Examples of autoisoptic curves

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Abstract. The locus k_α of all points where two different tangents of a planar curve k meet at a constant angle α is called the isoptic curve of k. We shall look for curves k that coincide with their isoptic curves k_α and call them autoisoptic curves. Describing a planar curve k by its support function d allows us to derive a system of two linear ordinary delay differential equations that have to be fulfilled by d in order to make k an autoisoptic curve. Examples of autoisoptic curves different from the only known examples, namely logarithmic spirals, shall be given. We do not provide the most general autoisoptic curves, since these involve ordinary delay differential equations with time dependent delays. We only treat the case of constant delays.

Keywords: isoptic curve, autoisoptic curve, autoevolutoid, spiraloid, support function, delay differential equation, Lambert W function

1 Introduction

From any planar curve various other curves can be deduced or constructed: Among them, we find the evolute, the family of involutes, offset curves (parallel curves), isoptic curves (see [7, 11]), equioptic curves (cf. [6, 8]), and many others. Allmost all of these curves somehow have practical applications in geometry, kinematics, and appear in different areas of physics, see *e.g.* [12].

Curves deduced from a curve k can even coincide with the original curve k, as is the case with autoevolutes (see [13]) which are their own evolutes. An autoevolute is also an autoisoptic curve, for there is a rigid right angle consisting of the curve's tangent T and normal N at any point P moving along k while N touches k in a further point $P^\star \neq P \in k$ (see Fig. 3). Among the autoisoptic curves, we find logarithmic spirals. However, deduced curves can be just congruent to the initial curves as is the case with (straight) cycloids. Further, there exist curves which are similar or congruent to their evolutes of a specific order, see [1, 2].

Recently, it was noted in [9] that logarithmic spirals are autoisoptic curves. In this note, we shall show that there are much more curves sharing this property with the logarithmic spirals. Section 2 describes how planar curves can be represented via support functions. We shall derive conditions on the support function of a planar curve in order to make it an autoisoptic curve in Section 3. This will lead to a system of linear ordinary delay differential equations (henceforth

ODDEs). Section 4 deals with exact solutions of a single equation out of the system of ODDEs. The curves obtained that way will be called autoevolutoides and could be seen as a generalization of the autoevolutes by Wunderlich (see [13]). In Section 5, we attack the main problem, *i.e.*, the solutions of both ODDEs. Unfortunately, there exist only approximate solutions. If $d: I \subset \mathbb{R} \to \mathbb{R}$ solves one of the equations, it will in general, not be an exact solution of the second equation. We will also present some examples in Section 5.

2 Curve representation via support function

We aim at a compact notation, and therefore, we define

$$\mathrm{c}_x := \cos x \quad \text{and} \quad \mathrm{s}_x := \sin x.$$

We deal with planar curves k and assume that their tangents T can be given by their equations in terms of Cartesian coordinates (x, y) as

$$T: \ -d(t) + x\,\mathrm{c}_t + y\,\mathrm{s}_t = 0 \tag{1}$$

where $d(t): I \subset \mathbb{R} \to \mathbb{R}$ is called the *support function* defined on some interval and gives the oriented distance from T to the origin of the coordinate system, see Fig. 1.

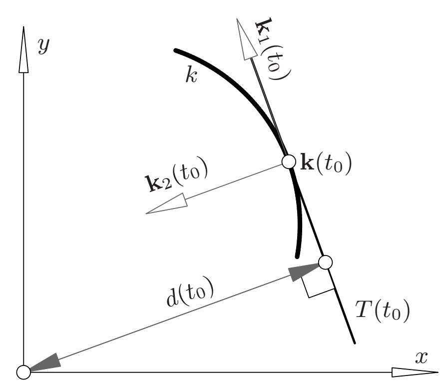

Fig. 1. The support function d of a curve k and its tangent T at t_0.

