Degenerate Cubic Surfaces and the Wallace-Simson-Theorem in Space

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ABSTRACT: Asking for the set of all points P in the Euclidean plane for which the feet of normals drawn to the sides of a triangle Δ are collinear, we find the points of the circumcircle u of Δ. In Euclidean three-space, the triangle is replaced by a skew quadrilateral and we ask for all points P whose normals have coplanar feet. It is known that the locus of such points P is a cubic surface $\mathscr{K}$, see [4, 5]. In this contribution, we ask for conditions on the quadrilateral such that $\mathscr{K}$ degenerates, *i.e.*, it becomes either the union of a quadric and a plane or the union of three planes. For that we develop algebraic conditions on the coefficients of $\mathscr{K}$'s equation such that it degenerates. Since there are some cases to be distinguished, we unfortunately do not find a single condition. These conditions are polynomials of relatively high degree, but can nevertheless be used and handled with a computer algebra system. Testing several types of skew quadrilaterals, we are able to give examples of skew quadrilaterals which determine degenerate cubic surfaces $\mathscr{K}$ as the locus of points P whose four normals to the sides of the quadrilateral have coplanar feet.

Keywords: skew quadrilateral, tetrahedron, cubic surface, degenerate surface, quadric, degeneracy conditions, computer algebra, commutative algebra, symmetry.

1. INTRODUCTION

It is well known that the zeros of a quadratic equation $\sum_{i+j\le 2} a_{ij}x^iy^j = 0$ in two variables, say x and y, are the points (x,y) of a conic including degenerate cases. If $a_{ij}\in\mathbb{R}$ and (x,y) are interpreted as Cartesian coordinates in the Euclidean plane, there are the following types of conics: the regular conics such as the ellipse (i.a), the parabola (i.b), the hyperbola (i.c), and the empty set (i.d); the singular conics such as a pair of lines (parallel or not) (ii.a), a single point (which is the intersection of a pair of complex conjugate lines - parallel or not) (ii.b), or a repeated line (line with multiplicity two) (iii), cf. Fig. 1.

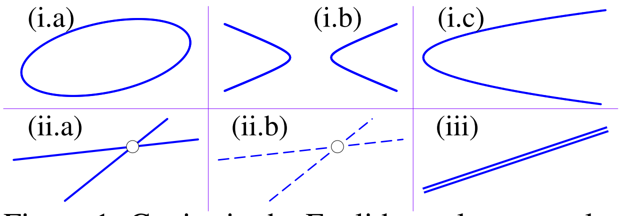

Figure 1: Conics in the Euclidean plane: regular (top row), singular or degenerate (bottom row).

The singular and regular conics can easily be characterized if we switch to homogeneous coordinates by letting $x = x_1x_0^{-1}$, $y = x_2x_0^{-1}$, $\mathbf{x} = (x_0,x_1,x_2)$, and writing the equation $\sum_{i+j\le 2} a_{ij}x^iy^j = 0$ in the form

$$c : (x_0,x_1,x_2)\begin{pmatrix} a_{00} & a_{01} & a_{02}\\ a_{01} & a_{11} & a_{12}\\ a_{02} & a_{12} & a_{22}\end{pmatrix}\begin{pmatrix}x_0\\ x_1\\ x_2\end{pmatrix} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = 0$$

where $\mathbf{A}$ is a symmetric 3×3-matrix with real entries. If $\mathbf{A}$ is regular, so is the conic c. If $\mathbf{A}$ is singular, we call the conic also singular or degenerate. The condition on $\mathbf{A}$ to be singular, or equivalently, the condition on c to be degenerate is simply

$$\det\mathbf{A} = 0.$$

Assuming that the conic c is degenerate (pair of lines or a repeated line), we can write down the homogeneous equation as a product of two linear homogeneous equations: $L \cdot M = (l_0x_0 + l_1x_1 + l_2x_2)(m_0x_0 + m_1x_1 + m_2x_2) = 0$ with arbitrary coefficients $l_i, m_i \in \mathbb{C}$ for $i \in \{0,1,2\}$. Though $a_{ij} \in \mathbb{R}$, it is useful to assume that $l_i, m_i \in \mathbb{C}$. For example $(x_1 + ix_2)(x_1 - ix_2) = x_1^2 + x_2^2 = 0$ is the real equation of a complex conjugate pair of lines containing only one real point $(1:0:0)$.

Comparing the coefficients of the monomials x_ix_j in the product $L \cdot M$ with those in the equation of c, yields a system of six equations:

$$l_im_i = a_{ii},\ i \in \{0,1,2\},$$
$$l_im_j + l_jm_i = 2a_{ij},\ (i,j) \in \{(0,1),(0,2),(1,2)\}.$$

From the latter system, l_i and m_i can be eliminated. This results in a condition on the coefficients a_{ij} that is to be satisfied in order to make them the coefficients of a singular quadratic form. The thus obtained condition equals

$$a_{00}a_{11}a_{22} + 2a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - \\ - a_{01}^2a_{22} - a_{02}^2a_{11} = \det \mathbf{A} = 0.$$

In the case of planar cubics, we have to distinguish between much more cases: A classification of planar cubics with respect to projective transformations in the real projective plane $\mathbb{P}^2$ yields four non-singular types. There are ellip-

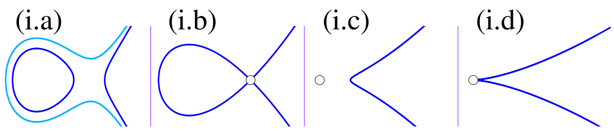

Figure 2: Non-degenerate cubics in the plane.

tic cubics, carrying no singular point (i.a) and the rational cubics, carrying precisely one singular point: an ordinary double point (i.b), or a cusp of the first kind (i.c), or an isolated double point (i.d), cf. Fig. 2. Up to projective transformations, there are eight different types of degenerate plane cubics: the union of a conic (carrying real points) with a line (three types depending on the number of common points) (ii.a), the union of an empty conic (with a real equation) and a real line (only one case) (ii.b), a triple of real lines (not incident with a single point) (iii.a), the union of a pair of complex conjugate lines with a real line (not incident with a single point) (iii.b), a triple of real and concurrent lines (iv.a), a triple of concurrent lines containing a complex conjugate pair of lines intersecting in a (real) point on the real line (iv.b), the union of a repeated line with a different real line (v), and a three-fold line (vi), see Fig. 3. The more we specialize the

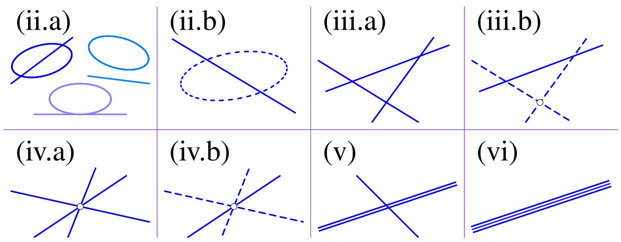

Figure 3: Degenerate cubics in the plane.

group of transformations to which the classification of planar cubics is made, the more cases are to be distinguished. However, we cannot expect to find a single condition on the coefficients of a cubic's equation. We make use of the fact that any univariate cubic polynomial with real coefficients has at least one real root. Thus, any trivariate cubic polynomial that factors has at least one factor of degree one.

