Circumparabolas in Chapple's Porism

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Abstract. We study the two-parameter manifold of parabolas circumscribed to triangles in a Poncelet porism between two circles (Chapple's porism). It turns out that the focal points of the parabolas in a certain one-parameter subfamily trace a straight line. The vertices of these parabolas move on rational cubic curves whose acnodes trace an ellipse centered at the poristic stationary triangle center which is the midpoint of the common incenter and the common circumcenter. The axes of the circumparabolas envelop a Steiner hypocycloid over the course of a porism. Varying the pivot point of the circumparabola, the one-parameter family of Steiner cycloids envelops two ellipses, one with the fixed common incenter and circumcenter as foci, the other one carrying the cusps of all cycloids.

Keywords: Circumparabola · triangle · porism · point orbit · envelope · focus · axis · vertex · quintic · septic.

1 Introduction

Overview of known results. The one-parameter family of triangles interscribed between a common circumcircle u and a common incircle i is known as *Chapple's porism*, see [5], referred below as the "poristic family". Within the past ten years, porisms in more general forms (including Chapple's) have been studied focused on various aspects: (i) poristic traces of triangle centers in [8–11, 21, 22], (ii) derivation of invariants by means of numerical experiments in [15, 17, 23], (iii) ellipticity of poristic traces of triangle centers and bicentric pairs [16], (iv) historical point of view [4]. Experiments in [24] in particular have motivated a detailed study of circumparabolas of the poristic family.

The results presented here try to verify the numerical and experimental results by means of algebraic techniques. We shall keep technical details aside and try to formulate proofs short and traceable. Therefore, we will sometimes not lay down all equations in detail, especially if they are of enormous length and high complexity.

Techniques. Since we are dealing with the contents of Euclidean geometry, we use Cartesian coordinates (x, y) for points. Whenever favorable, we will switch to homogeneous coordinates $x_0 : x_1 : x_2$ by setting $x = x_1 x_0^{-1}$ and $y = x_2 x_0^{-1}$ with $x_0 \neq 0$. Thereby, we will perform the projective closure of the Euclidean plane (with the ideal line $\omega :\ x_0 = 0$) and we shall also allow homogeneous coordinates to be complex. This allows us to describe a pair of complex conjugate points

$$I = 0 : 1 : \mathrm{i}, \quad J = \overline{I} = 0 : 1 : -\mathrm{i}$$

on the ideal line ω, called the *absolute points* of Euclidean geometry (see [14]). Each Euclidean circle contains both I and J, and each conic through I and J is a Euclidean circle. Further, a line in the projectively-closed complex extension of the Euclidean plane is called *isotropic* if it contains either I or J.

We choose a Cartesian coordinate frame such that the equations of the circumcircle u and the incircle i read

$$u :\ (x-d)^2 + y^2 = R^2, \quad i :\ x^2 + y^2 = r^2. \tag{1}$$

It is natural to parametrize u by means of trigonometric functions as

$$\mathbf{u} = (R\cos\tau + d, R\sin\tau), \quad \text{with } \tau \in \mathbb{R}. \tag{2}$$

In what follows, rational parametrizations of point orbits are essential (if available). Whenever computations are carried out by a computer algebra system (CAS), rational parametrizations are preferred. Therefore, trigonometric functions will be replaced with their rational equivalents

$$\cos\tau = \frac{1-T^2}{1+T^2}, \quad \sin\tau = \frac{2T}{1+T^2}. \tag{3}$$

Present Contributions. In Section 2, we first parametrize the poristic family. Then, we determine the ideal points of the circumparabolas with the help of the isogonal conjugation. Subsequently, we derive the equations of the circumparabolas of all triangles in the poristic family. Section 3 is devoted to the traces of the parabolas' foci, the envelopes of their axes which turn out to be Steiner's hypocycloids. The latter envelop an ellipse, while the cusps trace another ellipse. Finally, we look at the traces of the vertices and other points.

2 Parametrization of the porism and circumparabolas

Vertices of the triangles. We assume that the circumcircle u and the incircle i of a poristic triangle family are given by their equations (1), where the inradius r, the circumradius R, and the central distance d (distance between the incenter X_1 and the circumcenter X_3) are related via the Euler triangle formula

$$d^2 = R^2 - 2Rr \tag{4}$$

which guarantees a porism (cf. [5, 18, 19]). Here and thereafter, triangle centers are labeled according to C. Kimberling's encyclopedia, cf. [19, 20]. Clearly,

$r, R \in \mathbb{R}^+$, and no restriction is imposed if we assume $d \in \mathbb{R}^+$ (the coordinate frame can always be chosen such that $d > 0$).