The vector function $\mathbf{k}_2 = (\mathrm{c}_t, \mathrm{s}_t): I \subset \mathbb{R} \to \mathrm{S}^1$ is a unit normal vector of T and, at the same time, the unit normal vector field of the curve k at the point of contact with T. If the support function d is at least r times differentiable, then the envelope k of all lines T described by (1) is at least an $r-1$ times differentiable curve. It is well-known that the points $\mathbf{k}(t_0)$ of the envelope k can be found by intersecting $T(t_0)$ with its first derivative $\frac{\mathrm{d}}{\mathrm{d}t}T(t_0) = \dot{T}(t_0)$ there.

Consequently, $\mathbf{k}(t) := T(t) \cap \dot{T}(t)$ yields a parametrization $\mathbf{k}(t): I \to \mathbb{R}$ of the envelope k which can be written in terms of the support function as

$$\mathbf{k}(t) = \left(d\, \mathrm{c}_t - \dot{d}\, \mathrm{s}_t\, , d\, \mathrm{s}_t + \dot{d}\, \mathrm{c}_t \right) \tag{2}$$

where $\dot{d}$ is the (first) derivative of d with respect to the parameter t.

From the equation of k's tangents (1) we can read off the unit normal vector field $\mathbf{k}_2$ of k as well as the unit tangent vector field $\mathbf{k}_1(t) = (\, \mathrm{s}_t\, , -\mathrm{c}_t\,)$. Here, the orientation of $\mathbf{k}_1$ is chosen such that a positive quarter turn of $\mathbf{k}_1$ yields $\mathbf{k}_2$.

The representation of k by means of its tangents (1) allows us to describe tangents T_α that enclose a certain prescribed angle $\alpha \in]0, \pi[$ with any given tangent T by simply replacing t with $t + \alpha$, and thus,

$$T_\alpha : \; - d_\alpha + x\, \mathrm{c}_{t+\alpha} + y\, \mathrm{s}_{t+\alpha} = 0, \tag{3}$$

provided that $t \in I$ and $t + \alpha \in I$ hold and

$$d_\alpha := d(t + \alpha).$$

The representation of planar curves via support functions has some advantages: Tangents and normals are known from the very beginning. Further, the curvature radius $\rho(t_0)$ of k at $\mathbf{k}(t_0)$ equals $\rho(t_0) = d(t_0) + \ddot{d}(t_0)$. Evolutes of any order can easily be given using (2), see [7, 11].

3 Delay differential equations for autoisoptic curves

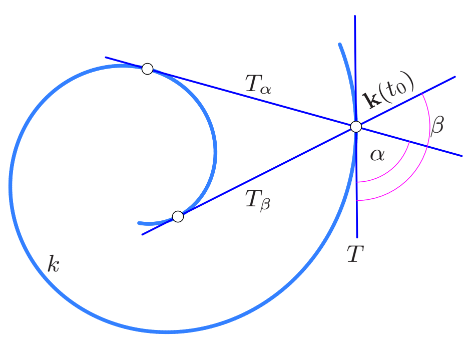

Fig. 2. The tangents of an autoisoptic curve and the enclosed angles.

Now, we shall find the conditions on the support function d in order to make k an autoisoptic curve. If k is an autoisoptic curve, then there are two different tangents T_α, T_β (with $\alpha \neq \beta$) touching k at $\mathbf{k}(t_0 + \alpha)$, $\mathbf{k}(t_0 + \beta)$ passing through the point $\mathbf{k}(t_0)$ as indicated in Fig. 2. Necessarily, the instances $t_0+\alpha$ and $t_0+\beta$ have to lie in I. Thus, the parametrization $\mathbf{k}$ from (2) annihilates the equations of both tangents T_α and T_β where the latter is obtained from (3) by replacing α with β. This gives two independent *linear ordinary delay differential equations* that read

$$\begin{aligned} d_\alpha - d\,\mathrm{c}_\alpha - \dot{d}\,\mathrm{s}_\alpha &= 0, \\ d_\beta - d\,\mathrm{c}_\beta - \dot{d}\,\mathrm{s}_\beta &= 0. \end{aligned} \tag{4}$$

The tangents T_α and T_β enclose the angle $\omega = |\alpha - \beta|$ ($\alpha \neq \beta$) and, if (4) is fulfilled, k is its own isoptic curve to the *optical angle* ω. We can state:

Theorem 1. *A curve $k \subset \mathbb{R}^2$ with support function $d: I \subset \mathbb{R} \to \mathbb{R}$ is an autoisoptic curve if, and only if, d solves* (4)*.*

Proof. We have to show that the conditions on d given in (4) are necessary and sufficient.