The computation of degeneracy conditions for multivariate inhomogeneous cubic polynomials uses the same technique as described above for the case of degenerate conics.

In the following, we shall leave the plane behind. We show how to find degeneracy conditions for cubic surfaces. The main part of the computation of degeneracy conditions is more or less technical. Furthermore, we are not aiming at a complete discussion of degenerate cubic surfaces. The conditions on the coefficients of cubic surfaces in Euclidean three-space $\mathbb{R}^3$ shall not be computed for their own sake. They shall be used for the discussion of a geometric problem which was addressed in [4–6].

In Sec. 2, we give the basic setup for our computations. Sec. 3 is dedicated to the computation of degeneracy conditions. To be honest, we shall not write down all the degeneracy conditions in full length, because some of them are very long. The use of these conditions needs a CAS. It is necessary to precompute these equations and store them in order to make use of them. Sometimes, we will only give the degree, the number of terms, and the degree of these polynomials considered as polynomials in certain variables. In Sec. 4, we apply the degeneracy conditions to the most general form of a skew quadrilateral. Unfortunately, this gives only rise to the Conjecture 4.1, since we were not able to finish the computations due to the lengths of the resultants to be built and because of limited memory. The computations were done with Maple 18© on a PC with an Intel© core iS-4460 with 3.2GHz and 7.8GB RAM.

In Sec. 5 we bring in the harvest and give examples of special skew quadrilaterals in $\mathbb{R}^3$. All these quadrilaterals have in common that their cubics $\mathscr{K}$ degenerate and for all points $P \in \mathscr{K}$ the four feet are coplanar. The degeneracy conditions derived in Sec. 3 allow us to find metric conditions on the lengths and angles in the skew quadrilateral such that $\mathscr{K}$ degenerates. This way of attacking the problem seems to be more efficient than just testing skew quadrilaterals whether $\mathscr{K}$ degenerates or not. Moreover, a CAS may not be able to factorize a trivariate polynomial with coefficients taken from some commutative field, because the factorization may need a proper field extension. The latter has to be found first and there are no algorithms for that.

2. THE PROBLEM - WALLACE-SIMSON

The result dealing with triangles in the Euclidean plane (also valid on the Euclidean unit sphere, see [2]) is due to W. Wallace and is often ascribed to R. Simson (see [1, 3]):

Lemma 2.1. *Let $\Delta = ABC$ be a triangle in the Euclidean plane and let u be its circumcircle. The feet of normals from P to Δ's side lines are collinear if, and only if, $P \in u$, see Fig. 4.*

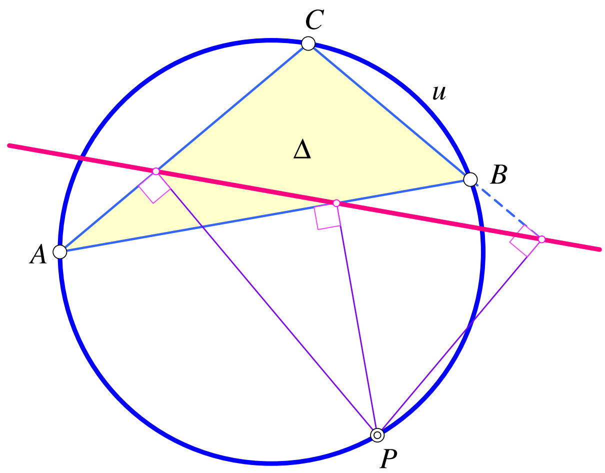

Figure 4: The three collinear feet of the normals from $P \in u$ to the sides of Δ.

In the following, we try a variation in Euclidean three-space $\mathbb{R}^3$: Assume that $ABCD$ is a skew quadrilateral in $\mathbb{R}^3$, *i.e.*, the vertices of a tetrahedron. Let further P be some point with Cartesian coordinates $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$. Then, we draw the normals $n_{[A,B]}$, $n_{[B,C]}$, ... from P to the lines $[A,B]$, $[B,C]$, ...which meet in the feet $F_{[A,B]}$, $F_{[B,C]}$, ... Now the question arises: Where to choose P such that the four pedal points are coplanar? According to [4] and [6], the points P gather on a cubic surface $\mathscr{K}$.

Our aim is to give conditions on the points A,B,C,D such that the cubic surface $\mathscr{K}$ degenerates, *i.e.*, it splits into a plane and a quadric.

It means no restriction to assume that the Cartesian coordinates of A,B,C,D are

$$\begin{aligned} \mathbf{a} &= (0,0,0),\ \mathbf{b} = (a,0,0), \\ \mathbf{c} &= (b,c,0),\ \mathbf{d} = (d,e,f), \end{aligned} \tag{1}$$

see Fig. 5. The choice of $A = (0,0,0)$ simplifies the computations. Choosing $B = (a,0,0)$ raises the computational symmetry rather than the choice $B = (1,0,0)$ simplifies it. The points A, B, C, D are not allowed to lie in a single plane, and thus, $\det(\mathbf{b},\mathbf{c},\mathbf{d}) = acf \neq 0$.

In almost all further cases, A will always coincide with the origin of the coordinate system. The coordinates of the remaining points will be chosen appropriately in order to benefit from symmetries whenever they occur.

The coordinates of the foot on $[A,B]$ are

$$F_{[A,B]} = \mathbf{a} + \alpha(\mathbf{b} - \mathbf{a}). \tag{2}$$

The parameter $\alpha \in \mathbb{R}$ is to be determined such that $[A,B] \perp [P, F_{[A,B]}]$. The same has to be done for the other feet with parameters β, γ, δ. Since A is represented by $\mathbf{o}$, we find

$$\begin{aligned} \alpha &= \frac{\langle \mathbf{x}, \mathbf{b} \rangle}{\|\mathbf{b}\|^2}, \ \beta = \frac{\langle \mathbf{x} - \mathbf{b}, \mathbf{c} - \mathbf{b} \rangle}{\|\mathbf{c} - \mathbf{b}\|^2}, \\ \gamma &= \frac{\langle \mathbf{x} - \mathbf{c}, \mathbf{d} - \mathbf{c} \rangle}{\|\mathbf{d} - \mathbf{c}\|^2}, \delta = \frac{\langle \mathbf{d} - \mathbf{x}, \mathbf{d} \rangle}{\|\mathbf{d}\|^2} \end{aligned} \tag{3}$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ is the canonical scalar product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ is the Euclidean length of a vector $\mathbf{u} \in \mathbb{R}^3$. Combining (2) and (3), we find the four feet as

$$\begin{aligned} F_{[A,B]} &= \alpha \mathbf{b}, \ \ F_{[B,C]} = \mathbf{b} + \beta(\mathbf{c} - \mathbf{b}), \\ F_{[C,D]} &= \mathbf{c} + \gamma(\mathbf{d} - \mathbf{c}), F_{[D,A]} = (1 - \delta)\mathbf{d}. \end{aligned} \tag{4}$$