First, we parametrize the one-parameter family of triangles $\Delta = P_1P_2P_3$ interscribed between u and i, *i.e.*, the poristic family. Let the triangle vertex P_1 be given by (2). The points P_2 and P_3 are found as the intersections of the tangents from P_1 to the incircle i with the circumcircle. This yields

$$\begin{aligned} P_{2,3} = {} & \frac{\sigma\delta}{2R(2dR\cos\tau + R^2 + d^2)^2} \cdot \\ & \cdot\big(\pm\mathcal{W}(R^2 - d^2)\sin\tau - 4dR^3\cos^2\tau - (R^4 + 6d^2R^2 + d^4)\cos\tau - 4d^3R, \\ & \quad \sin\tau(d^4 - R^4 - 4d^2R^2 - 4dR^3\cos\tau) \mp \mathcal{W}((R^2 + d^2)\cos\tau + 2dR)\big) \end{aligned} \tag{5}$$

where $\mathcal{W} := \sqrt{8dR^3\cos\tau + 3R^4 + 6R^2d^2 - d^4}$ and

$$\sigma := R + d, \quad \delta = R - d.$$

Ideal points of the circumparabolas, isogonal conjugation. Each triangle $\Delta = P_1P_2P_3$ in the Euclidean plane can serve as the fundamental triangle of a special quadratic Cremona transformation, called the *isogonal transformation*, cf. [14, 19]. Any point Q (not on the sidelines of Δ) is mapped to its isogonal conjugate $\iota(Q)$ by intersecting the reflections of the Cevians $[Q, P_i]$ in Δ's (interior) angle bisectors $[X_1, P_i]$. It is easily shown that if $Q \neq P_i$ is chosen on Δ's circumcircle u, $\iota(Q)$ is an ideal point (point at infinity). Further, ι is quadratic, *i.e.*, it maps lines (not incident with any P_i) to conics passing through all fundamental points P_i. Therefore, a tangent of the circumcircle is mapped to a parabola circumscribed to the fundamental triangle. Some circumparabolas of a certain triangle in the poristic family are shown in Fig. 1.

Equations of the circumparabolas related to the poristic family. Having defined the isogonal transformation, we can now determine the equations of the one-parameter family of circumparabolas for each triangle in the poristic family. In fact, we are about to determine a two-parameter family of parabolas: The first parameter T determines one particular triangle in the poristic family. The second parameter U determines one particular parabola circumscribed to Δ.

We may assume that the tangents to the circumcircle touch the circumcircle at a point $Q \in u$ which can be given by means of rational coordinate functions as $Q(U) = \mathbf{u}(U)$ (cf. (2)) with some real parameter U.

Since $X_1 = (0, 0)$ (center of i from (1)), we find the directions of the axes of all circumparabolas by reflecting $[P_1, Q]$ in $[P_1, X_1]$. This yields the direction vector $\mathbf{a}$ of the circumparabolas, or if we use homogeneous coordinates, the ideal points $A(U, T) = \iota(Q)$ of all circumparabolas as

$$\begin{aligned} A(U,T) = 0 : {} & T^3\delta^2 - \delta(\delta + 2\sigma)T^2U - \sigma(2\delta + \sigma)T + \sigma^2U : \\ & : -T^3U\delta^2 - \delta(\delta + 2\sigma)T^2 + \sigma(2\delta + \sigma)TU + \sigma^2. \end{aligned} \tag{6}$$

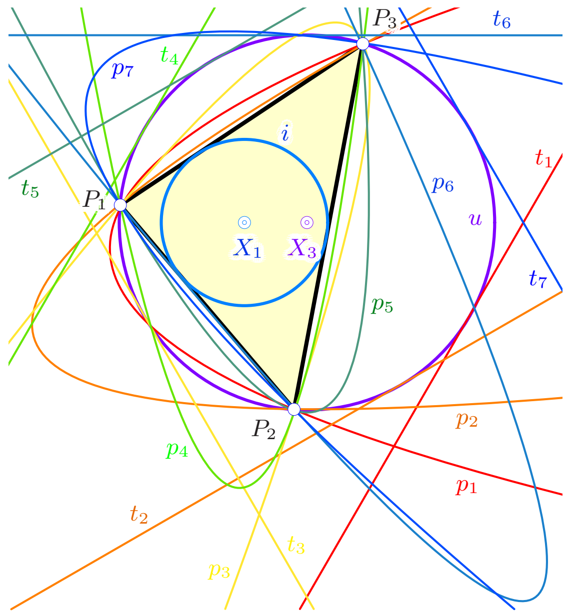

Fig. 1. Circumparabolas p_i of a triangle $P_1P_2P_3$ as isogonal images of the tangents t_i.

The points P_1, P_2, P_3, and A determine a unique parabola since the tangent at A is known: It is the line at infinity. A homogeneous equation of the circumparabolas will have the form

$$p: \ a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + a_{11} + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0 \tag{7}$$

with coefficients $a_{ij} \in \mathbb{R}[\delta, \sigma, U, T]$. In order to make p a parabola, the coefficients a_{ij} have to satisfy

$$a_{11}a_{22} - a_{12}^2 = 0, \tag{8}$$

since this is the condition on p to touch the ideal line $\omega: \ x_0 = 0$.