From the computation of (4), it is clear that these are necessary conditions.

In order to show that (4) are also sufficient in order to make k (parametrized by (2)) an autoisoptic curve, we prove that the intersection $\overline{\mathbf{k}}(t)$ of $T_\alpha(t)$ and $T_\beta(t)$ is identical with $\mathbf{k}(t)$ (for all $t \in I$). We find $\overline{\mathbf{k}}(t) = T_\alpha(t) \cap T_\beta(t)$ as

$$\overline{\mathbf{k}}(t) = \tfrac{1}{\mathrm{s}_{\alpha-\beta}} \left(d_\alpha\, \mathrm{s}_{t+\beta} - d_\beta\, \mathrm{s}_{t+\alpha}\, , d_\beta\, \mathrm{c}_{t+\alpha} - d_\alpha\, \mathrm{c}_{t+\beta} \right).$$

The points $\mathbf{k}(t_0)$ and $\overline{\mathbf{k}}(t_0)$ are identical if, and only if, $\|\mathbf{k}(t_0) - \overline{\mathbf{k}}(t_0)\| = 0$ which results in

$$\begin{gathered} (d^2 + \dot{d}^2)\,\mathrm{s}^2_{\alpha-\beta} + d^2_\alpha + d^2_\beta + dd_\alpha(\,\mathrm{c}_{2\beta-\alpha} - \mathrm{c}_a\,) + dd_\beta(\,\mathrm{c}_{2\alpha-\beta} - \mathrm{c}_b\,) - \\ -2\,\mathrm{s}_{\alpha-\beta}\,\dot{d}(d_\alpha\,\mathrm{c}_\beta - d_\beta\,\mathrm{c}_\alpha\,) = 0. \end{gathered} \tag{5}$$

Indeed, (5) is satisfied if both equations from (4) hold for any $t_0 \in I$. □

We shall point out that the necessary, but by no means sufficient criterion $\det\big(\mathbf{k}(t_0), \overline{\mathbf{k}}(t_0)\big) = 0$, *i.e.*, the linear dependency of $\mathbf{k}(t_0)$ and $\overline{\mathbf{k}}(t_0)$ which reads

$$d(d_\alpha\,\mathrm{c}_\beta - d_\beta\,\mathrm{c}_\alpha\,) - \dot{d}(d_\beta\,\mathrm{s}_\alpha - d_\alpha\,\mathrm{s}_\beta\,) = 0$$

also holds true, provided that (4) is true, since then the left-hand-side of latter equation also vanishes.

Remark 1. The most general case is covered if we assume that $\alpha: I \to \mathbb{R}$ and $\beta: I \to \mathbb{R}$ are real-valued functions on the desired interval I. Then, (4) are linear ODDEs with time-dependent delays which makes a solution more involved. We shall not discuss this case here.

4 Exact solutions

At first, we have to emphasize that if $d: I \subset \mathbb{R} \to \mathbb{R}$ is a solution of (4), then all scalar multiples $\lambda \cdot d : I \to \mathbb{R}$ ($\lambda \in \mathbb{R}$) are also solutions of (4). Moreover, if $d, e: I \to \mathbb{R}$ are two different solutions of (4), then any linear combination $\delta \cdot d + \varepsilon \cdot e: I \to \mathbb{R}$ (with $\delta, \varepsilon \in \mathbb{R}$) also solves (4). Thus, it is sufficient to find a collection of basic functions that solve (4) and use them as a basis in order to generate more general solutions as linear combinations.

