Examples of isoptic ruled surfaces

Boris Odehnal

University of Applied Arts Vienna, Vienna, Austria
boris.odehnal@uni-ak.ac.at

Abstract. The isoptic curve c_α of a planar curve c is defined as the locus of points where pairs of tangents of c intersect at an angle of $\alpha \in (0, \pi)$. The definition of isoptic curves (in the plane) cannot be carried over to three-dimensional spaces. We present a generalization of isoptic curves to a special class of ruled surfaces. For that purpose, we assume that a developable (torsal) ruled surface R is given. Since R is enveloped by its one-parameter family of tangent planes, we can ask for pairs of tangent planes that enclose a fixed angle $\alpha \in (0, \pi)$. The lines of intersection of all such pairs of tangent planes will then be defined as the *isoptic ruled surface* J_α of R. Especially, if $\alpha = \frac{\pi}{2}$, we shall call $J_{\frac{\pi}{2}}$ the *orthoptic ruled surface*. We shall give some general results on isoptic ruled surfaces together with some examples.

Keywords: ruled surface, isoptic surface, developable surface

1 Introduction

In planar geometry, the locus c_α of points from which a given curve c is seen under a constant angle α is called the α-isoptic of c. These curves have gained a lot of attention and were intensively studied at least in the case of conics, see [8, 12] for the Euclidean plane. In [7], a tool for the treatment of isoptics in the hyperbolic plane was developed. Considering the level sets of isoptic curves (for varying optical angle α) of pairs of conics enables us to compute equioptic curves, *i.e.*, sets of points from which two different curves can be seen at the same angle. For conics and other algebraic curves this was done in [10]. Curves with a simple kinematic generation also allow for an exact computation of isoptic curves. This is especially the case for trochoidal curves, see the results in [13, 14]. The determination of isotptic curves with a special predefined shape is a much more complicated task (see [16, 17]). Curves which are there own isoptics can only be determined approximately (cf. [11]) and contain the special case of *autoevolutes* (as described in [15]).

The notion of an isoptic curve to a planar curve cannot be carried over to three-dimensional Euclidean space $\mathbb{R}^3$ in the same way. For a generic space curve c, two arbitrary tangents will, in general, be skew. In computer graphics isoptic surfaces have been defined by means of integral measures (see, *e.g.*, [1]). Thus, the isoptic surfaces become the loci of all vertices of cones that are tangent to a given surface and define a certain spatial angle measure. This leads to a well-defined surface which can only be found numerically and point-wise. Therefore, we miss an analytic description, *i.e.*, an equation or a parametrization. The absence of an explicit description of these isoptic surfaces does not support further study of the surfaces.

We shall go another way: We restrict ourselves to the class of developable surfaces. Any developable surface is either a cylinder, a cone, or the envelope of the one-parameter family of osculating planes of a non-planar space curve. This has the advantage that we are able to define an optical angle as the angle between two tangent planes of the underlying surface. Moreover, since a developable surface is the envelope of a one-parameter family of planes, we only have to look for the locus of intersection lines of pairs of tangent planes that enclose a fixed angle. In [4], a spatial analogue to the Theorem of the Angle of Circumference is presented. The ruled quadrics and quartics appearing as the locus of intersection lines of planes from two pencils forming a fixed angle can be seen as the *equioptic surfaces* of two straight lines (the axes of the pencils). The isoptic surfaces of developable ruled surfaces are ruled surfaces and an analytical representation can easily be computed, at least from the theoretical point of view.

The paper is organized as follows: In Section 2, we provide some general results on isoptic ruled surfaces of cones and cylinders. Section 3 is devoted to the study of isoptic ruled surfaces which are invariant with respect to special groups of transformations such as the group of Euclidean motions and the group of equiform motions. The huge variety of algebraic developables and their isoptic ruled surfaces will only briefly be discussed in Section 4. Finally, in Section 5, we shall provide some ideas for future research.

2 General results, elementary surfaces

Throughout this paper, we assume that a developable ruled surface R is given. R is the envelope of the one-parameter family of osculating planes of a non-planar space curve g, and at the same time, it is swept by the tangents of g, since any two infinitely close osculating planes of g intersect in a tangent of g. The curve g is called the *curve of regression* of R and the osculating planes of g are tangent planes of R, see [6].

