarle's formula, see [6]

$$K = -\frac{\delta^2}{(\delta^2 + v^2)^2} < 0.$$

Note that K is a function in v only at any fixed generator. Another differential geometric property is the geodesic curvature of a curve on a ruled surface

$$\kappa_g = \frac{\det(n, \dot{c}, \ddot{c})}{\|\dot{c}\|^2}, \tag{11}$$

where n denotes the normal of the surface.

4.2 Curve flows on ruled surfaces

We treat curve flows on ruled surfaces in the direction of the rulings. Therefore, the Gaussian flow equation we use can be expressed as

$$\frac{\partial c}{\partial t} = e \cdot \frac{\partial}{\partial v} \ln K = e \cdot \frac{\partial K}{\partial v} \frac{1}{K}, \tag{12}$$

with e being the spherical image of the ruled surface. Hence, we examine the Gaussian curvature on every ruling. The flow forces the curve in the direction of increasing Gaussian curvature.
Another flow equation we are dealing with is the geodesic flow, which can be written as:

$$\frac{\partial c}{\partial t} = \kappa_g e, \tag{13}$$

where κ_g is the geodesic curvature of c with respect to the ruled surface. In this case the curve points are forced in the direction of decreasing geodesic curvature. At this point the Gaussian curvature is calculated analytically, because a parametrization of the ruled surfaces is known. In contrast to this, the geodesic curvature is approximated. The aim is to show that every curve converges to the striction curve under the Gaussian flow, or to a geodesic curve under the geodesic flow, respectively. After that there is no difficulty in using a discrete analogue for the Gaussian curvature. The advantage of this method is that for every set of given lines the striction curve can be approximated by the Gaussian flow applied to an arbitrary discrete curve on this discrete ruled surfaces.

4.3 Convergence

Now we formulate our hypothesis as a Theorem.

Theorem 1. The Gaussian curvature flow of a curve on a ruled surfaces in the direction of the ruling (12) converges to the striction curve.

Proof: The quotient $\frac{K_v}{K}$ and therefore the flow of the Gaussian curvature can be expressed by

$$\frac{K_v}{K} = -4 \frac{\langle f_u, f_{uv} \rangle}{\langle f_u, f_u - \langle f_u, f_v \rangle f_v \rangle}.$$

This equation can be interpreted geometrically. The projection of the derivative of the curve is compared with the derivative of the spherical image of the ruled surface. If the numerator is equal to zero, the flow stops and the striction curve is reached at this point

$$\langle f_u, f_{uv} \rangle = \langle c_u + v e_u, e_u \rangle \stackrel{!}{=} 0 \Rightarrow v = -\frac{\langle c_u, e_u \rangle}{\langle e_u, e_u \rangle}.$$

The parameter v corresponds to the parameter of the striction curve, see equation (10). □

Remark 2: We suspect a curve under the geodesic flow on the ruled surface in the direction of the ruling to converge to the shortest closed geodesic on the ruled surface. This conjecture is based on observations we made. Currently, we have no proof.

4.4 Application

Now we present a collection of examples for curve flows on ruled surfaces. The first example is an elliptic hyperboloid of one sheet, that is given by the parametrization:

$$f(u,v) = \begin{pmatrix} a\cos u \\ b\sin u \\ 0 \end{pmatrix} + v \begin{pmatrix} -a\sin u \\ b\cos u \\ c \end{pmatrix}.$$

In the following, we choose $a = 2, b = 3$ and $c = 3.7$. It is well-known, that the degree of the striction curve (10) of a hyperboloid of one sheet is equal to 4 and has no singularities. We have to remark that the Gaussian flow we are using is not a flow defined by properties of the curve. It just depends on the surface. Thus, it could be called a *pseudoflow*. The effect of this flow is visualized in Figure 9 (left). In Figure 9 (right) the development of an arbitrary curve under the geodesic flow is visualized.
All singular points of a ruled surface are contained in the striction curve. The Gaussian curvature in these points is not defined. This is the reason why the flow can develop singularities. If the calculated curve gets too close to singularities, difficulties in the computation may occur.

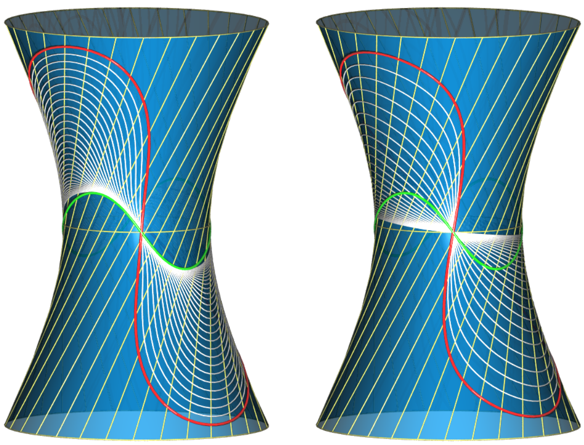

Figure 9: Gaussian flow on hyperboloid (left), Geodesic flow (right), initial curve (red), evolution curves (white), striction curve (green)

Our next example is chosen specifically. The ruled surface is parametrized by

$$f(u,v) = \begin{pmatrix} a\sin u \\ 0 \\ c \end{pmatrix} + v \begin{pmatrix} -a\sin u \\ b\cos u \\ -2c \end{pmatrix}.$$

Its striction curve has four singular points. For the parameters a, b and c we choose, $a = 4$, $b = 4$, $c = 4$. The visualizations of both flows are given in Figure 10.

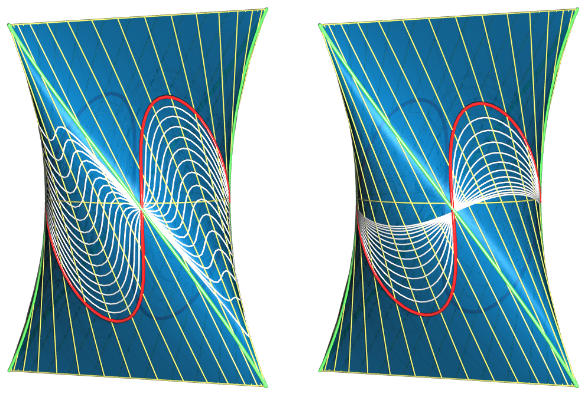

Figure 10: Gaussian flow (left), geodesic flow (right), initial curve (red), evolution curves (white), striction curve (green)

5. FURTHER RESEARCH

It seems that the geodesic flow converges to the shortest geodesic curve on the ruled surface. A proof of this conjecture remains an open problems. Furthermore, we do not use a discrete analogue of the Gaussian curvature, which also could be of interest.

6. CONCLUSION

In this paper, we repeated basic differential geometric properties of curves in $\mathbb{R}^3$. After that, we derived flow equations from the heat equation and showed different implementation methods. Due to numerical instabilities different approximation methods for the curvature were presented. Furthermore, we gave the definition of a ruled surface and declared all properties we needed. Finally, curve flows on ruled surfaces were discussed. It turned out that curves under the Gaussian curvature flow converge to the striction curve of the ruled surface. Curves undergoing the geodesic flow converge to the shortest geodesic on the ruled surface.

ACKNOWLEDGMENTS

This work was supported by the research project "Line Geometry for Lightweight Structures", funded by the DFG (German Research Foundation) as part of the SPP 1542.

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