4.1 One equation

Let us consider only one of the equations given in (4). Usually, such an equation is solved for the derivative of the undetermined function d which gives

$$\dot{d} = \frac{1}{\mathrm{s}_\alpha} d_\alpha - \frac{\mathrm{c}_\alpha}{\mathrm{s}_\alpha} d. \tag{6}$$

According to [4, 5], the exact solution of (6) is the infinite sum

$$d(t) = \sum_{j=-\infty}^{\infty} q_j \mathrm{e}^{\mathrm{W}(j,a_1)\cdot t} \tag{7}$$

with $a_1 = \frac{1}{\mathrm{s}_\alpha}$ and $\mathrm{W}(j, a_1)$ denoting the j-th branch of the Lambert W function. While the principal branch $\mathrm{W}(0, x)$ of the Lambert W function is a real value as long as $x > -\mathrm{e}^{-1}$, the values $\mathrm{W}(j, x)$ and $\mathrm{W}(-\mathrm{j}, \mathrm{x})$ are complex conjugate for any $x > -\mathrm{e}^{-1}$ and any $j \in \mathbb{Z}^\star$. If the argument of the W function turns out to be less than $-\mathrm{e}^{-1}$, we have to be aware of the fact that $\mathrm{W}(-j, x) = \overline{\mathrm{W}(j-1, x)}$ holds for all $j \in \mathbb{Z}$. We will keep this in mind, when we derive general solutions in Section 4.3.

Therefore, the coefficients q_j in (7) should be determined properly in order to obtain a real function $d(t)$.

Considering just one equation means that either $\beta = \alpha$ or $\beta = 0$ which is equivalent to $T_\beta = T_\alpha$ or $T_\beta = T$. These curves are characterized by sending only one further tangent T_α from each point to the curve itself, and therefore, they can be viewed as *autoevolutoides*.

4.2 Autoevolutes: $\boldsymbol{\beta = \alpha = \frac{\pi}{2}}$

Assuming $\beta = \alpha = \frac{\pi}{2}$ causes $T_\alpha = T_\beta = \dot{T}$ and (4) reduces to a single equation:

$$d_{\frac{\pi}{2}} - \dot{d} = 0 \tag{8}$$

which can also be achieved by $\alpha = 0$, $\beta = \frac{\pi}{2}$, *i.e.*, $T_\beta = \dot{T}$ and $T_\alpha = T$. Still, this equation characterizes the support function of autoisoptic curves. However, in this case two different tangents of the curve k meet at a right angle in the point $P = \mathbf{k}(t_0)$ on k and one of the tangents is k's normal at P which happens to be tangent to k precisely at $P^\star$, the center of curvature of k at P, see Fig. 3. Thus, k is its own evolute, and therefore, k is called an *autoevolute*. These curves are intensively studied in [13] and a kinematic generation is also given there. The

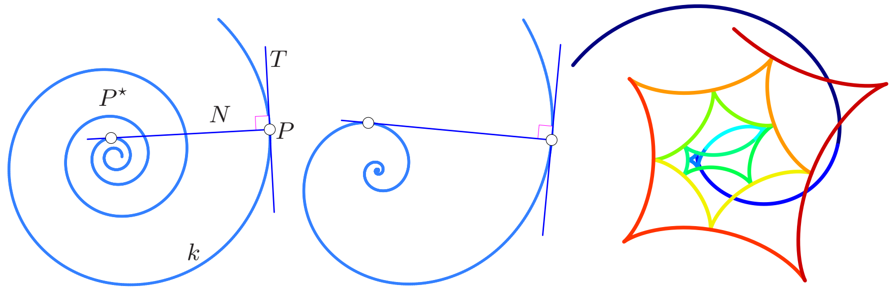

Fig. 3. Left: Logarithmic spirals are autoevolutes and autoisoptic curves as well. Right: An autoevolute composed of four elementary solutions. The inner part is of hypospiraloid type which turns to a curve that looks like a logarithmic spiral.

equation (8) can also be found in [13]. Starting from basic solutions which are logarithmic spirals, Wunderlich created the general solutions of (8) by means of linear combinations of elementary solutions. Naturally, the general forms of autoevolutes are spiraloid curves (see [11]) and sometimes a transition from a hypospiraloid to an epispiraloid can be observed with autoevolutes. This fact is illustrated in Fig. 3 and will also occur at general forms of autoisoptic curves.