The following definition shall be the starting point of our investigations:

Definition 1. *Let R be a developable ruled surface and let $\alpha \in (0, \frac{\pi}{2})$ be the optical angle. Then, the α-isoptic J_α of R is the locus of intersection lines of all pairs of tangent planes of R that enclose the angle α.*

From the above definition it is clear that the α-isoptic of R is a ruled surface.

The most simple examples of developable ruled surfaces are cylinders and cones. We can give the first nearly trivial results on isoptic ruled surfaces:

Theorem 1. *(1) The isoptic ruled surfaces of cylinders are cylinders.*

(2) The isoptic ruled surfaces of cones are cones.

Proof. (1) Let us assume that Λ is a cylinder with generators parallel to the z-axis. (The underlying Cartesian coordinate system, can always be chosen that way.) Then, the tangent planes of Λ (that contain the generators of contact) are also parallel to the z-axis. The $[x, y]$-plane meets Λ along a cross section l that can be used as a directrix of Λ. Further, the traces of the tangent planes of Λ in the $[x, y]$-plane are the tangents of l. The α-isoptic l_α of l is the locus of all points in the $[x, y]$-plane where tangents of l meet at the angle α and the lines parallel to the z-axis through the points of l_α form the isoptic ruled surface Λ_α of Λ. Obviously, Λ_α is a cylinder (parallel to Λ).

(2) Assume that Γ is a cone centered at some point, say O. The tangent planes of Γ pass through the cone's vertex O. Therefore, the intersection lines of any pair of tangent planes of Γ pass through O. This holds particularly true for those tangent planes that enclose the fixed angle α. Thus, the isoptic ruled surface Γ_α of Γ is also a cone centered at O. □

Remark 1.

1. According to Theorem 1, the isoptic ruled surface of a cylinder Λ of revolution is the cylinder Λ_α of revolution erected above the isoptic l_α of an orthogonal cross section l of Λ: Since any orthogonal cross section l of Λ is a circle, its planar isoptic l_α is a circle, too. Therefore, Λ_α is also a cylinder of revolution. If $r > 0$ is the radius of Λ, and thus, of l, then it is elementary to verify that
$$r_\alpha = r \operatorname{cosec} \frac{\alpha}{2}$$
is the radius of the isoptic circle l_α of l, and consequently, it is the radius of the isoptic cylinder Λ_α.
2. The isoptic ruled surface Γ_α of a cone of revolution Γ with an angle of aperture $0 < 2\omega < \pi$ is also a cone of revolution. The angle $2\omega_\alpha$ of aperture of the isoptic cone Γ_α is related to ω by
$$\cos\omega_\alpha = \sqrt{\frac{\cos 2\omega + \cos\alpha}{1 + \cos\alpha}}. \tag{1}$$
This is can be verified as follows: Assume that Γ is given by the equation $\Gamma:\ x^2 + y^2 - \frac{z^2}{k^2} = 0$, where $k = \operatorname{ctg}\omega$. Now, the tangent planes of Γ have the equations $\tau(v):\ k(\cos vx + \sin vy) - z = 0$ with $v \in (0, 2\pi)$. The rulings e of the isoptic cone are the intersection lines of two different tangent planes. It means no restriction to assume that $e = \tau(v) \cap \tau(-v)$ for a yet undetermined v. Thus, e is parallel to $\mathbf{e} = (1, 0, k\cos v)$. The condition $\sphericalangle(\tau(v), \tau(-v)) = \alpha$ yields
$$\cos\alpha = \frac{1 + k^2\cos 2v}{1 + k^2} = \frac{1 - k^2 + 2\cos^2 v}{1 + k^2}.$$
We arrive at (1), since
$$\cos\omega_\alpha = \cos\sphericalangle(\mathbf{e}, (0, 0, 1)) = \frac{k\cos v}{\sqrt{1 + k^2\cos^2 v}}.$$

