CONCHOIDAL RULED SURFACES

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ABSTRACT: This paper aims at the definition of a line-geometric conchoid transform for ruled surfaces in three-dimensional Euclidean space. The presented definition of the transform preserves ruled surfaces as well as rational parametrizations of ruled surfaces. Here we give examples of conchoids to some simple ruled surfaces and collect some properties of the thereby obtained surfaces. Upper bounds on algebraic degrees of conchoidal ruled surfaces are also derived.

Keywords: Conchoid, ruled surface, regulus, cross ratio, affine ratio, parametrization

1. INTRODUCTION

The construction of a conchoid of a plane curve c is well known: We choose a fixed point F and a real value d. Assume X is a point on c, then we find a point X_d on the line $[F,X]$ such that the signed distance $\overline{XX_d}$ equals d, see [1, 2]. The set c_d of all points X_d as X varies in c is called the conchoid of c with focus F at distance d. An example is displayed in Figure 1. Especially, the conchoid of Nicomedes, *i.e.*, the conchoid of a line, plays an important role for problems of cube duplication, angle trisection, heptagon construction, see [2, 3].

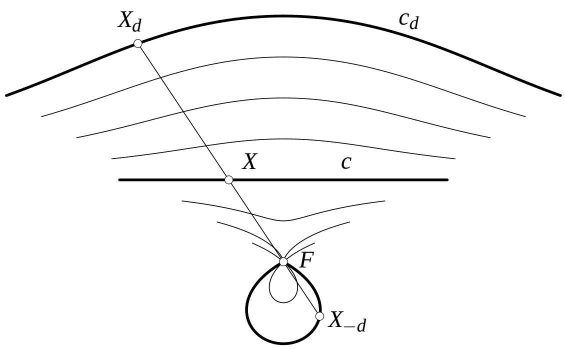

Figure 1: Conchoid c_d of a line c with unsigned distance d and focus F.

In the last three years conchoids have been rediscovered for CAGD, see [4], [5, 6]. The case of conchoids to surfaces has undergone intensive study in [7]. In the latter work the authors focus on rational parametrizations since these are useful in CAGD. Unfortunately, the method presented in [7] does not preserve ruled surfaces.

Our work is dedicated to the definition of conchoids of ruled surfaces which themselves are ruled surfaces. The conchoidal ruled surface of a ruled surface is obtained by applying the conchoidal transform to any of its generators. The presented kind of line-geometric conchoid transform uses the affine ratio, *i.e.*, it is invariant under affine transformations, but it also allows a generalization to a projectively invariant conchoid transform. Both constructions, the affine and the projective version, do not only preserve ruled surfaces (in contrast to the approach in [7]), they also preserve rational parametrizations of ruled surfaces.

The outline of this paper is organized as follows. In Section 2, we provide some well-know facts on ruled surfaces as needed for the definition of the line-geometric conchoid transform in Section 3. In Section 4, we give examples of conchoids to some simple ruled surfaces and collect some properties of the thereby obtained surfaces in Section 5.

2. PRELIMINARIES

Assume $I \subset \mathbb{R}$ is an open interval, $\mathbf{a} : I \to \mathbb{R}^3$ is a curve and $\mathbf{e} : I \to S^2$ is a unit vector field. A ruled surface $\mathscr{L}$ in three-dimensional Euclidean space $\mathbb{R}^3$ is parametrized by

$$\mathbf{r}(u,v) = \mathbf{a}(u) + v\,\mathbf{e}(u) \tag{1}$$

with $(u,v) \in I \times \mathbb{R}$. The curve $\mathbf{a}$ is referred to as a directrix and $\mathbf{e}$ is the spherical image of $\mathscr{L}$. Furthermore, the curves $\mathbf{r}(u_0,v)$ for fixed $u_0 \in I$ are called rulings of $\mathscr{L}$. We denote the canonical scalar product of arbitrary vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^3$ by $\mathbf{u}\cdot\mathbf{v}$. The thus induced cross product $\mathbf{u}\times\mathbf{v}$ yields a vector which is orthogonal to both, $\mathbf{u}$ and $\mathbf{v}$, respectively.