Four points P, Q, R, S with coordinate vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ are coplanar if, and only if,

$$\begin{aligned} &\det(\mathbf{p}, \mathbf{q}, \mathbf{s}) + \det(\mathbf{q}, \mathbf{r}, \mathbf{s}) + \\ &+ \det(\mathbf{r}, \mathbf{p}, \mathbf{s}) - \det(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0. \end{aligned} \tag{5}$$

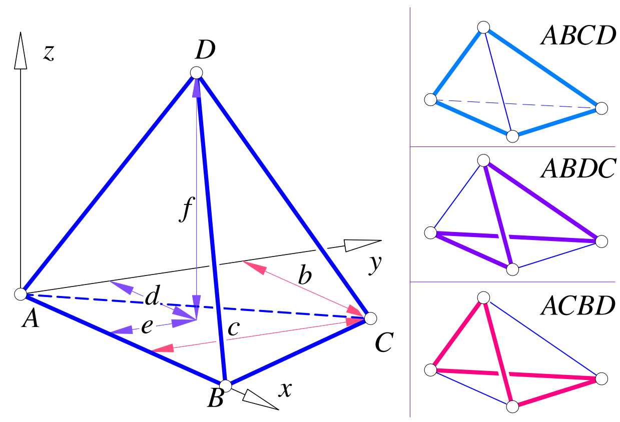

Figure 5: The tetrahedron $ABCD$ and the three different skew quadrilaterals.

If we insert the feet (4) into (5), we obtain a condition on α, β, γ, δ which can be written in terms of the elementary symmetric polynomials

$$\begin{gathered} \varepsilon_0 = 1, \ \varepsilon_1 = \alpha + \beta + \gamma + \delta, \\ \varepsilon_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta, \\ \varepsilon_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta \end{gathered}$$

as

$$\mathscr{K} : \ \varepsilon_0 - \varepsilon_1 + \varepsilon_2 - \varepsilon_3 = 0 \tag{6}$$

where $\det(\mathbf{b}, \mathbf{c}, \mathbf{d}) = acf \neq 0$ is canceled. The equation (6) is indeed an equation of a cubic surface since the elementary symmetric polynomial $\varepsilon_4 = \alpha\beta\gamma\delta$ does not show up and α, β, γ, and δ are linear in $\mathbf{x}$.

It is easy to verify that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ annihilate (6), and therefore, the points A, B, C, D are located on the cubic surface $\mathscr{K}$.

The cubic equation (6) is one of the main ingredients of the following computations. In the next section, we try to elaborate conditions on a general cubic equation in three unknowns x, y, z such that the cubic equation factors, and thus, the corresponding cubic surface degenerates.

3. THE DEGENERACY CONDITIONS

A cubic surface in three-space can degenerate in several ways but all these have one thing in common: a plane splits off. Therefore, we only discuss the case that the cubic becomes the union of a plane and some quadratic variety, whatever it may look like.

Allthough the initial example for the computation of a degeneracy condition starts with the homogeneous equations of the involved components, we now use the inhomogeneous equations, because it is sufficient to do so. We use the technique sketched in Sec. 1 and assume that the cubic surface $\mathscr{K}$ has the equation

$$\mathscr{K} : \sum_{r+s+t \leq 3} k_{r,s,t} x^r y^s z^t = 0 \tag{7}$$

in terms of Cartesian coordinates x, y, z. Then, we assume that $\mathscr{K}$ is degenerate and splits into a plane $\mathscr{P}$ and the set $\mathscr{Q}$ of zeros of a trivariate polynomial of degree 2 with the respective equations

$$\begin{aligned} \mathscr{P}:&\ l_0+l_1x+l_2y+l_3z=0,\\ \mathscr{Q}:&\sum_{r+s+t\le 2} q_{r,s,t}x^ry^sz^t=0. \end{aligned} \tag{8}$$

We do not care whether $\mathscr{Q}$ is a regular or singular quadric. Since $A\in\mathscr{K}$ shall always coincide with the origin of the coordinate frame, we have

$$k_{000}=0. \tag{9}$$

There are two cases which have to be treated separately: (A) The component $\mathscr{P}$ contains the point A, and thus, $l_0=0$. (B) The component $\mathscr{Q}$ passes through the point A, and hence, $q_{000}=0$. Note that A equals the origin of the coordinate system.

Case (A): If $\mathscr{K}=\mathscr{P}\cup\mathscr{Q}$, then (7) is the product of the quadratic and linear equations in (8). We compare the coefficients and arrive at the system of equations

$$\begin{gathered} l_1q_{i-1,0,0}=k_{i,0,0}, l_2q_{0,i-1,0}=k_{0,i,0},\\ l_3q_{0,0,i-1}=k_{0,0,i},\\ l_1q_{110}+l_2q_{200}=k_{210},\ l_1q_{020}+l_2q_{110}=k_{120},\\ l_1q_{010}+l_2q_{100}=k_{110},\ l_1q_{101}+l_3q_{200}=k_{201},\\ l_1q_{002}+l_3q_{101}=k_{102},\ l_1q_{001}+l_3q_{100}=k_{101},\\ l_2q_{011}+l_3q_{020}=k_{021},\ l_2q_{002}+l_3q_{011}=k_{012},\\ l_2q_{001}+l_3q_{010}=k_{011},\\ l_1q_{011}+l_2q_{101}+l_3q_{110}=k_{111}, \end{gathered} \tag{10}$$

with $i\in\{1,2,3\}$. From (10), we eliminate l_i and q_{ijk} and obtain the conditions on the coefficients k_{rst} in $\mathscr{K}$'s equation such that $\mathscr{K}$ is the union of a quadric and a plane

$$\begin{gathered} k_{010}^3k_{300}-k_{010}^2k_{100}k_{210}+k_{010}k_{100}^2k_{120}-k_{030}k_{100}^3=0,\\ k_{001}^3k_{300}-k_{001}^2k_{100}k_{201}+k_{001}k_{100}^2k_{102}-k_{003}k_{100}^3=0,\\ k_{001}^3k_{030}-k_{001}^2k_{010}k_{021}+k_{001}k_{010}^2k_{012}-k_{003}k_{010}^3=0,\\ k_{010}^2k_{200}-k_{010}k_{100}k_{110}+k_{020}k_{100}^2=0,\\ k_{001}^2k_{200}-k_{001}k_{100}k_{101}+k_{002}k_{100}^2=0,\\ k_{001}^2k_{020}-k_{001}k_{010}k_{011}+k_{002}k_{010}^2=0,\\ -2k_{001}k_{010}^3k_{300}+k_{00,1}k_{010}^2k_{100}k_{210}-k_{001}k_{030}k_{100}^3\\ +k_{010}^3k_{100}k_{201}-k_{010}^2k_{100}^2k_{111}+k_{010}k_{021}k_{100}^3=0. \end{gathered} \tag{11}$$

Note that (9) has to be fulfilled too.