The logarithmic spiral is by no means a trivial example of an autoisoptic curve. Due to its invariance under certain one-parameter subgroups of the group of equiform motions, it is an autoisoptic curve to infinitely many optical angles at the same time (see Fig. 4). This corresponds to the infinitely many branches of the Lambert W function returning either infinitely many spiral parameters p when we insert $d(t) = \mathrm{e}^{p \cdot t}$ with $p \in \mathbb{R}^\star$, or returning infinitely many optical angles $\alpha \in \mathbb{R}$, when fixing $p \in \mathbb{R}^\star$.

4.3 Autoevolutoides $\boldsymbol{\alpha = \beta \neq \frac{\pi}{2}}$

Now, the angle α enclosed by the second curve tangent T_α and T that passes through $P = T \cap \dot{T}$ shall differ from $\pi/2$. Evolutoides in general, may also be generated by moving a straight line along a given curve k such that it encloses the fixed angle α with the tangents of the curve. If there exists an envelope of all these lines, it could be called the α-*evolutoid*. However, a generic evolutoid will not be an autoevolutoid. We have:

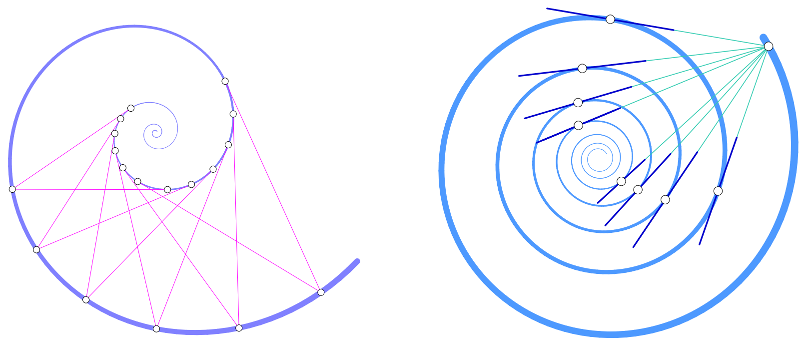

Fig. 4. Left: The equiform motion that generates the logarithmic spiral moves the fixed and rigid angle along the curve. Right: The logarithmic spiral is by no means a trivial example of an autoisoptic curve: It is an autoisoptic to infinitely many optical angles at the same time.

Theorem 2. *The support function d of autoevolutoides,* i.e., *curves k that send from each point $P = T \cap \dot{T} \in k$ a further tangent $T_\alpha \neq T, \dot{T}$ with $\sphericalangle(T, T_\alpha) = \alpha$ to k, solve the first equation of* (4), *and,* vice versa.

We do not have to show that these conditions are sufficient and necessary, since we have done so in the proof of Thm. 1.

The support function of a logarithmic spiral equals $d(t) = \mathrm{e}^{pt}$ (with spiral parameter $p \in \mathbb{R}^\star$ and $t \in \mathbb{R}$) and leads to a huge variety of examples of autoevolutes and autoevolutoides which are also autoisoptic curves.

We can try the ansatz $d(t) = \mathrm{e}^{pt}$ with $p \in \mathbb{R}^\star$ and with (4) we find

$$\mathrm{e}^{p\alpha} - \mathrm{c}_\alpha - p\,\mathrm{s}_\alpha = 0 \tag{9}$$

which relates the optical angle α and the spiral parameter p. Moreover, (9) is the *characteristic equation* of the ODDE (6). The solutions of (9) are called the eigenvalues of the ODDE and the spectral analysis becomes more involved, since (9) is transcendental and has infinitely many zeros for arbitrary $\alpha \neq j\pi$ (with $j \in \mathbb{Z}$).