In order to simplify computations, we replace the directrix $\mathbf{a}$ by the vector-valued function $\widehat{\mathbf{e}} := \mathbf{a}\times\mathbf{e}$ which is obviously independent of the choice of $\mathbf{a}$ on $\mathscr{L}$. For any $u_0 \in I$, the vector $\widehat{\mathbf{e}}(u_0) =: \widehat{\mathbf{e}}_0$ is the momentum vector of the ruling $L_0 \in \mathscr{L}$. Moreover, $\mathbf{e}\times\widehat{\mathbf{e}}$ parametrizes the set of pedal points on $\mathscr{L}$ and therefore it is a directrix. The vector

$$(\mathbf{e}_0,\widehat{\mathbf{e}}_0)^{\mathrm{T}} = (e_1,e_2,e_3,e_4,e_5,e_6)^{\mathrm{T}} \quad \text{with} \quad \mathbf{e}_0\cdot\mathbf{e}_0 = 1,\quad \mathbf{e}_0\cdot\widehat{\mathbf{e}}_0 = 0 \tag{2}$$

is usually called the *spear coordinates vector* of the oriented line L_0. Conversely, any 6-tuple $(\mathbf{f},\widehat{\mathbf{f}})^{\mathrm{T}} \in \mathbb{R}^6$ satisfying Equation (2) uniquely defines an oriented line in $\mathbb{R}^3$. In the following, we identify any $L \subset \mathbb{R}^3$ with its unique spear coordinates and write $\mu(L,L) := 2\mathbf{e}\cdot\widehat{\mathbf{e}} = 0$ in Equation (2). Two lines $L_1 = (\mathbf{e}_1,\widehat{\mathbf{e}}_1)$, $L_2 = (\mathbf{e}_2,\widehat{\mathbf{e}}_2)$ are intersecting or parallel exactly if their spear coordinates fulfill $\mu(L_1,L_2) := \mathbf{e}_1\cdot\widehat{\mathbf{e}}_2 + \widehat{\mathbf{e}}_1\cdot\mathbf{e}_2 = 0$, and vice versa, see [8].

Example 1: We examine the set of rulings of an orthogonal hyperbolic paraboloid, *i.e.*, a regulus $\mathscr{L}$. It can be defined using the z-axis and the line F given by $\mathbf{f}(u) = (e, u\cot\alpha, u)^{\mathrm{T}}$ (see Figure 2): The regulus $\mathscr{L}$ is the set of all lines L being parallel to the $[x,y]$-plane, and intersecting both lines. Its spear coordinates are

$$L(u) = \frac{1}{\lambda}(e, \cot\alpha\, u, 0, -\cot\alpha\, u^2, eu, 0)^{\mathrm{T}} \tag{3}$$

with $u \in \mathbb{R}$ and $\lambda = \sqrt{e^2 + (\cot\alpha)^2u^2}$. One can easily check that $\mathscr{L}$ forms an orthogonal hyperbolic paraboloid Φ with equation $xz = ye\tan\alpha$. The x-axis is contained in $\mathscr{L}$. It is the generator $L_0 := L(0)$ through the vertex of Φ.[1]

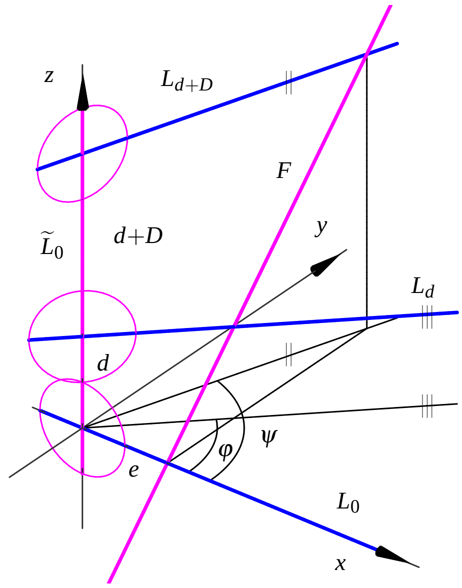

Figure 2: Regulus $\mathscr{L}$ of an orthogonal hyperbolic paraboloid: bijective correspondence between points of contact and the tangent planes (discs) along the vertex-generator $\widetilde{L}_0$ of the opposite regulus $\widetilde{\mathscr{L}}$.