We shall point out that the isoptic ruled surface of a cone is in a close relation to isoptic curves in elliptic geometry: The cone Γ mentioned in the proof of Theorem 1 defines a spherical curve γ by intersecting all rulings of Γ with the Euclidean unit sphere S^2 centered at O. (Conversely, any curve $\gamma \in \mathrm{S}^2$ defines a unique cone.) The tangent planes of Γ meet S^2 along great circles, *i.e.*, the straight lines in spherical geometry. Any pair of tangent planes of Γ that encloses the angle α meets S^2 along two great circles of S^2 that intersect at the angle α. Hence, the rulings of Γ_α intersect S^2 in the points of the *spherical isoptic* γ_α (cf. [2]) of the *spherical image* γ of Γ, see Figure 1.

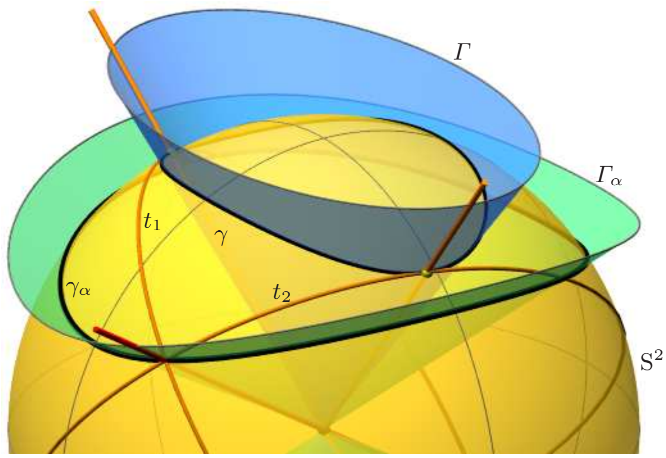

Fig. 1. The spherical isoptic curve γ_α of the spherical curve γ is the intersection of the isoptic cone Γ_α of Γ. Note that the spherical tangents (great circles) t_1 and t_2 of γ meet at a point of the spherical isoptic γ_α.

3 Helical and spiral developables

While helices are paths of points under one-parameter subgroups of the group of Euclidean motions, helical surfaces are generated by applying such a one-parameter subgroup to a curve. Thus, a helical surface is invariant under the generating subgroup. This will simplify the computation of the isoptic ruled surfaces in this case. This holds similarly true for curves and surfaces which are invariant under one-parameter subgroups of the equiform motion group. Here, the path curves are called cylindro-conical spirals and the invariant surfaces are called spiral surfaces, cf. [5, 9].

The fact that helical and spiral developables are invariant with respect to the generating subgroups has an important consequence for the respective isoptic ruled surfaces:

Theorem 2. *The isoptic ruled surfaces of helical and spiral developables are helical and spiral ruled surfaces, respectively.*

Proof. The isoptic ruled surface of a helical developable R is the locus of intersection lines of pairs of tangent planes of R that enclose the fixed angle $\alpha \in (0, \frac{\pi}{2})$. Assume that τ_1 and τ_2 are two such planes that intersect along the line j (and clearly, $\sphericalangle(\tau_1, \tau_2) = \alpha$). Applying the generating helical motion to τ_1, τ_2, and j, leaves the angle between τ_1 and τ_2 unchanged and j sweeps a helical ruled surface. Similar arguments hold for spiral developables and their isoptics. □

At this point, we shall emphasize that the isoptic ruled surface to a helical or a spiral developable may consist of infinitely many branches. Once we fix the tangent plane τ_1, we will find infinitely many tangent planes τ_2 of the given helical (spiral) developable which are orthogonal to τ_1. Therefore, the isoptic ruled surfaces of helical and spiral developables can never be algebraic. During the computation of the isoptic ruled surfaces, we will have to solve transcendental equations in order to find the parameter values corresponding to τ_2.