Inserting the rational equivalents of P_1, P_2 and P_3 from (5), and (6) into (7) and using (8), we find

$$\begin{aligned} a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22} = {} & \delta\sigma(\sigma + \delta U^2)(1+T^2)(\delta^2T^2+\sigma^2)^2 : \\ & : \ (\sigma^2 - \delta^2U^2)(1+T^2)(\delta^2T^2+\sigma^2)^2 : 2\delta\sigma U(1+T^2)(\delta^2T^2+\sigma^2)^2 : \\ & : \ -(\delta+\sigma)(\delta^2T^3U + \delta(\delta+2\sigma)T^2 - \sigma(2\delta+\sigma)TU - \sigma^2)^2 : \\ & : -2(\delta+\sigma)(\delta^2T^3U + \delta(\delta+2\sigma)T^2 - \sigma(2\delta+\sigma)TU - \sigma^2)\cdot \\ & \quad \cdot (\delta^2T^3 - \delta(\delta+2\sigma)T^2U - \sigma(2\delta+\sigma)T + \sigma^2U) \\ & : \ -(\delta+\sigma)(\delta^2T^3 - \delta(\delta+2\sigma)T^2U - \sigma(2\delta+\sigma)T + \sigma^2U)^2. \end{aligned} \tag{9}$$

Note that the one-parameter family of circumparabolas of any triangle Δ (in the poristic family) is itself a conic in the Veronese model $V_2^2 \in \mathbb{P}^5$: Inserting the homogeneous coordinates of the three vertices P_1, P_2, P_3 for x_0, x_1, x_2 in (7) yields three hyperplanes in $\mathbb{P}^5$ which are to be intersected with the quadratic cone determined by (8).

3 Properties of the circumparabolas

Focal traces. According to VON STAUDT, a point F is a focus of a planar algebraic curve if the tangents from F to c are a pair of complex conjugate isotropic lines, cf. [1, 6, 14].

Now, it is rather elementary to determine the tangents t_I and t_J from I and J to p that differ from the line at infinity. Then, the one and only (real) focus of p is the point $F = t_I \cap t_J$ with homogeneous coordinates

$$\begin{aligned}
&f_0 : f_1 : f_2 = 4(\delta+\sigma)(U-T)(1+U^2)\cdot \\
&\cdot\Big(\delta^3(\delta+2\sigma)T^2U^2 - \delta^2\sigma^2T^2 + 2\delta\sigma(\delta+\sigma)^2TU - \delta^2\sigma^2U^2 + (2\delta+\sigma)\sigma^3\Big) : \\
&: \delta^4(\delta^2-4\sigma^2)T^3U^4 - 2\delta^3\sigma(2\delta^2+7\delta\sigma+4\sigma^2)T^3U^2 - \delta^2\sigma^2(4\delta^2+4\delta\sigma-\sigma^2)T^3 - \\
&\quad - \delta^3(\delta+2\sigma)(\delta^2-4\delta\sigma-4\sigma^2)T^2U^5 + 2\delta^2\sigma(\delta+2\sigma)(4\delta^2+7\delta\sigma+2\sigma^2)T^2U^3 + \\
&\quad + \delta\sigma^2(\delta+2\sigma)(4\delta^2-\sigma^2)T^2U + \delta^2\sigma(\sigma+2\delta)(4\sigma^2-\delta^2)TU^4 + \\
&\quad + 2\delta\sigma^2(\sigma+2\delta)(2\delta^2+7\delta\sigma+4\sigma^2)TU^2 + \sigma^3(\sigma+2\delta)(4\delta^2+4\delta\sigma-\sigma^2)T + \\
&\quad + \delta^2\sigma^2(\delta^2-4\delta\sigma-4\sigma^2)U^5 - 2\delta\sigma^3(4\delta^2+7\delta\sigma+2\sigma^2)U^3 - \sigma^4(4\delta^2-\sigma^2)U : \\
&: \delta^4(\delta+2\sigma)^2T^3U^5 + 2\delta^3\sigma^2(\delta+2\sigma)T^3U^3 + \delta^2\sigma^2(4\delta^2+8\delta\sigma+\sigma^2)T^3U + \\
&\quad + \delta^3(\delta+2\sigma)(\delta^2+8\delta\sigma+4\sigma^2)T^2U^4 + 2\delta^3\sigma(\delta+2\sigma)(\sigma+2\delta)T^2U^2 + \\
&\quad + \delta\sigma^2(\delta+2\sigma)(\sigma+2\delta)^2T^2 - \delta^2\sigma(\sigma+2\delta)(\delta+2\sigma)^2TU^5 - \\
&\quad - 2\delta\sigma^3(\delta+2\sigma)(\sigma+2\delta)TU^3 - \sigma^3(\sigma+2\delta)(4\delta^2+8\delta\sigma+\sigma^2)TU - \\
&\quad - \delta^2\sigma^2(\delta^2+8\delta\sigma+4\sigma^2)U^4 - 2\delta^2\sigma^3(\sigma+2\delta)U^2 - \sigma^4(\sigma+2\delta)^2.
\end{aligned} \tag{10}$$

where U is as defined in Eq. (6). Now we are able to show:

Theorem 1. *Over Chapple's porism, and a fixed U, the foci of circumparabolas of triangles in the poristic family trace a straight line given by*

$$\begin{aligned}
\mathcal{F}:\ & 4(\delta+\sigma)(1+U^2)(\delta^2U^2-\sigma^2)x - 8\delta\sigma(\delta+\sigma)U(1+U^2)y + \\
&+\delta^2(\delta^2-4\delta\sigma-4\sigma^2)U^4 - 2\delta\sigma(2\delta^2+3\delta\sigma+2\sigma^2)U^2 - \sigma^2(4\delta^2+4\delta\sigma-\sigma^2) = 0.
\end{aligned} \tag{11}$$

Proof. With (10), we have a parametrization of the trace of the foci of the circumparabolas of all triangles in the poristic family. If we eliminate T from (10), we obtain an equation of the trace of the focus of the circumparabolas. This yields (11) which is the equation of a straight line. □

Lines $\mathcal{F}$ from (11) envelop a sextic curve. This can be easily shown by eliminating parameter U from $\mathcal{F}$ and its derivative $\frac{\mathrm{d}}{\mathrm{d}U}\mathcal{F}$.

The parametrization of the foci allows us to verify a result established in [13]:

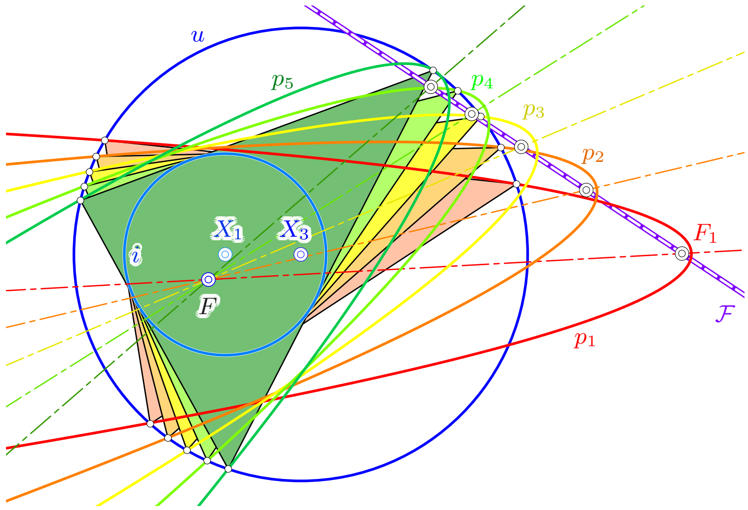

Fig. 2. Over poristic triangles (variable T), if U (and thus, Q) is fixed, the focus of circumparabolas (for fixed U) move along a straight line while the axes pass through a fixed point F.

Theorem 2. *The locus of the focus of all circumparabolas of three points is a circular quintic.*

Proof. In order to find the equation of the locus $\mathcal{F}$ of the foci of all circumparabolas, we have to eliminate U from the affine parametrization $(f_1 f_0^{-1}, f_2 f_0^{-1})$. (without loss of generality, the computation is simplified if we fix T, or even set it equal to 0). □

The rather long equation for $\mathcal{F}$ can be found (for a special type of coordinatization) in [13]. Fig. 3 shows the focus curve $\mathcal{F}$ for one particular triangle. Indeed, this result is not related to porisms. However, each triangle in the poristic family defines its own quintic.

Parabolas' axes. With the direction (6) of the axis and the focus (10), the axes a of all parabolas are well-determined, yielding

$$\begin{aligned} a:\ & 2(\delta+\sigma)(1+U^2)\Big((\delta^2T^3U+\delta(\delta+2\sigma)T^2-\sigma(2\delta+\sigma)TU-\sigma^2)x+ \\ & \qquad + (\delta^2T^3-\delta(\delta+2\sigma)T^2U-\sigma(2\delta+\sigma)T+U\sigma^2)y\Big)+ \\ & + \delta^4T^3U^3-\delta^2\sigma(2\delta+\sigma)T^3U+\delta^2(\delta+2\sigma)^2T^2U^2-\delta\sigma^2(\delta+2\sigma)T^2+ \\ & - \delta^2\sigma(2\delta+\sigma)TU^3+\sigma^2(2\delta+\sigma)^2TU-(\delta^2\sigma^2+2\delta\sigma^3)U^2+\sigma^4=0. \end{aligned} \tag{12}$$

This allows us to compute the set of vertices of circumparabolas over triangles in the poristic family. Thereby, we can verify another well-known result (see [12]):