For reguli $\mathscr{L}$ in projective three-dimensional space, one can use the cross ratio for parametrization. For any four lines $L_\infty \in \mathscr{L}$ with $i \in \{0,1,2,3\}$, it is defined as the cross ratio of the points of intersection p_i with an arbitrary line of the opposite regulus, *i.e.*,

$$\mathrm{cr}(L_0,L_1,L_2,L_3) = \mathrm{cr}(p_0,p_1,p_2,p_3).$$

Reguli of hyperbolic paraboloids in three-dimensional space have one generator at infinity. Here the cross ratio simplifies to the affine ratio $\mathrm{ar}(\ldots)$ of three proper lines. Consequently, the parameter is an affine one, *i.e.*, the affine ratio.

[1] The z-axis is the generator $\widetilde{L}_0$ of the opposite regulus $\widetilde{\mathscr{L}}$ of Φ through the vertex. $\widetilde{\mathscr{L}}$ is the set of lines intersecting all lines of $\mathscr{L}$.

Example 2: We continue Example 1. In order to obtain the generator L_∞ at infinity of $\mathscr{L}$, we substitute $u = t_1/t_0$ and multiply Equation (3) by t_0^2. And let $(t_0 : t_1) = (0 : 1)$, which results in $L_\infty := L(0:1) = (0,0,0,1,0,0)^{\mathrm{T}}$. It is the line at infinity of all planes $\alpha \perp L_0$. In terms of spear coordinates this reads $L_\infty = (\mathbf{o}, \mathbf{e}_0)^{\mathrm{T}}$.

The (constant) distribution parameter along the vertex generator $\widetilde{L}_0$ relates its points to their tangent planes of Φ, see [8]. This parameter is defined by $\delta_a = -u/\tan\varphi$, where φ is the oriented angle enclosed by the tangent planes at $u_0 = 0$ and $u_1 = d$. For the set of proper lines of $\mathscr{L}$, we obtain

$$\operatorname{cr}(L_{d+D}, L_d, L_0, L_\infty) = \frac{d+D}{d} = \frac{\tan\psi}{\tan\varphi} \qquad (4)$$

which is set to $\operatorname{ar}(L_{d+D}, L_d, L_0)$, see Figure 2.

3. CONCHOIDAL RULED SURFACES

We assign the definition of conchoidal surfaces to ruled surfaces. On the contrary to Peternell et al. [7], our definition should preserve the type of the surface, *i.e.*, a conchoid of a ruled surface is again a ruled surface.

Definition 1: Let A be a line, which neither has an intersection with $\mathscr{L}$ nor is orthogonal to any generator $L(u_0) \in \mathscr{L}$. There is a uniquely defined regulus $\mathscr{R}_0 := \mathscr{R}(u_0)$ on A and $L(u_0)$ of an orthogonal hyperbolic paraboloid with $A \in \mathscr{R}_0$ and $L(u_0)$ being its vertex-generator. Let $C(u_0) \in \mathscr{R}_0$ with

$$\operatorname{ar}(C, A, L)(u_0) = \delta \qquad (5)$$

for a fixed $\delta \in \mathbb{R}$. The set of lines $\mathscr{C}_\delta := \{C(u) : u \in I\}$ is again a ruled surface for any $\delta \in \mathbb{R}$. It shall be called the *conchoidal ruled surface* $\mathscr{C}_\delta$ of $\mathscr{L}$ with respect to A.[2]

Remark 1: For $\delta = 0$, the surface $\mathscr{L} = \mathscr{C}_0$ is contained in the set of conchoidal ruled surfaces,

[2]The definition of conchoidal ruled surfaces by means of the cross ratio allows some freedom: For example, one could also use the right hand side of Equation (4) with $D = \text{const.}$ in Equation (5).

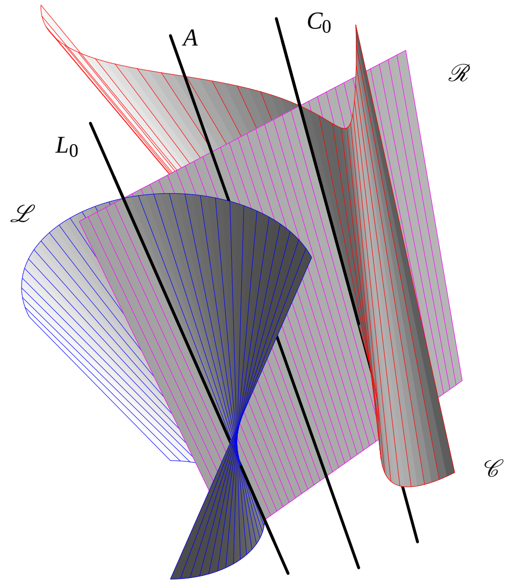

Figure 3: Conchoidal ruled surface $\mathscr{C}$ of a ruled surface $\mathscr{L}$ with focal line A and a regulus $\mathscr{R}$.

independent of the position of A. It can be called the *base ruled surface*. In the case $\delta = 1$, all lines of $\mathscr{C}_1$ coincide with A, which shall not be considered in the following.