Case (B): In this case, the polynomials comprising the set of degeneracy conditions will be to long to be written down explicitly. Just in order to give an idea of the shape of these polynomials, we give a list $L(p)$ for each polynomial p that contains the following entries

$$L(p)=[d,n,[v_1,d_1],\dots,[v_k,d_k]]$$

with $d=\deg p$, n is the number of terms (if p is expanded), and $[v_i,d_i]$ gives the degree of p considered as a polynomial in the variable v_i. The equations (11) in Case (A) yield:

$$\begin{gathered} [4,4,[k_{010},3],[k_{030},1],[k_{100},3],[k_{120},1],[k_{210},1],[k_{300},1]],\\ [4,4,[k_{001},3],[k_{003},1],[k_{100},3],[k_{102},1],[k_{201},1],[k_{300},1]],\\ [4,4,[k_{001},3],[k_{003},1],[k_{010},3],[k_{012},1],[k_{021},1],[k_{030},1]],\\ [3,3,[k_{010},2],[k_{020},1],[k_{100},2],[k_{110},1],[k_{200},1]],\\ [3,3,[k_{001},2],[k_{002},1],[k_{100},2],[k_{101},1],[k_{200},1]],\\ [3,3,[k_{001},2],[k_{002},1],[k_{010},2],[k_{011},1],[k_{020},1]],\\ [5,6,[k_{001},1],[k_{010},3],[k_{021},1],[k_{030},1],\dots\\ \dots[k_{100},3],[k_{111},1],[k_{201},1],[k_{210},1],[k_{300},1]]. \end{gathered}$$

In the case (B) with $l_0\neq 0$ and $q_{000}=0$, *i.e.*, $A\in\mathscr{Q}$ but $A\notin\mathscr{P}$, we find degeneracy conditions which contain (9) and seven further equations of the folllowing shape:

$$\begin{gathered} [12,116,[k_{010},6],[k_{020},6],[k_{030},4],[k_{100},6],[k_{120},4],\dots\\ \dots[k_{200},6],[k_{210},4],[k_{300},4]],\\ [12,116,[k_{001},6],[k_{002},6],[k_{003},4],[k_{100},6],[k_{102},4],\dots\\ \dots[k_{200},6],[k_{201},4],[k_{300},4]],\\ [12,116,[k_{001},6],[k_{002},6],[k_{003},4],[k_{010},6],[k_{012},4],\dots\\ \dots[k_{020},6],[k_{021},4],[k_{030},4]],\\ [10,84,[k_{010},6],[k_{020},2],[k_{030},2],[k_{100},6],[k_{110},4],\dots\\ \dots[k_{200},6],[k_{210},4],[k_{300},4]],\\ [10,84,[k_{001},6],[k_{002},2],[k_{003},2],[k_{100},6],[k_{101},4],\dots\\ \dots[k_{200},6],[k_{201},4],[k_{300},4]],\\ [10,84,[k_{001},6],[k_{002},2],[k_{003},2],[k_{010},6],[k_{011},4],\dots\\ \dots[k_{020},6],[k_{021},4],[k_{030},4]],\\ [28,23470,[k_{001},4],[k_{002},4],[k_{003},4],[k_{010},12],\dots\\ \dots[k_{020},12],[k_{021},8],[k_{030},8],[k_{100},12],[k_{111},8],\dots\\ \dots[k_{200},12],[k_{201},8],[k_{210},8],[k_{300},8]]. \end{gathered}$$

At this point, we have to recall that the degeneracy conditions are necessary conditions. We do not know if they are sufficient for a trivariate cubic polynomial with coefficients k_{rst} to be a product of two polynomials of degree 1 and 2.

4. APPLICATION TO SKEW QUADS

With the preparations from Sec. 2 and 3, we are able to attack the main problem, *i.e.*, the search for skew quadrilaterals that lead to degenerate cubic surfaces $\mathscr{K}$ all of whose points send normals with coplanar feet to the sides of the quadrilateral.

In [4], it was recognized that symmetries of the tetrahedron *ABCD* may lead to a degenerate cubic surface $\mathscr{K}$. The mere choice of a tetrahedron yields 4!=24 differently labelled quadrilaterals. Infact, there are only three labellings that lead to different cubic surfaces $\mathscr{K}$ since it doesn't matter if we start counting at a particular vertex or if we traverse the quadrilateral in the opposite direction. So, there are three representatives, one of each orbit: We distinguish between the three quadrilaterals *ABCD*, *ABDC*, and *ACBD*, cf. Fig. 5. We shall call these the three *orbits*. The respective cubics shown in Fig. 6 share at least the four given points A, B, C, D.

Testing the degeneracy conditions by inserting the coefficients of the equations of $\mathscr{K}$ for either orbit gives rise to the following

Conjecture 4.1. *If the tetrahedron ABCD with vertices* (1) *has no symmetries and shows no right angles between any pair of edges (whether skew or not), then none of the three cubic surfaces* $\mathscr{K}$ *from* (6) *associated with the three types of skew quadrilaterals (ABCD, ABDC, ACBD) degenerates.*

Justification: A conjecture needs no proof, only a verification or a falsification. The reasons why the above statement is only a conjecture shall be given here. Firstly, we are not able to compute all necessary resultants (at least at the moment) due to their lengths, degrees, and the complexity of the computations. We shall explain the details below. Secondly, the degeneracy conditions given in Sec. 3 are necessary conditions on the coefficients of a trivariate cubic polynomial in order to make it a factorizable polynomial. Unfortunately, we do not know if these conditions are sufficient.

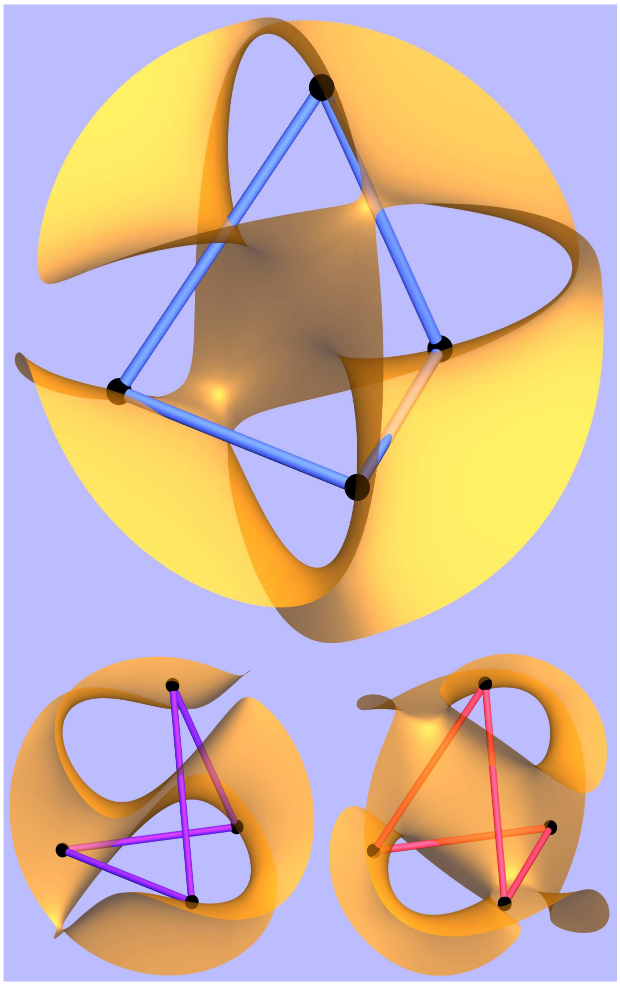

Figure 6: Three quadrilaterals and three cubics.