Theorem 3. *The locus of all vertices of circumparabolas of a triangle is a circular septic curve.*

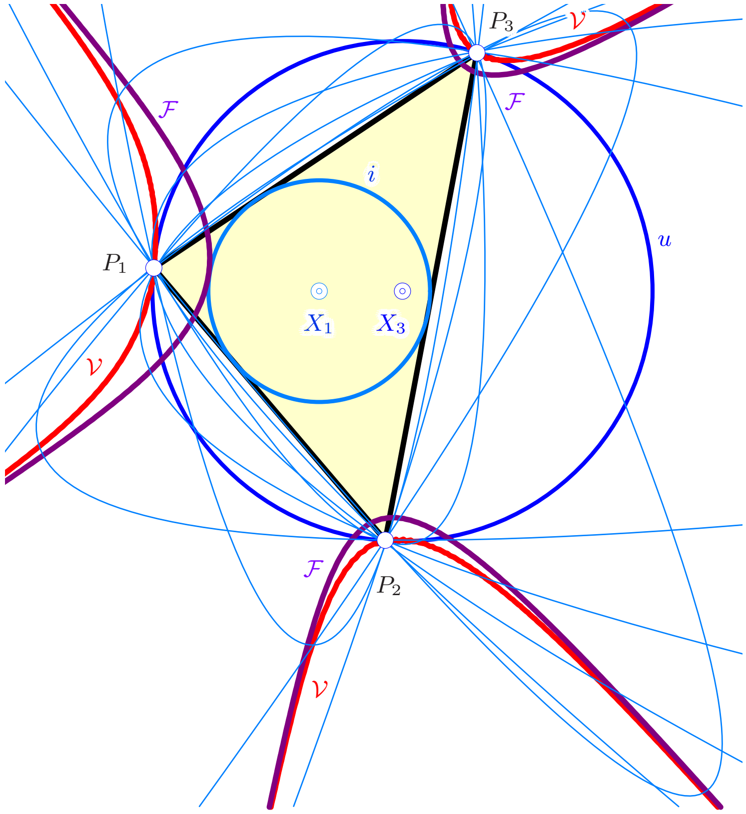

Fig. 3. The vertices of all circumparabolas lie on a septic curve $\mathcal{V}$ (red); while their foci lie on a quintic curve $\mathcal{F}$ (violet).

The envelope of axes over the poristic family. The equations of the axes (12) depend on two parameters: (i) The parameter T describing $P_1 \in u$ and the other two vertices of poristic triangles. (ii) The parameter U determining $Q \in u$, and therefore, a parametrization of the family of circumparabolas. This allows us to consider the axes (12) as two independent one-parameter families of lines, each of which envelops a certain curve.

Referring to Fig. 2, we first show that:

Theorem 4. *For a fixed pivot $Q \in u$,* i.e., *a fixed ideal point $\iota(Q)$, the axes of circumparabolas of triangles in the poristic family pass through a fixed point F. The set of all points F (while Q traverses u) is an ellipse e_i with the semi-axes*

$$a_i = \frac{\delta^2+\sigma^2}{4(\delta+\sigma)} = \frac{R^2+d^2}{4R}, \quad b_i = \frac{\delta\sigma}{2(\delta+\sigma)} = \frac{R^2-d^2}{4R} \tag{13}$$

centered at $X_{1358} = \left(\frac{d}{2}, 0\right)$.

Proof. Eliminating the *poristic parameter* T from the two equations a (given in (12)) and $a_T := \frac{\mathrm{d}}{\mathrm{d}T}a$ yields the poristic envelope. Over the real numbers, the resultant of a and a_T factors into 3 polynomials, one of degree 1

$$\mathcal{L}:\ 2(\delta+\sigma)U(1+U^2)\big(Ux+y\big)+U(U^2\delta^2-2\delta\sigma-\sigma^2)=0 \tag{14}$$

and two of degree 2. The two quadratic polynomials describe two pairs of complex conjugate lines (one of which is an isotropic pair) emanating from the real point

$$F=\frac{1}{2(\delta+\sigma)(1+U^2)}\begin{pmatrix}\sigma^2-U^2\delta^2\\ 2\delta\sigma U\end{pmatrix}. \tag{15}$$

This point is incident with the line $\mathcal{L}$, and over the course of all pivot points Q traces the circumcircle u, while the point F itself moves on the ellipse

$$e_i:\ 4\delta^2\sigma^2(\delta+\sigma)^2x^2+2\delta^2\sigma^2(\delta-\sigma)(\delta+\sigma)^2x+(\delta+\sigma)^2(\delta^2+\sigma^2)^2y^2-\delta^4\sigma^4=0 \tag{16}$$

which has semi-axes (13) and is centered at X_{1358}. □

The ellipse e_i has for real foci the incenter $X_1=(0,0)$ and circumcenter $X_3=(d,0)$ common to all triangles in the porism. It is therefore centered at X_{1385}. (For details on relations between said triangle centers see [20].) It can be also be shown:

Theorem 5. *The axes of all circumparabolas of a fixed triangle $P_1P_2P_3$ envelop a Steiner cycloid.*

Proof. Computing the envelope of the axes for a fixed triangle can be done by eliminating parameter T from a (given in (12)) and $a_U=\frac{\mathrm{d}}{\mathrm{d}U}a$. This results in a quartic factor $\mathcal{Q}$ and a linear factor $\mathcal{M}$. $\mathcal{Q}=0$ describes a Steiner cycloid (rational, bicyclic, quartic curve with three cusps of the first kind, cf. [2,3,6,7, 25]) (it is rather technical to show that $\mathcal{Q}=0$ indeed describes a one-parameter family of Steiner cycloids). The linear factor

$$\begin{aligned}\mathcal{M}:\ 2T(\delta+\sigma)(\delta^2T^2-2\delta\sigma-\sigma^2)x-2(\delta+\sigma)(\delta(\delta+2\sigma)T^2-\sigma^2)y+\\ +\delta^2T(\delta^2T^2-2\delta\sigma-\sigma^2)=0\end{aligned}$$

is the equation of a line which is tangent to the cycloid $\mathcal{Q}=0$ for all $T\in\mathbb{R}$. □

Fig. 4 shows the Steiner cycloid enveloped by the axes of all circumparabolas of a certain triangle in the poristic family.

Envelopes of Steiner cycloids and a further porism.

Theorem 6. *The envelope of all Steiner cycloids over the poristic family consists of two ellipses e_i, e_c: e_i (given in (16)) is internally tangent to all cycloids, while e_c carries the cusps of the cycloids. Like e_i, e_c is also centered at X_{1385} and has the following semi-axes:*

$$a_c=\frac{\delta^2+4\delta\sigma+\sigma^2}{4(\delta+\sigma)}=\frac{3R^2-d^2}{4R},\quad b_c=\frac{\delta^2+\delta\sigma+\sigma^2}{2(\delta+\sigma)}=\frac{3R^2+d^2}{4R}. \tag{17}$$

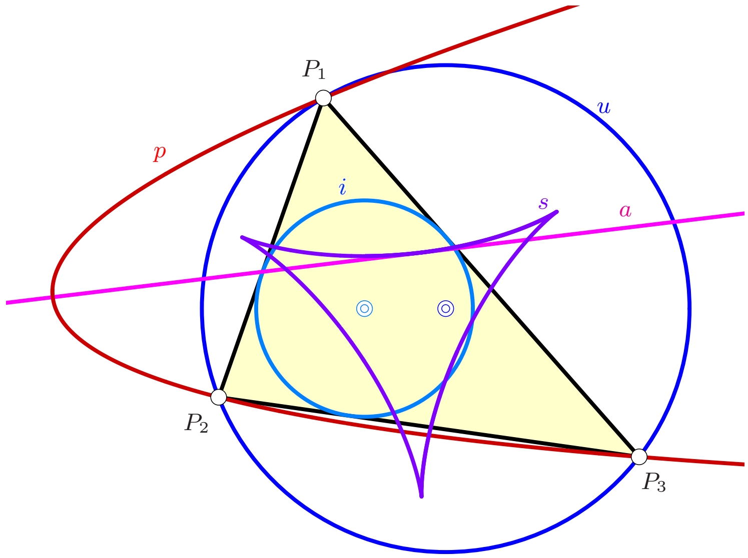

Fig. 4. While the circumparabola p traverses the family of all circumparabolas of a fixed triangle in the poristic family, its axes envelop a Steiner cycloid s.

Proof. The quartic polynomial $\mathcal{Q}$ given in the proof of Theorem 5 depends on the porism parameter T. Therefore, it describes a one-parameter family of Steiner cycloids whose envelope is given by $\mathcal{Q} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{Q} = 0$. Eliminating T from the latter two equations, we obtain an implicit equation for the envelope factoring into two ellipses with the semi-axes (13) and (17). $\square$

Fig. 5 shows some Steiner cycloids together with the trace e_c of the cusps which is obviously traced thrice in the course of one poristic round. Fig. 5 also shows the ellipse e_i enveloped and touched by the Steiner cycloids. Since the cycloids' cusps on the outer ellipse e_c are singular points (of multiplicity two) considered on the cycloids, they contribute to the envelope of the cycloids.

Theorem 6 does not only describe the envelope of the Steiner cycloids, but it also allows us to formulate another kind of porism:

Consider two nested and concentric ellipses e_i and e_c such that the major axis of e_i lies on the minor axis of e_c. If it is possible to draw a Steiner cycloid with cusps on e_c such that the cycloid touches the inner ellipse e_i three times for one particular starting point C (cusp) on e_c, then the same is possible for any choice of $C \in e_c$.