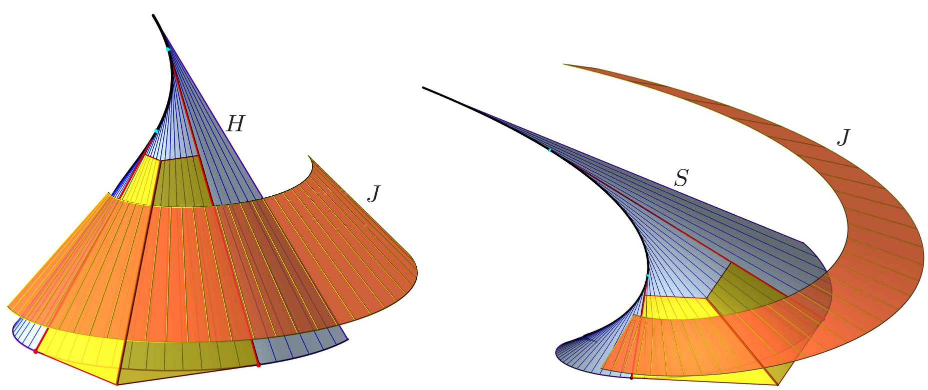

Fig. 2. Isoptic ruled surfaces J: to a helical developable H (left) and a spiral developable S (right). In both cases, a portion of J between two horizontal planes is displayed.

3.1 Helical developables

We can define a helical developable R by prescribing its curve g of regression. For that purpose, we assume $\mathbf{g}(t) = (r\cos t, r\sin t, pt)$ with $t \in \mathbb{R}$ is a parametrization of the curve of regression and $r, p \in \mathbb{R}^+$ are the radius and the pitch, respectively. Obviously, this curve is generated by the helical motion with the z-axis for its axis and the pitch p.

The one-parameter family of tangent planes of R equals the one-parameter family of g's osculating planes. The binormals $\mathbf{g}_3$ of g are the normals of the osculating planes, and thus, they are parallel to $\dot{\mathbf{g}} \times \ddot{\mathbf{g}}$, where dots indicates differentiation with respect to the parameter t and $\times$ is the canonical exterior product of two vectors in $\mathbb{R}^3$. Hence, the binormal vector field along g reads

$$\mathbf{g}_3 = \frac{1}{\sqrt{p^2+r^2}} \begin{pmatrix} p\sin t \\ -p\cos t \\ r \end{pmatrix}.$$

For each $t \in \mathbb{R}$, we can give the equations of the osculating planes σ as

$$\sigma(t): \ p\sin t\, x - p\cos t\, y + rz = prt.$$

Now, we are looking for pairs of osculating planes of the helical developable R that enclose the angle α and assume that $u \in \mathbb{R} \setminus \{0\}$, and thus, u and $-u$ are two different parameters. Then, $\sigma(-u) = \tau_1$ and $\sigma(u) = \tau_2$ are two different tangent planes of R and intersect along the straight line

$$\mathbf{e} = \left(\frac{ru}{\sin u}, w, \frac{pw\cos u}{r}\right), \quad \text{with w} \in \mathbb{R}. \tag{2}$$

The angle criterion $\sphericalangle(\tau_1, \tau_2) = \alpha$ equals $\langle \mathbf{g}_3(u), \mathbf{g}_3(-u)\rangle = \cos\alpha$ (with $\langle \cdot, \cdot \rangle$ denoting the canonical scalar product of two vectors in $\mathbb{R}^3$) and yields the following condition on the parameter u

$$\cos\alpha = \frac{p^2\cos 2u + r^2}{p^2+r^2} \tag{3}$$

and with (2) we have the parametrization of exactly one line that is the intersection of a pair of tangent planes of R that enclose the angle α. Especially, the orthoptic ruled surfaces are defined by $\cos 2u = -\frac{r^2}{p^2}$. Applying the underlying (generating) helical motion with the pitch p yields a parametrization of the isoptic helical ruled surface J_α.

Developable isoptic helical ruled surfaces

Among the isoptic helical ruled surfaces J_α, we may find developable ones. For that purpose, we use the following fact: Let $\mathbf{g} = (d, w, kw)$ with fixed $d, k \in \mathbb{R}$ (and $w \in \mathbb{R}$) be the parametrization of a straight line g. We apply the helical motion with pitch p to g and the helical ruled surface S swept by g is developable if, and only, if

$$p = dk. \tag{4}$$

From (2) we can read off $d = \frac{ru}{\sin u}$ and $k = \frac{p\cos u}{r}$. The product of these values has be equal to the parameter p of the helical motion. This yields

$$u\cos u - \sin u = 0 \Longleftrightarrow u = \tan u, \tag{5}$$

which is obviously independent of the parameter p. Since (5) has infinitely many real solutions (cf. Figure 3, left), we can state:

Theorem 3. *To each helical developable R there exist infinitely many torsal (developable) helical ruled surfaces which are isoptic ruled surfaces of R. The optical angles can be obtained by inserting the solutions of* (5) *into* (3).