There is also a good reason why we describe these computations though they do not enable us to verify (or falsify) the conjecture. During the computations we get an idea what are possible conditions on the vertices A, B, C, and D such that the cubic surfaces $\mathscr{K}$ degenerate.

In the following, we do as if we were proving a theorem. We compute the cubics (6) and extract the coefficients from the respective equations. Note that the coefficients are polynomials in $a, \ldots, f$ whose degree is at most 6. Then, we insert them into the equations of the degeneracy conditions (11) for case (A). For the sake of completeness, we also insert these coefficients into the degeneracy conditions for case (B), the ones we haven't written down explicitly because of their lengths. In any case, the eighth equation $k_{000} = 0$ is automatically fulfilled because of the choice $\mathbf{a} = (0,0,0)$.

Case (A): After inserting the coefficients into the degeneracy conditions for case (A), we arrive at three sets of polynomial conditions on a, …, f: one for each orbit. In any of the three subcases, we observe that some equations are fulfilled automatically.

Some linear, quadratic, cubic, and even quartic factors can be canceled for they are not allowed to be zero: If a, b, c, d, e, or f equals zero, we have $A = B$; $\sphericalangle BAC = \frac{\pi}{2}$; A, B, C are collinear; $\sphericalangle BAD = \frac{\pi}{2}$; $\sphericalangle [A,B,C],[A,B,D] = \frac{\pi}{2}$; or $ABCD$ is planar. So we can cancel any powers of $a,\dots,f$ as well as any sum of even powers of $a,\dots,f$ from the degeneracy conditions. As we shall see, when handling the polynomial degeneracy conditions, there are far more such factors: $a-b \neq 0$ otherwise $\sphericalangle ABC = \frac{\pi}{2}$; $A \neq D$ is equivalent to $d^2+e^2+f^2 \neq 0$; $\sphericalangle ACB = \frac{\pi}{2}$ corresponds to $ab-b^2-c^2 = 0$. Further, $C \neq D \iff (b-d)^2+(c-e)^2+f^2 \neq 0$; $\sphericalangle ADC = \frac{\pi}{2}$ if, and only if, $bd+ce-d^2-e^2-f^2 = 0$ and $\sphericalangle ACD = \frac{\pi}{2}$ if, and only if, $b(b-d)+c(c-e) = 0$; $A \neq C$ if $b^2+c^2 \neq 0$. If the angle along $[B,C]$ is right, then $b(e^2+f^2)-cde = 0$. The normal vector to the plane $[A,C,D]$ is not the zero vector, and thus, $b^2e^2+b^2f^2+c^2e^2 \neq 0$. Since none of the variables $a,\dots,f$ is zero, any sum of squares of these variables cannot be zero. In total there are more than 60 such non-vanishing factors that can be canceled from the evaluated degeneracy conditions.

In order to find solutions to the initial problem, *i.e.*, quadrilaterals depending on a, …, f with degenerate cubic surface $\mathscr{K}$, we start to eliminate variables from the remaining equations. Maple's `Groebner` package renders the function `InterReduce(P,T,p)` that computes from a list P of polynomials according to some monomial order T (eventually with respect to some positive characteristic p) a list of polynomials defining the same ideal as P such that no polynomial can be reduced by the leading term of another polynomial. We cancel the non-vanishing factors and reduce the polynomials. This simplifies the representation of the polynomial ideal and drops the degree of the basis. Tab. 1 shows the characteristics of the remaining polynomials for any of the three orbits.

ABCD
$[5, 11, [a,1], [b,3], [c,3], [d,2], [e,2], [f,2]]$
$[12, 651, [a,4], [b,9], [c,9], [d,6], [e,9], [f,8]]$
$[15, 1561, [a,5], [b,10], [c,10], [d,9], [e,11], [f,10]]$
$[16, 1979, [a,6], [b,10], [c,10], [d,9], [e,12], [f,12]]$
ABDC
$[12, 604, [a,4], [b,6], [c,6], [d,8], [e,9], [f,6]]$
$[15, 1706, [a,4], [b,8], [c,8], [d,9], [e,11], [f,8]]$
$[16, 2118, [a,4], [b,8], [c,9], [d,11], [e,12], [f,8]]$
ACBD
$[5, 25, [a,2], [b,3], [c,2], [d,3], [e,2], [f,2]]$
$[7, 51, [a,4], [b,4], [c,4], [d,4], [e,4], [f,4]]$
$[9, 111, [a,5], [b,5], [c,4], [d,5], [e,4], [f,4]]$
$[11, 142, [a,5], [b,7], [c,6], [d,4], [e,4], [f,4]]$
$[11, 331, [a,6], [b,6], [c,5], [d,6], [e,5], [f,4]]$
$[16, 1094, [a,7], [b,8], [c,8], [d,8], [e,8], [f,6]]$
$[23, 4112, [a,9], [b,12], [c,11], [d,11], [e,11], [f,8]]$

Table 1: Degrees of reduced evaluated degeneracy conditions with non-vanishing factors canceled in case (A) for all three orbits.

From the remaining 4, 3, 7 equations in *Case (A)*, we start to eliminate $a,\dots,f$ in order to find solutions, *i.e.*, tetrahedrons (skew quadrilaterals) without symmetries or right angles but with degenerate cubic surfaces $\mathscr{K}$. Until now, all factors of resultants that show up during the computation turn out to correspond to solutions with symmetries or right angles. In many cases we were not able to finish the computations due to the circumstances described in Sec. 1.

Case (B): If now A lies in the quadratic part, then we use the second kind of degeneracy conditions. In this case, the computations are not nearly as fruitful as in the first case. We cancel the non-vanishing factors and reduce the polynomials and arrive at polynomials whose characteristics can be seen in Tab. 2. The computational

ABCD
[39, 82424, [a, 12], [b, 21], [c, 23], [d, 24], [e, 24], [f, 18]]
[43, 110994, [a, 12], [b, 25], [c, 27], [d, 25], [e, 26], [f, 22]]
[≤168, ?, [a, ?], [b, ?], [c, ?], [d, ?], [e, ?], [f, ?]]
ABDC
[39, 80604, [a, 13], [b, 24], [c, 24], [d, 21], [e, 23], [f, 16]]
[43, 117506, [a, 13], [b, 25], [c, 26], [d, 25], [e, 27], [f, 20]]
ACBD
[23, 6099, [a, 10], [b, 12], [c, 11], [d, 14], [e, 12], [f, 10]]
[27, 6775, [a, 9], [b, 15], [c, 12], [d, 16], [e, 14], [f, 14]]
[30, 11356, [a, 11], [b, 17], [c, 12], [d, 18], [e, 16], [f, 16]]
[30, 17065, [a, 12], [b, 16], [c, 16], [d, 19], [e, 16], [f, 14]]
[45, 103418, [a, 14], [b, 25], [c, 22], [d, 25], [e, 23], [f, 18]]
[53, 186696, [a, 16], [b, 29], [c, 26], [d, 29], [e, 27], [f, 22]]
[≤168, ?, [a, ?], [b, ?], [c, ?], [d, ?], [e, ?], [f, ?]]