Of course, ellipses e_c and e_i must satisfy certain conditions (maybe not as simple as those in [14, Thm. 9.5.4, p. 432]) in order to guarantee the existence of such a porism to exist.

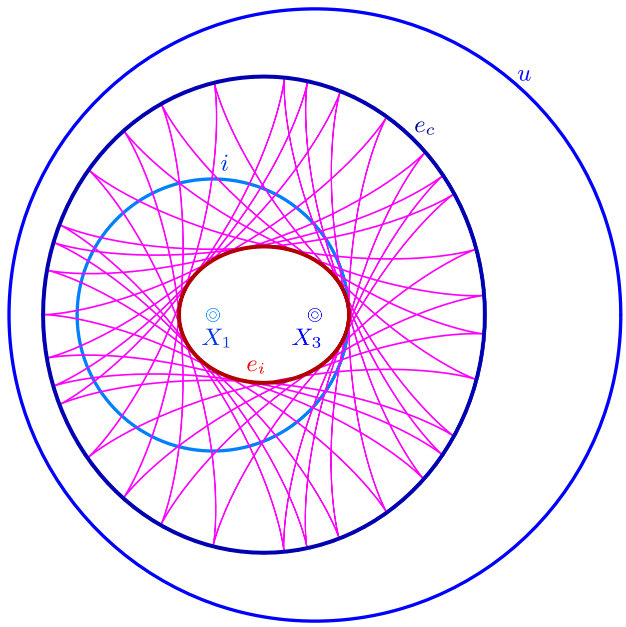

Fig. 5. Over poristic triangles, the Steiner cycloids as envelopes of the axes of the circumparabolas envelop two ellipses e_i and e_c. The triangles' incenter X_1 and circumcenter are the real foci of e_i.

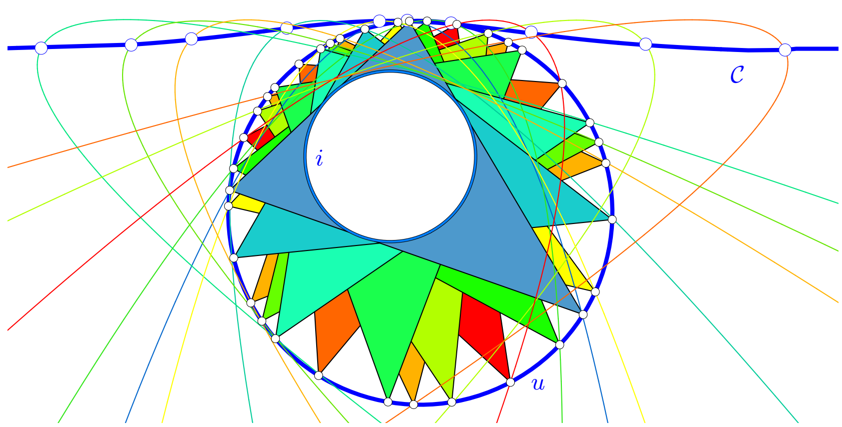

Fig. 6. The poristic trace of the vertices of the circumparabolas is a rational cubic $\mathcal{C}$.

Vertices and other points. With the equations of the parabolas (7), (9), and the equations of their axes (12), we can find the vertices. The intersection of a parabola with its axis yields only one proper point, the vertex V. The homogeneous coordinates $v_0 : v_1 : v_2$ of V are bi-variate polynomials $v_i(T, U)$ of bi-degree $(9,7)$ and reads:

$$
\begin{aligned}
&v_0 : v_1 : v_2 = 4(\delta+\sigma)(T-U)(1+T^2)(1+U^2)^2(\delta^2T^2+\sigma^2)^2\cdot\\
&\cdot\Big(\delta^3(\delta+2\sigma)T^2U^2-\delta^2\sigma^2T^2+2\delta\sigma(\delta+\sigma)^2TU-\delta^2\sigma^2U^2+\sigma^3(2\delta+\sigma)\Big):\\
&:\delta^8(4\sigma^2-\delta^2)T^9U^6+\ldots+\delta^7(\delta+2\sigma)(\delta^2-4\delta\sigma-4\sigma^2)U^7T^8+\ldots:\\
&:-\delta^8(\delta+2\sigma)^2T^9U^7+\ldots+\delta^7(\delta+2\sigma)(\delta^2-4\delta\sigma-4\sigma^2)T^8U^6+\ldots.
\end{aligned}
\tag{18}
$$

The parametrization (18) allows us to verify the following:

Theorem 7. *Over poristic triangles $\Delta = P_1P_2P_3$, the vertices of all circumparabolas move on a cubic curve.*