Any regulus $\mathscr{R}$ can be parametrized using the cross ratio δ of a generator $R \in \mathscr{R}$ with respect to three distinct generators of the regulus. Labelling these lines by R_j with $j \in \{0,1,2\}$, it reads

$$R = \frac{1}{\lambda}\left((1-\delta)\,\mu(R_1,R_2)\,R_0 + \delta\,\mu(R_0,R_2)\,R_1 + (\delta-1)\delta\,\mu(R_0,R_1)\,R_2\right). \qquad (6)$$

For $\delta = 0$ $(1,\infty)$, we obtain R_0 (R_1, R_2). One further easily verfies that Equation (6) fulfills $\mu(R,R) = 0$ for any $\delta = \delta_0$. With the proper choice of λ, we can also achieve $\mathbf{e}\cdot\mathbf{e} = 1$, see Equation (2). Hence, Equation (6) gives the spear coordinates of a ruled surface. As R can be written in Bèzier form with control vectors depending on R_0, R_1, R_2, it describes a quadratic ruled surface. It is a regulus of a ruled quadric, if and only if, all factors $\mu(R_i, R_j)$ in Equation (6) are different from zero.

We apply Definition 1 and use Equation (6) in order to parametrize $\mathscr{C}_\delta$ by letting $R_0 = L_0$, $R_1 = A$, $R_2 = L_\infty(u_0)$. Here $L_\infty(u_0)$ denotes the line at infinity with the coordinate vector $L_\infty(u_0) = (\mathbf{o}, \mathbf{e})^{\mathrm{T}}(u_0)$, c.f. Example 2. We obtain

$$C(u) = \frac{1}{\lambda}\,((1-\delta)\,\mu(A,L_\infty)\,L + \delta\,\mu(L,L_\infty)\,A \\ +(\delta-1)\delta\,\mu(L,A)\,L_\infty)\,(u) \tag{7}$$

where $C(u)$ are the spear coordinates of $\mathscr{C}_\delta$.

4. SOME EXAMPLES

We present examples of conchoids of some simple ruled surfaces such as cylinders and reguli. We also show that some properties of the base surface are preserved on the associated conchoidal ruled surface while others do not. In order to discuss all conchoidal ruled surfaces of a given ruled surface, we have to distinguish all positions of A with respect to $\mathscr{L}$.

4.1 Cylinder of revolution

Assume we are given a cylinder of revolution $\mathscr{L}$. With respect to a Cartesian coordinate system it can be parametrized by

$$L(u) := (0,0,1,\sin u, -\cos u, 0)^{\mathrm{T}},\ u \in [0,2\pi).$$

Here the axis of $\mathscr{L}$ coincides with the z-axis and the directrix in the plane $z = 0$ is the circle with radius $r = 1$.

As all generators are parallel, there are only three cases to be discussed: The line A can be tangent to $\mathscr{L}$, it can be intersecting, or non-intersecting. We have to exclude $A \perp L$ and $A \parallel L$ for $L \in \mathscr{L}$. Without loss of genarality $A = (0,a,b,0,-eb,ea)^{\mathrm{T}}$ with $e \geq 0$ and $a^2+b^2 = 1$, where $ab \neq 0$ is assumed. Then the conchoidal ruled surfaces $\mathscr{C}_\delta$ to $\mathscr{L}$ and A can be parametrized with help of Equation (7). We obtain

$$C(u) = (1-\delta)\frac{b}{\lambda}\,(0,0,1,\sin u, -\cos u, 0)^{\mathrm{T}} \\ + \frac{\delta}{\lambda}\,(0,a,b,0,-eb,ea)^{\mathrm{T}} \\ + (\delta-1)\frac{\delta a}{\lambda}(e-\cos u)\,(0,0,0,0,0,1)^{\mathrm{T}} \tag{8}$$

with $L_\infty = (0,0,0,0,0,1)^{\mathrm{T}}$ for all $u \in I$. If $e < 1$ there exist two $u_0 \in [0,2\pi)$ for which the last summand in Equation (8) vanishes; if $e = 1$ ($e > 1$) there exists one (no such) value u_0. Further one can easily see that $\mathscr{C}_\delta$ is again a cylinder for the first spear vector equals $\mathbf{c}(u_0) = \mathbf{c}_0 = \text{const.}$ for all u_0.