Table 2: Degrees of reduced evaluated degeneracy conditions with non-vanishing factors canceled in case (B) for all three orbits.

problems in the *Case (B)* are even worse than in the *Case (A)*. The seventh condition which is of degree 28 from the beginning becomes a polynomial of degree 168 when we insert the coefficients of the cubic equations. Even the expansion fails, and thus, the reduction fails too. ♢

5. EXAMPLES

From the previous section we have learned that skew quadrilaterals without symmetry or right angles may not lead to degenerate cubic surfaces $\mathscr{K}$. So it makes sense to study special classes of quadrilaterals. For some special choices of skew quadrilaterals we shall compute the cubic surfaces $\mathscr{K}$, insert their coefficients into the degeneracy conditions. This yields conditions on the coordinates $a,\dots,f$ of the quadrilaterals' vertices such that the surfaces $\mathscr{K}$ degenerate.

5.1 Quads with one plane of symmetry

A tetrahedron *ABCD* with one plane of symmetry is given by

$$A=(0,0,0),\quad B=(a,\ b,0),\qquad C=(c,0,d),\quad D=(a,-b,0).$$

The only plane of symmetry of *ABCD* is $\pi_3:\ y=0$, provided that $c^2-2ac+d^2\neq 0$. In this par-

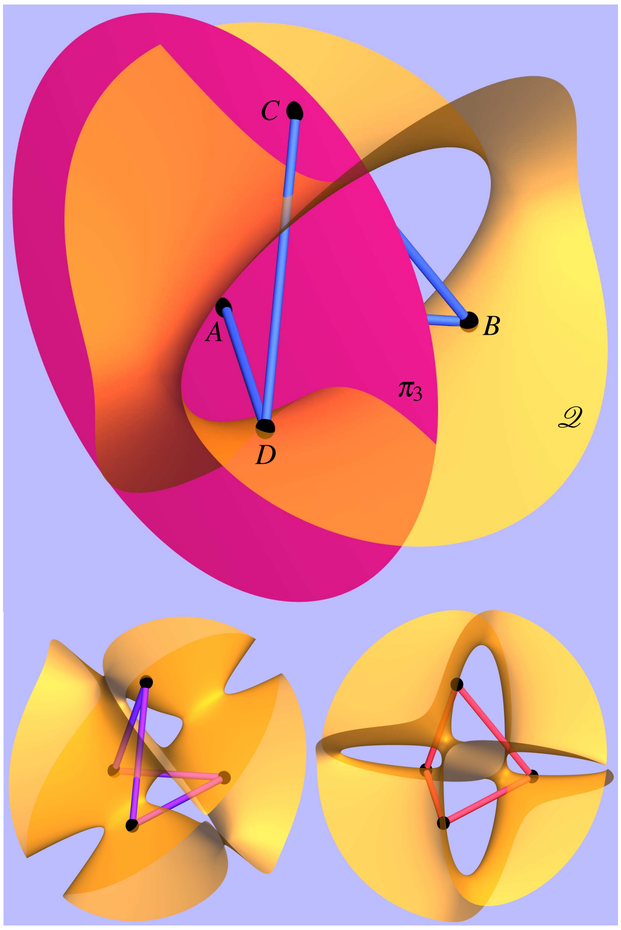

Figure 7: One plane of symmetry (top) and two degenerate cubics (bottom).

ticular case, the cubic corresponding to *ABCD* splits into the plane π_3 and a quadric $\mathscr{Q}$ with the equation

$$\begin{aligned}\mathscr{Q}:\ &(b^2c^2-a^2d^2-2ab^2c)x^2+b^2(c^2+d^2-2ac)y^2\\ &+d^2(a^2+b^2)z^2-2d(a-c)(a^2+b^2)xz\\ &-(a^2+b^2)(a^2c-2ac^2+c^3-2ad^2-b^2c+cd^2)x\\ &+d(a^2+b^2)(a^2+b^2-c^2-d^2)z\\ &+(a^2+b^2)(ac-c^2-d^2)(a^2-ac+b^2)=0.\end{aligned}$$

The quadric $\mathscr{Q}$ is centered at

$$M = \frac{1}{2ad}\left(dl_1^2, 0, al_2^2 - cl_1^2\right)$$

with $l_1 := \overline{AB} = \overline{AD}$ and $l_2 := \overline{BC} = \overline{CD}$. Its longest (real) axis is parallel to $(0,1,0) \parallel [B,D]$.

The quadric $\mathscr{Q}$ splits into a pair of planes

$$\mathscr{Q}:\ (cz-dx)(cx+dz) = 0$$

if $2ac - c^2 - d^2 = 0$. In this case the, the point C can be choosen on a circle in the plane π_3 centered at $(a,0,0) = \mathscr{P} \cap [B,D]$ with radius a and the quadrilateral has a second plane of symmetry, namely the bisector of the segment CD.

In any case, the points $A, C \in \mathscr{P}$, whereas $B, D \in \mathscr{Q}$. In case of $a^2 - ac + b^2 = 0$, all vertices of the quadrilateral lie in $\mathscr{Q}$.

5.2 Quads with two planes of symmetry

We specialize the quadrilaterals of the previous case by assuming that $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DA}$ which yields an equilateral skew quadrilateral as long as $d \neq 0$. Here, $ABCD$ has four equally long edges. If further $\overline{AC} = \overline{AB}$, the quadrilateral $ACBD$ has four equally long edges. Finally, the case $\overline{AC} = \overline{BD} = \overline{AB}$ all three skew quadrilaterals $ABCD$, $ABDC$, and $ACBD$ are equilateral. The last case is more or less trivial and is added at the end of the section in order to be complete.

The tetrahedron $ABCD$ is not a regular tetrahedron unless $\overline{AC} = \overline{BD} = \overline{AB}$ which shall be excluded for the moment. The equilateral skew quadrilateral $ABCD$ can be obtained by folding a planar rhombus along a diagonal $[B,D]$. The vertices of the skew quadrilateral shall now be

$$\begin{array}{rclrcl} A &=& (0,0,0), & B &=& (a,b,0),\\ C &=& \left(\frac{2a}{1+t^2}, 0, \frac{2at}{1+t^2}\right), & D &=& (a,-b,0) \end{array} \qquad (12)$$

with $t \in \mathbb{R} \setminus \{0\}$. If $t=0$, the quadrilateral is planar. Instead of the rational expressions for C's coordinates we could have taken the parametrization $c(\omega) = (a(1+\cos\omega), 0, a\sin\omega)$ with $\omega \in]0,\pi[$ but we prefer the rational expressions because of their computational advantages. Note that $\omega \neq 0, \pi$ otherwise $ABCD$ is planar.