Proof. We consider $(v_1v_0^{-1}, v_2v_0^{-1})$ as a parametrization of a curve depending on the poristic parameter T. The elimination (by means of resultant) of T from $x = v_1v_0^{-1}$ and $y = v_2v_0^{-1}$ yields a product of a cubic polynomial

$$
\begin{aligned}
&4(\delta+\sigma)^2(1+U^2)^2(x^2+y^2)\big((\sigma^2-\delta^2U^2)x+2\delta\sigma Uy\big)-\\
&-(1+U^2)(\delta+\sigma)\Big(\big(4\delta^2(\delta^2-\delta\sigma-\sigma^2)U^4-4\delta\sigma(\delta^2+3\delta\sigma+\sigma^2)U^2-\\
&-4\sigma^2(\delta^2+\delta\sigma-\sigma^2)\big)x^2+12\delta\sigma(\sigma^2-U^2\delta^2)Uxy+\\
&+\big((\delta^2-4\delta\sigma-4\sigma^2)\delta^2U^4-2\delta\sigma(2\delta^2-3\delta\sigma+2\sigma^2)U^2-\sigma^2(4\delta^2+4\delta\sigma-\sigma^2)\big)y^2\Big)-\\
&-\Big((\delta^2-4\delta\sigma-4\sigma^2)\delta^2U^4-2\delta\sigma(2\delta^2+3\delta\sigma+2\sigma^2)U^2-\sigma^2(4\delta^2+4\delta\sigma-\sigma^2)\Big)\cdot\\
&\cdot\Big(\delta^2U^2x-2\delta\sigma Uy-\sigma^2x\Big)+\delta\sigma(\delta U^2+\sigma)(\delta^2U^2+\sigma^2)^2=0
\end{aligned}
\tag{19}
$$

and degree one polynomial

$$
\begin{aligned}
&-4(\delta+\sigma)(1+U^2)\Big(\delta^4(\delta+2\sigma)^2U^{10}+\delta^2(\delta+2\sigma)(4\delta^3+10\delta^2\sigma+\delta\sigma^2-2\sigma^3)U^8+\\
&+\delta(\delta^5+6\delta^4\sigma+4\delta^3\sigma^2-20\delta^2\sigma^3-21\delta\sigma^4-4\sigma^5)U^6+\\
&+\sigma(4\delta^5+21\delta^4\sigma+20\delta^3\sigma^2-4\delta^2\sigma^3-6\delta\sigma^4-\sigma^5)U^4+\\
&+\sigma^2(\sigma+2\delta)(2\delta^3-\delta^2\sigma-10\delta\sigma^2-4\sigma^3)U^2-\sigma^4(\sigma+2\delta)^2\Big)x+\\
&+4(\delta+\sigma)U(1+U^2)\Big(\delta^3(\delta+2\sigma)(\delta^2+2\delta\sigma+4\sigma^2)U^8-\delta^2(\delta^4+4\delta^3\sigma-9\delta^2\sigma^2-\\
&-32\delta\sigma^3-8\sigma^4)U^6+\delta\sigma(4\delta^4+39\delta^3\sigma+88\delta^2\sigma^2+39\delta\sigma^3+4\sigma^4)U^4+\\
&+\sigma^2(8\delta^4+32\delta^3\sigma+9\delta^2\sigma^2-4\delta\sigma^3-\sigma^4)U^2+\sigma^3(\sigma+2\delta)(4\delta^2+2\delta\sigma+\sigma^2)\Big)y+\\
&+4\delta\sigma^4(\delta+\sigma)(\sigma+2\delta)^2+4\delta^4\sigma(\delta+\sigma)(\delta+2\sigma)^2U^{12}-\\
&-\delta^3(\delta+2\sigma)(3\delta^4-6\delta^3\sigma-56\delta^2\sigma^2-60\delta\sigma^3-16\sigma^4)U^{10}+\\
&+4\delta^2\sigma(6\delta^5+45\delta^4\sigma+110\delta^3\sigma^2+106\delta^2\sigma^3+43\delta\sigma^4+5\sigma^5)U^8+\\
&+2\delta\sigma(2\delta^6+34\delta^5\sigma+136\delta^4\sigma^2+199\delta^3\sigma^3+136\delta^2\sigma^4+34\delta\sigma^5+2\sigma^6)U^6+\\
&+4\delta\sigma^2(5\delta^5+43\delta^4\sigma+106\delta^3\sigma^2+110\delta^2\sigma^3+45\delta\sigma^4+6\sigma^5)U^4+\\
&+\sigma^3(\sigma+2\delta)(16\delta^4+60\delta^3\sigma+56\delta^2\sigma^2+6\delta\sigma^3-3\sigma^4)U^2=0.
\end{aligned}
\tag{20}
$$

The cubic polynomial is annihilated by the vertices coordinates (18) and describes a one-parameter family of rational circular cubic curves. Their singularities are the points given in (15) which are isolated double points and located on the ellipse e_i with the equation (13). □

In a similar way, we can show that there are some further points on the parabolas' axes that trace cubic curves. Even the envelopes of the parabolas' directrices are rational cubic curves. Thm. 7 is illustrated in Fig. 6.

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