Explicit solutions of (9) for p can be given in terms of the Lambert W function:

$$p = -\mathrm{ctg}\alpha - \tfrac{1}{\alpha}\mathrm{W}\left(j, -\tfrac{\alpha}{\mathrm{s}_\alpha}\exp(-\alpha \cdot \mathrm{ctg}\alpha)\right) \tag{10}$$

where $\mathrm{W}(j, x)$ denotes the j-th branch of the Lambert W function. The principal branch corresponds to $j = 0$ and is usually simply denoted by $\mathrm{W}(x)$.

Any complex solution for p of (9) has a conjugate solution. As described earlier, linear combinations of any solutions of the ODDE are also solutions to the same ODDE, the real sum of any pair of complex conjugate solutions results in a real solution. However, the difference of a complex conjugate pair is a purely imaginary number and - by dividing it by 2i - it gives also rise to a real number.

Replacing the principal branch of W in (10) by the j-th branch, we find a complex spiral parameter for a logarithmic spiral as

$$p_j = -\mathrm{ctg}\alpha - \tfrac{1}{\alpha}\mathrm{W}\left(j, -\tfrac{\alpha}{\mathrm{s}_\alpha}\exp(-\alpha\cdot\mathrm{ctg}\alpha)\right) = q_j + r_j\mathrm{i}.$$

Assuming that $-\frac{\alpha}{\mathrm{s}_\alpha}\exp(-\alpha\cdot\mathrm{ctg}\alpha) < -\mathrm{e}^{-1}$, we have $p_{-j} = \overline{p_j} = q_j - \mathrm{i}r_j$.
The elementary solutions of (4) are $\sigma_j = \exp(p_j t) = \exp(q_j t)(\,\mathrm{c}_{r_j t} + \mathrm{i}\,\mathrm{s}_{r_j t}\,)$. Since linear combinations of elementary solutions produce further solutions of (4), we can build the sum as well as the difference - the latter is to be divided by i - in order to get real solutions of (4). Thus, we obtain the real elementary solutions

$$2a_j = \sigma_j + \sigma_{-j} \quad \text{and} \quad 2b_j = -\mathrm{i}(\sigma_j - \sigma_{-j})$$

which read

$$a_j = \exp(p_j t)\cos(q_j t) \quad \text{and} \quad b_j = \exp(p_j t)\sin(q_j t). \tag{11}$$

It would not be necessary to divide the sums and differences of elementary solutions by 2, since this causes only a scaling of the basis functions. The real solutions of (4) in the most general form depend on two one-parameter families $(\lambda_j)_{j\in I}$ and $(\mu_j)_{j\in I}$ of real coefficients (with $I \in \mathbb{N}$ being some index set):

$$d(t) = \sum_{j=1}^{N} \lambda_j a_j + \mu_j b_j \tag{12}$$

where $N := \#I$ shall be called the *order* of the autoisoptic curve.

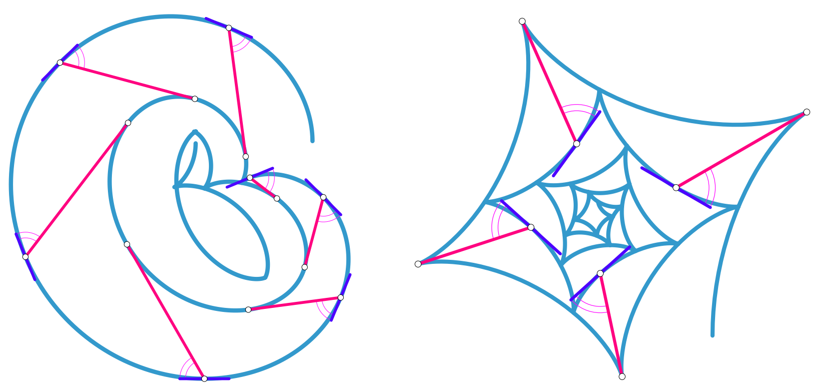

Fig. 5. Autoisoptic curves that are also autoevolutoides: At some points $\mathbf{k}(t_j)$ on the curve, the tangents there and the tangents to the points $\mathbf{k}(t_j+\alpha)$ are drawn $\alpha = -\frac{8}{3}\pi$ and $t \in [-3\pi, 9\pi]$ (left), $\alpha = -\frac{7}{3}\pi$ and $t \in [-8\pi, 4\pi]$ (right).