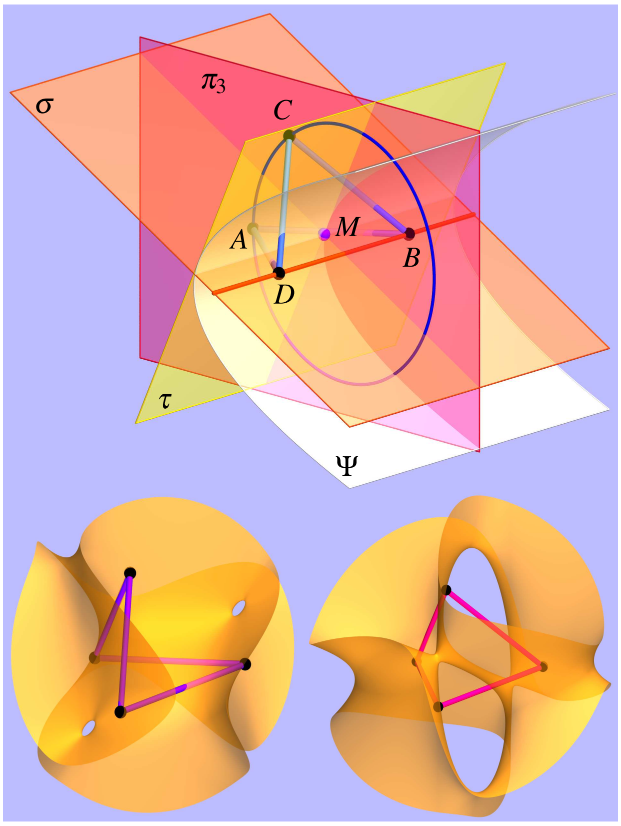

Figure 8: Skew quadrilateral with two planes of symmetry and a triple of planes as degenerate cubic $\mathscr{K}$ (top). The cubics for $ABDC$ and $ACBD$ do not degenerate (bottom).

With (6) and (12) we find the equation of the cubic surface $\mathscr{K}$ corresponding to the quadrilateral $ABCD$ (after canceling constant and non-vanishing factors) as

$$\begin{gathered}\mathscr{K}:\ y(x+tz-a)\cdot\\ \cdot(2at(tx-z)-(a^2+b^2)t^2+a^2-b^2)=0.\end{gathered}$$

Obviously, the cubic surface $\mathscr{K}$ is the union of three planes, see Fig. 8. It is worth to have a closer look at these planes. The factor y gives the equation of the $[x,z]$-plane π_3, if set equal to zero. This is a plane of symmetry of the quadrilateral $ABCD$ and comes as no big surprise, see the previous case in Sec. 5.1. The quadratic part of $\mathscr{K}$ corresponds to a pair of planes

$$\begin{gathered}\sigma:\ x+tz-a=0,\\ \tau:\ 2at(tx-z)-(a^2+b^2)t^2+a^2-b^2=0.\end{gathered}$$

The triple (π_3, σ, τ) consists of three mutually orthogonal planes all concurrent in the point

$$M = \frac{1}{2}\left(\frac{a^2+b^2}{a}, 0, \frac{a^2-b^2}{at}\right).$$

Note that $t \neq 0$. If t traces $\mathbb{R}\setminus\{0\}$, the plane σ traverses the pencil about the line $s = (a, \lambda, 0)$ (with $\lambda \in \mathbb{R}$). The one-parameter family of planes τ is the set of tangent planes of the parabolic cylinder

$$\Psi:\ a^2z^2 - 2a(a^2-b^2)x + a^4 - b^4 = 0.$$

In the case $a = b$ the parabolic cylinder also degenerates and becomes a pair of planes through a real line. Therefore, the family of planes τ is then a pencil of planes.

The cubic surfaces corresponding to *ABDC* and *ACBD* do not degenerate, see Fig. 8. However, this could not be expected since the planes of symmetry of *ABCD* will in general not coincide with those of *ABDC* or *ACBD*.

5.3 Quadrilaterals with axial symmetry

A skew quadrilateral *ABCD* with axial symmetry can be given by

$$\begin{aligned} A &= (0,0,0),\ B = (a+c, b+d, 0),\\ C &= (a,b,h),\ D = (c,d,h) \end{aligned} \tag{13}$$

where only the coordinate vectors of B and C differ from those in (1). From (13) and (6) we find the degenerate cubics for *ABCD*, *ABDC*, and *ACBD* being unions of planes and quadrics: $\mathscr{K}_i = \mathscr{P}_i \cup \mathscr{Q}_i$ (with $i \in \{1,2,3\}$. The planes have the equations

$$\mathscr{P}_1:\ d = 2y,\ \mathscr{P}_2:\ a = 2x,\ \mathscr{P}_3:\ h = 2z. \tag{14}$$

The three quadratic components share the center $M = \frac{1}{2}(a,d,h)$ which lies in all the three planes. The equations of the quadrics are

$$\begin{aligned} \mathscr{Q}_1 &: a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{D}_{ah}(d-y)y +\\ &\quad + h^2\mathrm{S}_{ad}(z-h)z = 0,\\ \mathscr{Q}_2 &: a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{S}_{ah}(d-y)y +\\ &\quad + h^2\mathrm{S}_{ad}(z-h)z = 0,\\ \mathscr{Q}_3 &: a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{S}_{ah}(y-d)y +\\ &\quad + h^2\mathrm{D}_{ad}(h-z)z = 0 \end{aligned} \tag{15}$$

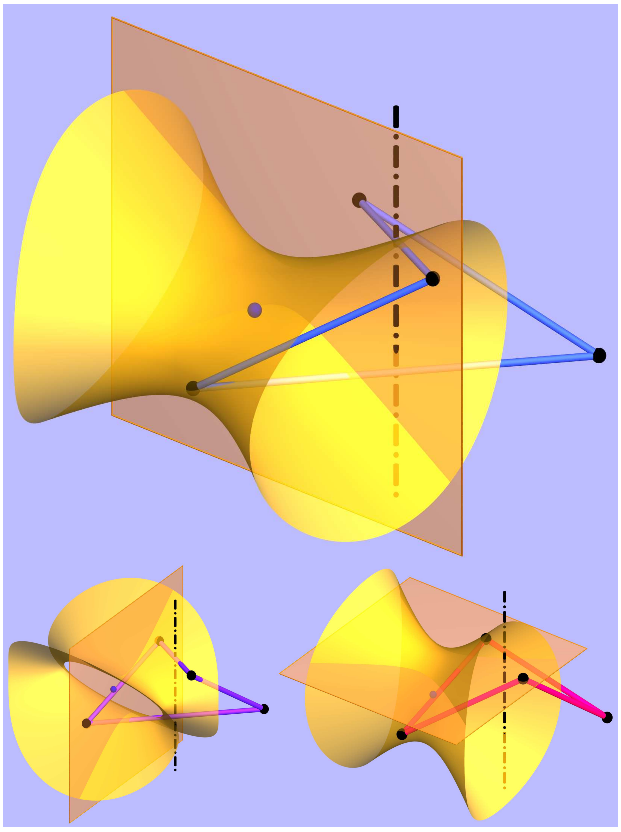

Figure 9: A skew quadrilateral with axial symmetry and the three degenerate cubic surfaces.

where $\mathrm{S}_{uv} := u^2 + v^2$ and $\mathrm{D}_{uv} := u^2 - v^2$. Note that b and c do not show up in (15), and therefore, they have no influence on the shape and degeneration of the cubic (quadratic) surfaces.