3.2 Spiral developables

Again, we can start with the curve of regression g: Let $\mathbf{g} = \exp(pt)(\cos t, \sin t, 1)$ with spiral parameter $p \neq 0$ and $t \in \mathbb{R}$ be a parametrization of g. In the same way as we have done in the case of helical developables, we compute the binormal vector field as

$$\mathbf{g}_3 = \frac{1}{\sqrt{(1+p^2)(1+2p^2)}} \begin{pmatrix} -p(p\cos t - \sin t) \\ -p(p\sin t + \cos t) \\ 1+p^2 \end{pmatrix},$$

which yields the equations of $\mathbf{g}$'s osculating planes

$$p(\sin t - p\cos t)x - p(\cos t + p\sin t)y + (1+p^2)z = \exp(pt).$$

Now, we let $u \in \mathbb{R}$ be some parameter. Then, $\tau_1 = \sigma(-u)$ and $\tau_2 = \sigma(u)$ are two tangent planes of the spiral developable. These planes intersect along the lines

$$\mathbf{e}(w) = \begin{pmatrix} \frac{\sinh pu}{p\sin u} \\ 0 \\ \frac{p\sinh pu\cos u + \cosh pu\sin u}{(1+p^2)\sin u} \end{pmatrix} + w \begin{pmatrix} p \\ 1 \\ p\cos u \end{pmatrix} \quad \text{with } w \in \mathbb{R}. \tag{6}$$

The analogue to the angle criterion given by (3) now reads

$$\cos\alpha = \frac{p^2\cos 2u + p^2 + 1}{2p^2+1}. \tag{7}$$

Orthoptic ruled surfaces to spiral developables show up if $\cos 2u = -1 - \frac{1}{p^2}$.

Developable isoptic spiral developables

In the case of spiral developables, we cannot apply such a simple criterion as the one available for the helical surfaces. In order to find the developable surfaces among the isoptic ruled surfaces of spiral developables, we apply the underlying generating spiral motion (uniform equiform motion) to the lines (6). This yields a rather complicated parametrization of a ruled spiral surface J_α. The developability of J_α is equivalent to the vanishing of the Gaussian curvature and results in the condition

$$p\,\mathrm{tg}\,u - \mathrm{tgh}\,pu = 0, \tag{8}$$

which relates the parameter(s) u chosen on the initial spiral developable with the spiral parameter (see Figure 3, right).

We can summarize:

Theorem 4. *Independent of the spiral parameter p, there exist infinitely many developable spiral isoptic ruled surfaces J_α to a given spiral developable R. The respective optical angles can be obtained from* (8) *and* (7).

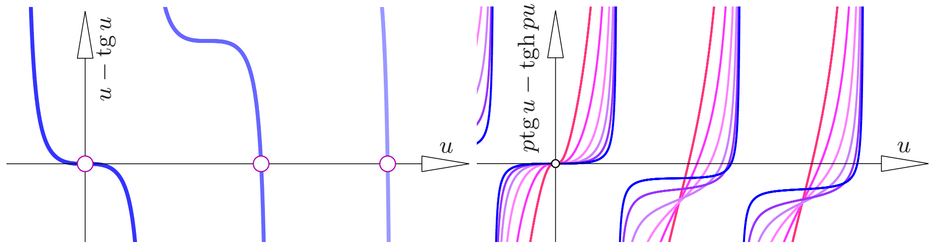

Fig. 3. Left: The zeros of (5) correspond to isoptic helical developables. Right: The zeros of (8) correspond to isoptic spiral developables. The curves for some spiral parameters p are displayed.