Fig. 5 (left) shows an example of an autoisoptic curve k of order three with $\alpha = -\frac{8}{3}\pi$. The tangents at points $\mathbf{k}(t_j)$ enclose the angle $\alpha = -\frac{8}{3}\pi \equiv \frac{1}{3}\pi \pmod{2\pi}$ with the tangents drawn to the point $\mathbf{k}(t_j+\alpha)$. The spiraloid behavior of the curve which is clear from its mathematical representation is clearly seen. Fig. 5 (right) shows an example of an autoisoptic curve of order three with $\alpha = -\frac{7}{3}\pi \equiv \frac{2}{3}\pi$ (mod 2π). Since this curve is also an autoevolutoid, it carries its cusps.

5 Two equations and approximations

According to Theorem 1, the support function d of an autoisoptic curve k has to fulfill both linear ODDEs given in (4). Since the eigenvalues of one equation, say the first, will differ from those of the second as long as $\alpha \neq \beta$, the elementary solutions of the second equation will be different from the first collection of elementary solutions.

Therefore, we claim that there exist only numerically defined approximative solutions of the system (4), at least for the case of constant delays α, β. Numerical solutions can, for example, be generated with Maple™(version 2016 and newer) or with Matlab™. We will not use the algorithms implemented in any software. It seems more appropriate to use standard techniques from approximation theory and functional analysis that can be found in [10]. Some of the algorithms can be found therein and shall not be explained in detail here.

5.1 Approximation in the space of elementary solutions

With (11) and (12) we can solve both ODDEs given in (4). This yields two different functions, since the elementary solutions of the two ODDEs will in general not agree. The coefficients λ_j and μ_j in (12) are still unknown and may be determined such that we obtain a function $d^\star : I \subset \mathbb{R} \to \mathbb{R}$ within the space of basic functions that is close to solve both ODDEs. Approximation techniques yield these coefficients and a finite linear combination of elementary functions that is close to solve both ODDEs (4). We shall give a few images of such numerically defined approximations in order to show the variability of these curves.

Fig. 8 shows that the support function $d^\star$ (approximating a bunch of elementary solutions of both ODDEs (4)) of the curve shown in Fig. 7 (left) is close to annihilate both ODDEs (4). The absolute values of $d^\star$ evaluated at both ODDEs are close to zero in both cases compared to the absolute values of the support function $d^\star$. Fig. 9 shows two more approximative solutions of (4), one showing hypospiraloid behavior (left), one showing epispiraloid behavior (right).

5.2 Approximation with power series

Assuming that there exist an analytic function $d : I \subset \mathbb{R} \to \mathbb{R}$ that solves both equations in (4), we can expand it in power series of the form

$$d(t) = \sum_{j=0}^{N} v_j t^j. \qquad (13)$$

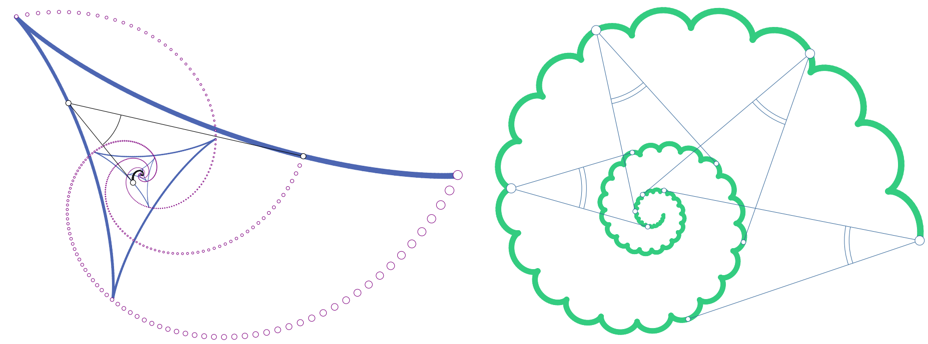

Fig. 6. Hypospiraloid (left) and epispiraloid (right) as autoisoptic curves.