In three further special cases, at least one of the quadrics degenerates and becomes the union of two planes:

$$\begin{aligned} a = h &\iff \mathscr{Q}_1:\ (x-z)(h-x-z) = 0,\\ h = d &\iff \mathscr{Q}_2:\ (y-z)(d-z-y) = 0,\\ d = a &\iff \mathscr{Q}_3:\ (x-y)(a-y-x) = 0. \end{aligned}$$

These latter cases also cover the very special case of the quadrilaterals taken from the regular tetrahedron shown in Sec. 5.6.

5.4 Orthoschemes

Among the tetrahedrons we find orthoschemes, *i.e.*, tetrahedrons with a chain of three subsequent orthogonal edges. Consequently, the faces of an orthoscheme are four right triangles. Any cuboid can be disssected into six orthoschemes, see Fig. 10. Up to orientation preserving affine

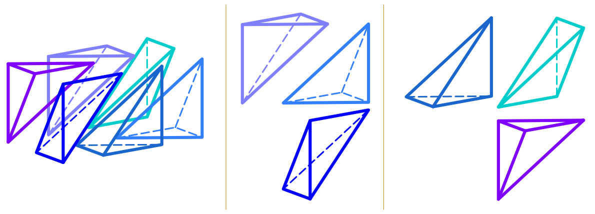

Figure 10: The six orthoschemes in cuboid arranged in two groups.

transformations, we find two different types of orthoschemes cut out of a cuboid. However, it is sufficient to treat only one orthoscheme that can be parametrized by (1). Let

$$\mathbf{a} = (0,0,0),\ \mathbf{b} = (a,0,0),$$
$$\mathbf{c} = (a,b,0),\ \mathbf{d} = (a,b,c)$$

and observe that replacing c by $-c$ yields a copy of this orthoscheme which can be obtained by applying an orientation reversing transformation to $ABCD$. The cubic surface $\mathscr{K}$ associated to the quadrilateral $ABDC$ degenerates if the orthoscheme shows an axial symmetry, and thus, $c = \pm a$ (or $b = \pm a$) and becomes (in the first case)

$$(x \mp z)(by^2 + axy \pm ayz \mp bxz - (a^2+b^2)y) = 0.$$

The planar component contains A and D, while B and C lie on the quadric with center $\frac{1}{2}(a,b,\pm a)$ which is a one-sheeted hyperboloid (which is never a quadric of revolution), see Fig. 11.

5.5 Corner of a cuboid

Cutting off a corner of a cuboid leads to a tetrahedron with vertices

$$\mathbf{a} = (0,0,0),\ \mathbf{b} = (a,0,0),$$
$$\mathbf{c} = (0,c,0),\ \mathbf{d} = (0,0,f)$$

where $acf \neq 0$. The cubic surfaces corresponding to the quadrilaterals $ABCD$, $ABDC$, $ACBD$ degenerate if $a = \pm f$, $a = \pm c$, $a = \pm c$, according

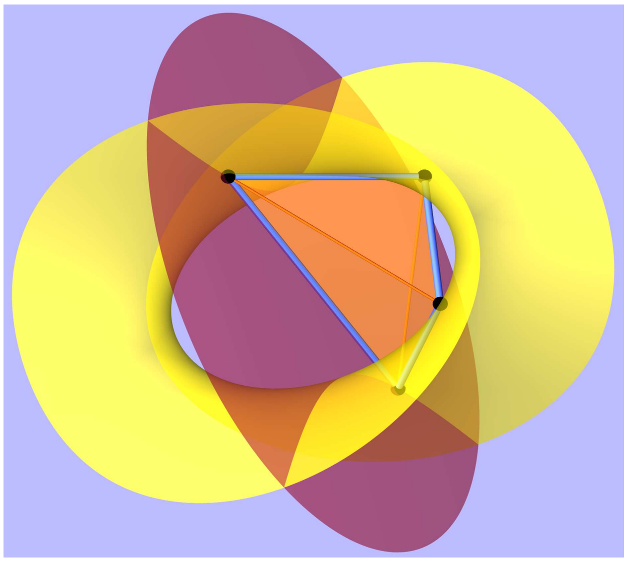

Figure 11: The degenerate cubic surface $\mathscr{K}$ of an orthoscheme with axial symmetry.

to the results given in Sec. 5.1, since then there exists at least one plane of symmetry. Then, the planar parts are

$$x \pm z = 0,\quad x \pm y = 0,\quad y \pm z = 0$$

while the quadrics are

$$cy^2 \mp fxy \mp cxz - fyz \pm cfx - (c^2-f^2)y + cfz - cf^2 = 0,$$
$$fz^2 \pm fxy \mp cxz - cyz \pm cfx + cfy - (f^2-c^2)z - c^2f = 0,$$
$$ax^2 - cxy \mp cxz \mp ayz - (a^2-c^2)x + acy \pm acz - ac^2 = 0,$$

with centers $\frac{1}{2}(\pm f, c, f)$, $\frac{1}{2}(\pm c, c, f)$, $\frac{1}{2}(a, c, \pm c)$.

If further $a = c = f$, then the tetrahedron has three congruent right faces and $\mathscr{K}$ for either orbit does not degenerate any further.

5.6 Regular quadrilaterals

A tetrahedron is called regular, if all faces are congruent equilateral triangles. On a regular tetrahedron we find three equilateral skew quadrilaterals. The coordinates of the vertices are (1) with $a = 1$, $b = d = \frac{1}{2}$, $c = \frac{\sqrt{3}}{2}$, $e = \frac{\sqrt{3}}{3}$, and $f = \frac{\sqrt{6}}{3}$. In this case, all three cubics degenerate simultaneously and become the onion of three planes. Any such triple of planes passes through the center of the tetrahedron. Especially for the case of a regular tetrahedron, it is advantageous to give up the coordinatization of the vertices (1). We assume that $A=(0,0,0)$, $B=(1,1,0), C=(0,1,1), D=(1,0,1)$.

Clearly, this tetrahedron is regular but the edge length equals $\sqrt{2}$. This doesn't matter since we are dealing with a problem which is invariant with respect to equiform transformations.

It turns out that the cubic surface $\mathscr{K}$ associated to *ABCD* is the union of three mutually orthogonal planes and has the equation

$$\mathscr{K}:\ (y-z)(2x-1)(y+z-1)=0.$$

The cubics associated to *ABDC* and *ACBD* can be obtained by changing the variables according to $x\to y$, $y\to z$, $z\to x$ once and twice. The three planes contain the two planes of symmetry of the quadrilateral.

6. DISCUSSION

The degeneracy conditions we have presented are necessary conditions on the coeffcients of a trivariate cubic polynomial such that it can be factorized. The sufficiency is not shown so far.

We haven't given a complete list of special tetrahedrons with degenerate cubic sufrcae $\mathscr{K}$ because it is unclear whether the degeneracy conditions are only necessary or not.

Due to the lack of power and capacity of the author's computer, solving the systems of equations emerging from the degeneracy conditions failed. It is not necessary to write down the degeneracy conditions in full length, but we should at least be able to insert the coefficients of the cubic equations and the solve the systems of equations.

The geometric problem that deals with the feet of normals from a point to the faces of a tetrahedron can be treated in the same way.

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