4 Algebraic isoptic ruled surfaces

In this section, we shall compute the isoptic ruled surfaces of algebraic developables that allow a rational or even polynomial parametrization. Even with this restriction, we will see the limits of the symbolic computational approach.

Assume that $\mathbf{g} = (g^1, g^2, g^3) : \mathbb{R} \to \mathbb{R}^3$ is a polynomial vector function (and thus, the parametrization of a space curve g in Euclidean three-space) which is at least two times differentiable. Let d be the algebraic degree of g, *i.e.*, $d = \max_i(\deg g^i)$.

Following the computations we have done in the previous sections, the dual curve $g^\star$ (or equivalently, the developable swept by g's tangents) can then be parametrized by

$$\mathbf{g}^\star = (\det(\mathbf{g}, \dot{\mathbf{g}}, \ddot{\mathbf{g}}), -\dot{\mathbf{g}} \times \ddot{\mathbf{g}}).$$

Actually, $\mathbf{g}^\star : \mathbb{R} \to \mathbb{R}^4$ is a parametrization of a curve in the projectively extended dual space of the Euclidean three-space. Counting the degrees of the derivatives, we can infer that $\deg g^\star \leq 3d - 3$. In many cases, this is an upper bound of the degree of $g^\star$.

The isoptic ruled surface J_α of the developable ruled surface $R = g^\star$ is the locus of intersection lines of planes in $g^\star$. Therefore, it can be expected that $\deg J_\alpha \leq 6d - 6$.

Let u and v be two different parameter values. The corresponding planes of $g^\star$ are the tangent planes $\tau_1 = \mathbf{g}^\star(u)$ and $\tau_2 = \mathbf{g}^\star(v)$. In order to shorten the notation, we shall write $\mathbf{g}_u := \mathbf{g}(u)$ and $\mathbf{g}_v := \mathbf{g}(v)$. Hence, the rulings e of the isoptic ruled surface have the Plücker coordinates

$$\mathbf{J} = ((\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u) \times (\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v), (\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u) \det(\mathbf{g}_v, \dot{\mathbf{g}}_v, \ddot{\mathbf{g}}_v) - (\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v) \det(\mathbf{g}_u, \dot{\mathbf{g}}_u, \ddot{\mathbf{g}}_u)) . \quad (9)$$

Formally, the polynomials $\mathbf{g}_u$ and $\mathbf{g}_v$ are the same polynomials. Thus, each coordinate function of $\mathbf{J}$ is divisible by the factor $u - v \neq 0$ (which expresses that $\tau_1 \neq \tau_2$), and therefore, $\deg \mathbf{J} \leq 5d - 6 - 1 = 5d - 7$.

From $\mathbf{J} = (\mathbf{j}, \bar{\mathbf{j}})$ we can change to a parametrization of the ruled surfaces described by $\mathbf{J}$. To be more precise, $\mathbf{J} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^6$ is the Plücker coordinate representation, of a congruence of chords of the curve g. Within this congruence, we find the isoptic ruled surfaces J_α of R by imposing the angle condition on the pairs (τ_1, τ_2) of tangent planes of R. Since, by definition, $\alpha = \sphericalangle(\tau_1, \tau_2)$, we have

$$\cos\alpha = \frac{\langle \dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u, \dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v \rangle}{\|\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u\| \cdot \|\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v\|}.$$

Squaring the latter equation in order to make it a purely polynomial condition, we find that the degrees of the numerator and the denominator of the right-hand side are $8d - 12$ at most. Further reductions of the degree will appear in the case of so-called *higher degree parabolae*, *i.e.*, curves of degree d which have a $(d-1)$-fold intersection with the plane at infinity.

An explicit computation of a parametrization $\mathbf{j}_\alpha$ of the isoptic ruled surface J_α consist of several steps of elimination, either by means of the Groebner basis algorithm or by means of resultants.

4.1 Some special polynomial curves

Cubic curves. The cubic space curves and their tangent developables serve as illustrative examples. From the projective point of view, there exists only one type of (non-planar) cubic space curve, cf. [3]. However, here the affine point of view is of more importance. Thus, we distinguish between the following four types of curves:

(i) *cubic ellipses* with one real ideal point (and a complex conjugate pair),
(ii) *cubic hyperbolae* with three different real ideal points,
(iii) *cubic parabolic hyperbolae* with two different real ideal points (one of multiplicity two), and
(iv) *cubic parabolae* osculating the plane at infinity.