It is easily verified that $(\mathbf{c} \times \widehat{\mathbf{c}})/\lambda^2(u)$ is a conic section and thus planar and rational of degree 2. So any $\mathscr{C}_\delta$ is a quadratic cylinder. An example is shown in Figure 4.

4.2 Regulus in a quadric of revolution

Given a regulus $\mathscr{L}$ of a one-sheeted hyperboloid of revolution. We suppose that its axis coincides with the z-axis of a Cartesian coordinate system and thus it can be parametrized by

$$L(u) = (-m\sin u, m\cos u, n, n\sin u, -n\cos u, m)^{\mathrm{T}}$$

$u \in [0,2\pi)$, where $m = \cos\alpha$, $n = \sin\alpha$, and α is the (constant) angle enclosed by the rulings L and the $[x,y]$-plane. Note that $\alpha \neq 0$ and $\alpha \neq \frac{\pi}{2}$. With the same argument as above, the line A can be chosen orthogonal to the x-axis. Therefore, A contains the point $B = (e,0,f)^{\mathrm{T}}$ with $f \in \mathbb{R}$, and thus we find $A = (0,a,b,-fa,-eb,ea)^{\mathrm{T}}$. Here we allow also the limit case $A \in \mathscr{L}$. There are further positions of A with regard to $\mathscr{L}$ to be discussed. It might be chosen skew to $\mathscr{L}$, or tangent, or intersecting, or parallel to one $L(u_0) \in \mathscr{L}$ (or even parallel to one generator of the opposite regulus).

We focus on two special cases: At first we take $A = L_0$, that is, $e = 1$, $f = 0$ and thus $m = a$, $n = b$. For any fixed δ the conchoidal ruled sur-

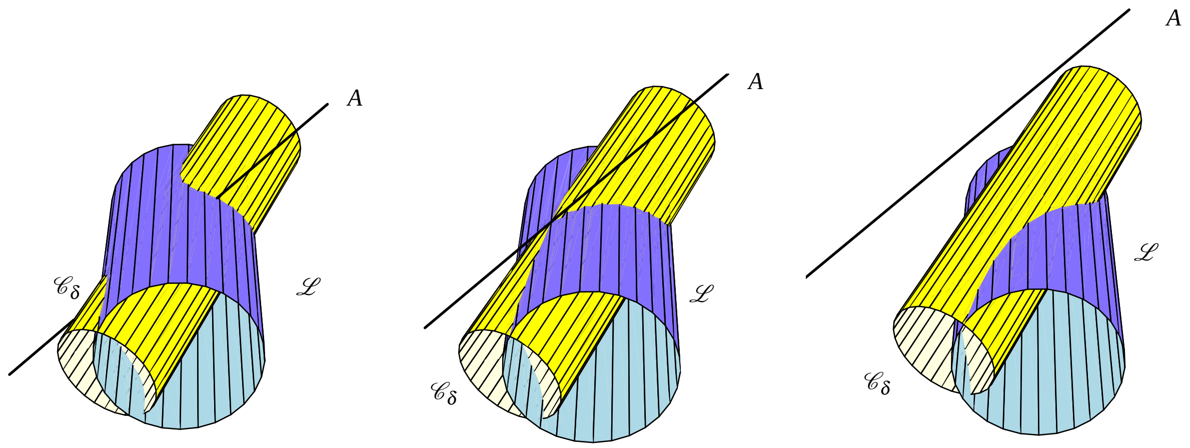

Figure 4: Conchoidal ruled surface $\mathscr{C}_\delta$ $\left(\delta = \frac{3}{10}\right)$ of a cylinder of revolution $\mathscr{L}$ for various positions of the line A, intersecting $\mathscr{L}$ in two points (left), touching it (center) at one generator and being skew to $\mathscr{L}$ (right).