In any of the four cases, the developable R is cubic as a one-parameter family of planes, quartic as a set of points. Thus, the intersection with any plane, and especially with the ideal plane ω is a quartic curve r_ω of degree four and class three. According to the above classification of cubic space curves, the shapes of the curves r_ω are

(i) a double line plus a pair of lines (eventually one line from the pair coincides with the double line making it a triple line,
(ii) a quartic with three cusps,
(iii) a cusped cubic with its inflection tangent, and
(iv) a conic with a double tangent.

In the cases (i) – (iii), the angle criterion

$$\cos\alpha = \langle \mathbf{g}_3(u), \mathbf{g}_3(v) \rangle \tag{10}$$

results in an algebraic equation of degree 12 relating the parameters u and v on the cubic curve. The thus described algebraic curves in the parameter domain are of genus 19 in any case. Things become simpler for the cubic parabola. Then, the angle criterion defines an octic curve of genus five in the $[u, v]$-plane.

The degrees of the isoptic ruled surface to arbitrary angles $\alpha \neq 0, \frac{\pi}{2}$ become relatively high. Nevertheless, parametrizations can be derived from (9) and (10) by eliminating either u or v.

It is not at all surprising that the orthoptic ruled surfaces are of less degree. For the orthoptics of cubic developables, only the numerator of (10) matters. This polynomial is a full square of a sextic polynomial in the cases (i) – (iii) and a quartic polynomial in the case (iv) of cubic parabolae.

Explicit and useful parametrizations of orthoptic ruled surfaces can be given in the case of the three-parameter family of cubic parabolae given by

$$\mathbf{p}(t) = \left(At, Bt^2, \frac{1}{3AC}(B^4 + C^2)t^3\right), \tag{11}$$

$t \in \mathbb{R}$ and $A, B, C \in \mathbb{R} \setminus \{0\}$ are shape parameters. In this particular case, the angle criterion (10) factors and yields

$$((B^4 + C^2)uv + A^2B^2)(B^2(B^4 + C^2)uv + A^2C^2) = 0 \tag{12}$$

which are the equations of two hyperbolae in the $[u, v]$-plane. This means that the mapping $u \mapsto v$ that assigns each parameter value u on $\mathbf{p}$ precisely that parameter value v that corresponds to the orthogonal osculating plane is a projective mapping. So we are bale to state

Theorem 5. *The orthoptic ruled surfaces of the tangent developables R of all cubic parabolae* (11) *consists of a pair of hyperbolic paraboloids.*

Proof. The fact that the orthoptic surface splits into two parts is caused by the splitting of the ortogonality condition (12). Each factor of (12) describes a linear rational mapping $u \mapsto v$, and thus, a projective automorphism on p and on the ideal curve on R (cf. [3]).

The fact that $J_{\frac{\pi}{2}}$ consists of a pair of hyperbolic paraboloids is best shown by computation. Following beaten tracks, we find the equations of the orthoptics:

$$\begin{aligned}
J_{1,\frac{\pi}{2}} &: 9C(B^4 + C^2)(B^2x + Cz)z + 3A^2B^3(B^4 + C^2)y + A^4B^6 = 0, \\
J_{2,\frac{\pi}{2}} &: 9B^4(B^4 + C^2)(B^2x + Cz)z + 3A^2BC^2(B^4 + C^2)y + A^4C^4 = 0.
\end{aligned}$$

It is left to the reader to verify that $J_{1,\frac{\pi}{2}}$ and $J_{2,\frac{\pi}{2}}$ are hyperbolic paraboloids. Hint: Look at the already factored quadratic terms. $\square$

Note that the two hyperbolic paraboloids in Theorem 5 coincide, if two factors of (12) are proportional. This is the case if, and only if, $C^2 = B^4$. A case with two different orthoptics is shown in Figure 4 (left).