face $\mathscr{C}_\delta$ of $\mathscr{L}$ and A is then parametrized by

$$C(u) = \frac{1}{\lambda}\,((1-\delta)(m^2\cos u + n^2)L + \delta L_0 + 2\delta(\delta-1)mn(1-\cos u)L_\infty) \tag{9}$$

where $L_\infty(u) = (0,0,0,-m\sin u, m\cos u, n)^{\mathrm{T}}$ represents the generator at infinity of $\mathscr{R}_u$, see Definition 1. For $u \to 0$, we get $C(0) \in \mathscr{C}_\delta$ with $C(0) = L_0$ for any δ as a limit position, see Equation (9). We replace $\sin u$ and $\cos u$ with rational expressions and obtain a parametrization of the conchoidal ruled surface which is obviously of degree four. An example is shown in Figure 5 (left).

Secondly, we choose A as the axis of $\mathscr{L}$, hence, $A = (0,0,1,0,0,0)^{\mathrm{T}}$. Here the parametrization reads

$$C(u) = \frac{1}{\lambda}\,((1-\delta)nL + \delta A + \delta(\delta-1)mL_\infty) \tag{10}$$

which evidently results in a rational parametrization with polynomial degree 2 if we replace $\sin u$ and $\cos u$ by rational expressions. Thus, we obtain a coaxial regulus for all δ, see Figure 5 (right).

5. SOME PROPERTIES

The above construction of conchoids of ruled surfaces results in ruled surfaces as is clear from Definition 1. It also preserves rational parametrizations, *i.e.*, if $\mathscr{L}$ has a rational parametrization then $\mathscr{C}_\delta$ also has one, see Equation (6). Counting the degrees in Equation (7) for a rational/algebraic L, we can state:

Theorem 1. *The conchoidal ruled surfaces of a given rational ruled surface with polynomial degree n are again rational ruled surfaces with polynomial degree of at most $2n$, independent on the choice of δ $(\delta \neq 1)$.*[3]

Theorem 1 allows us to estimate the algebraic degree of conchoidal ruled surface obtained in the way explained in Definition 1. Rational ruled surfaces lead to rational ruled surfaces. Actual degrees will not exceed the upper bound given in Theorem 1, but may sometimes be lower. We apply our construction of conchoids to some special rules surfaces in order to find some examples which is an objective of this paper. In the following we discuss some further properties of conchoidal ruled surfaces.

[3]The case $\delta = 1$ is discussed in Remark 1.

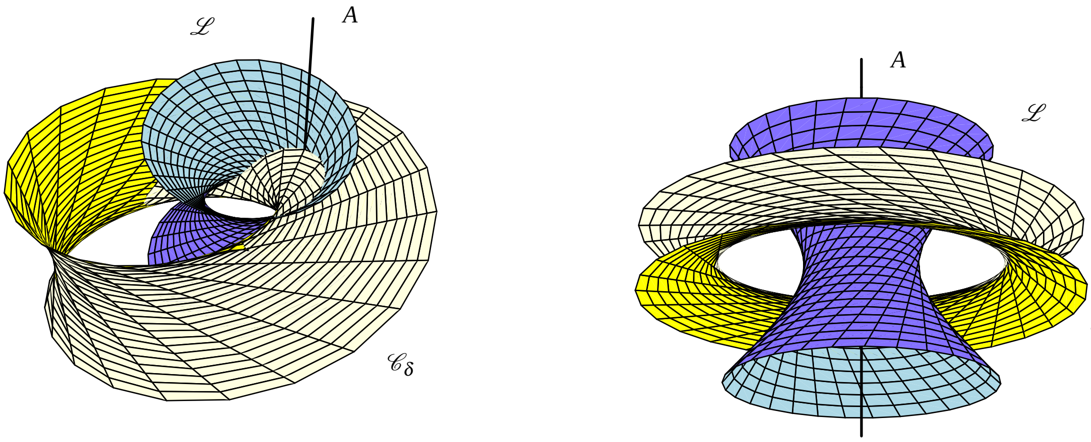

Figure 5: Conchoidal ruled surfaces $\mathscr{C}_\delta$ of a regulus of revolution $\mathscr{L}$, where the line A coincides with one of its generators (left) respectively with its axis.