A singular quartic curve. Isoptic ruled surfaces and especially orthoptic ruled surfaces can be computed to tangent developables R even if their curves g of regression carry singularities. We shall have a look a the following example:

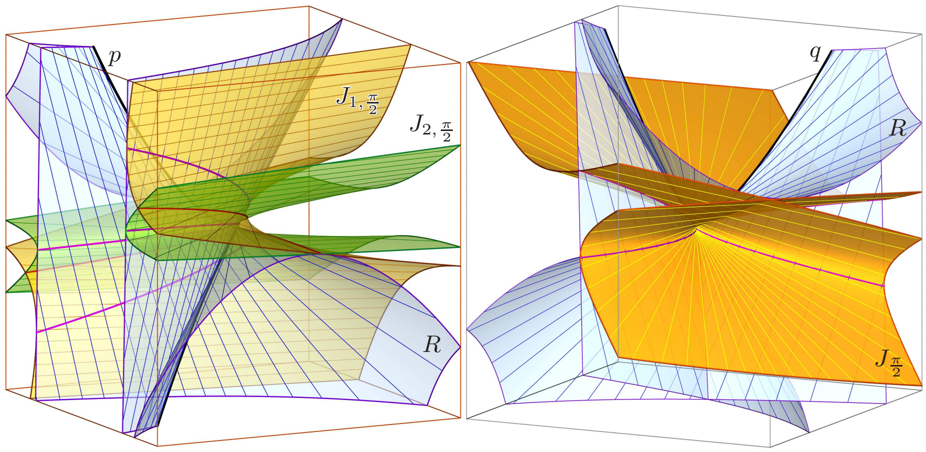

Fig. 4. Isoptic ruled surfaces: Left: two orthoptic hyperbolic paraboloids mentioned in Theorem 5. Right: cubic isoptic with multiplicity two to the quartic developable from Theorem 6.

The parametrization of a quartic space curve of the 2^{nd} kind (cf. [3]) with a cusp seems to be artificial at first glance:

$$\mathbf{p}(t) = \left(\frac{1}{2}At^2, \frac{1}{3}Bt^3, \frac{B^2(1+C^2)}{8A(1-C^2)}t^4\right), \quad t \in \mathbb{R}, \tag{13}$$

$A, B \in \mathbb{R} \setminus \{0\}$ and $C \in \mathbb{R}$ are shape parameters. However, it allows us to show that the orthoptic surface has a very special shape:

Theorem 6. *The orthoptic surfaces of the quartics in the three-parameter family of tangent developables of the singular quartics of the* 2^{nd} *kind given by* (13) *split into a pair of cubic surfaces.*

Proof. In this case, the angle criterion (10) splits into two quadratic factors:

$$((B^2(1+C^2)uv + 2A^2(1+C)^2)((B^2(1+C^2)uv + 2A^2(1-C)^2) = 0$$

Again, there arise two projective mappings $u \mapsto v$ which together with (9) yield parametrizations of the orthoptics. A subsequent implicitization confirms the theorem. □

Figure 4 (right) shows the tangent developable of precisely that curve q in (13) where the two cubic orthoptics coincide. This occurs if, and only if, $C = 0$.

5 Conclusion, future work

The class of isoptic ruled surfaces of developables invariant under certain groups of motions is not very rich. However, this is not the case with algebraic developables. There is a huge variety of algebraic space curves and corresponding tangent developables that allow for a computation of orthoptics and general isoptics. Clearly, we are restricted to low degree examples, but this could be a minor flaw. For example: The computation of isoptics to the developables of cubic parabolas is sufficient if we want to study the behaviour of isoptic ruled surfaces in the vicinity of a generic regular generator of a developable, since the local cubic approximant of a space curve in the vicinity of a regular, non-inflection and non-handle point is a cubic parabola.

The isoptic ruled surfaces of the tangent developables of quartic and quintic space curves are as useful as the ones related to cubics. In the vicinity of a handle point, a generic curve can be approximated by (t, t^2, t^4), while the two types of spatial inflection points allow approximations in the form (t, t^3, t^4) and (t, t^3, t^5), respectively. The computation of isoptics and orthoptics in these cases is comparably simple.

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