Remark 2: Based on Definition 1 also the construction of conchoidal line-manifolds/-sets of k-parametric manifolds/sets of lines $(k \leq 3)$ is possible.[4] Both properties, preserving the basic element and the rational parametrization, evidently hold true also for manifolds/sets of lines.

We have already seen that the conchoidal ruled surface of a cylinder might again be a cylinder. Extending this to general cylinders, we can state:

Proposition 2. *Given any cylinder $\mathscr{L}$, and any line A which is neither parallel nor orthogonal to a generator $L \in \mathscr{L}$. Then all conchoidal surfaces $\mathscr{C}_\delta$ are again cylinders.*

Proof. Analogously to the example in Section 4, we compute the direction vector $\mathbf{c}$ of $\mathscr{C}_\delta$. It is easy to see that it does not depend on the parameter u of $L(u)$ for the vectors $\mathbf{e}$ with $\mathbf{e} \cdot \mathbf{e} = 1$ and $\mathbf{a}$ do not. $\square$

In general, a regular generator $L(u_0) \in \mathscr{L}$ is torsal, *i.e.*, locally either that of a cylinder, or of a cone, or of a tangent surface of a grad curve, if and only if $\mu(\dot{L},\dot{L}) = 0$ (the dot denotes the derivative with respect to u), see [8]. In contrast to Proposition 2, we obtain here that the corresponding generator $C_0 \in \mathscr{C}_\delta$ of a torsal, but non-cylindric, generator $L_0 \in \mathscr{L}$ needs not to be torsal.

[4] For example, conchoids of a bundle of lines or of a linear line congruence, which might have impact in geometric optics.

Proposition 3. *The generator $C_0 := C(u_0)$ of a conchoidal ruled surface $\mathscr{C}_\delta$ associated with a torsal and non-cylindric generator $L_0 \in \mathscr{L}$ is torsal, if and only if, $\mu(\dot{C}_0,\dot{C}_0) = 0$, which yields at u_0*

$$\mu(A,\dot{L}_\infty)\,\mu(A,\dot{L}) + \mu(A,L_\infty)\,\mu(A,L) = 0. \quad (11)$$

Proof. We define $C^\star = \lambda C$ with Equation (7). Without loss of generality, we may examine $\mu(\dot{C}_0^\star,\dot{C}_0^\star) = 0$ for all $u_0 \in I$ because $\dot{C}^\star = \dot{\lambda} C + \lambda \dot{C}$ and $\mu(C,\dot{C}) = 0$. Then the derivative of $C^\star$ with respect to u can be given as

$$\begin{aligned}\dot{C}^\star = {} & (1-\delta)\left(\mu(A,\dot{L}_\infty)L + \mu(A,L_\infty)\dot{L}\right) \\ & + \delta\left(\mu(\dot{L},L_\infty) + \mu(L,\dot{L}_\infty)\right)A \\ & + (\delta-1)\delta\left(\mu(\dot{L},A)L_\infty + \mu(L,A)\dot{L}_\infty\right).\end{aligned}$$

Calculating the expression $\mu(\dot{C}^\star,\dot{C}^\star)$, we may assume $\mu(L,L_\infty) = \mathbf{e}\cdot\mathbf{e} = 1$ and thus have $\mu(\dot{L},L_\infty) + \mu(L,\dot{L}_\infty) = 2\mu(\dot{L},L_\infty) = 0$. Analogously, the identity $\mu(\dot{L},L) = 0$ holds. If we additionally reparametrize $\dot{C}^\star(u)$ such that $\mu(\dot{C}^\star,\dot{C}^\star_\infty)(s) = 1$, *i.e.*, changing to the arc length parametrization of the spherical image of C, we obtain

$$\begin{aligned}\mu(\dot{C}_0^\star,\dot{C}_0^\star) &= -(1-\delta)^2\delta\left(\mu(A,\dot{L}_\infty)\,\mu(A,\dot{L})\right.\\ &\quad \left.+\,\mu(A,L_\infty)\,\mu(A,L)\right)(u_0)\\ &= 0\end{aligned}$$

at $u_0 \in I$ and thus verify Equation (11).

□

6. CONCLUSIONS

We proposed a definition of conchoid transform for straight line sets in three-dimensional space. This transform acts within the space of lines, *e.g.* ruled surfaces are mapped to ruled surfaces. We gave some examples of conchoids to some simple ruled surfaces and collected some properties of the thereby obtained ruled surfaces.

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