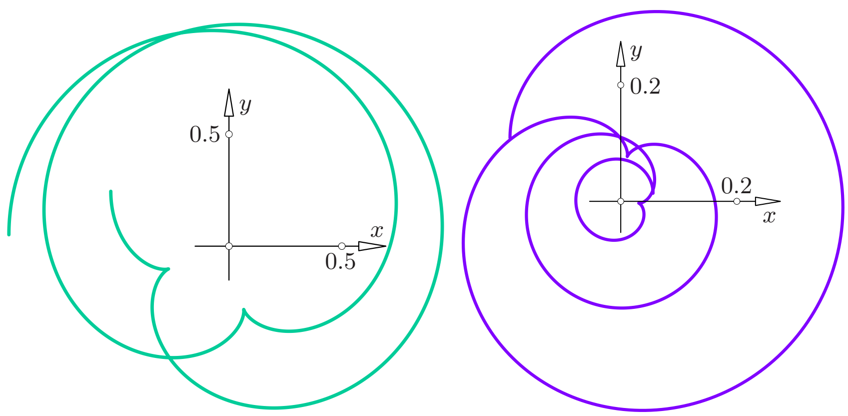

Fig. 7. Two approximations in the space of basic elementary solutions.

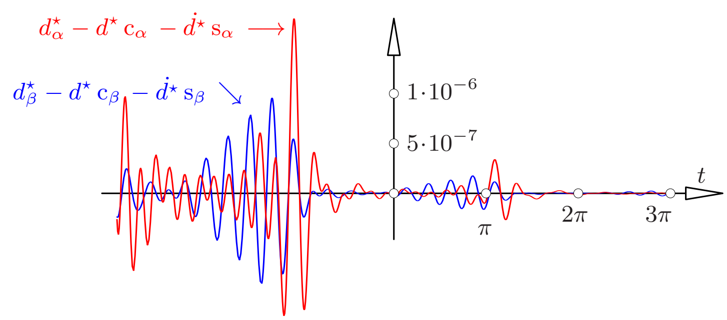

Fig. 8. The support function $d^\star$ of the curve in Fig. 7 (left) is a very good approximation of the solutions of both ODDEs.

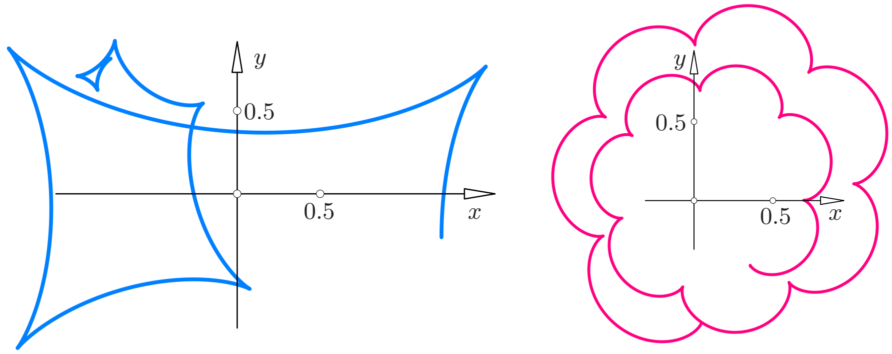

Fig. 9. Two further approximations in the space of basic elementary solutions.

Inserting this into (4), we can derive a system of linear equations in the yet unknown coefficients v_j. As a matter of fact, it is mere technique to solve this system of equations. Numerical instability caused by exaggerated accuracy, *i.e.*, the choice of a too large N, also produces strange results. In the solutions can be trusted in a vicinity of $t = 0$ (provided the series are developed there).

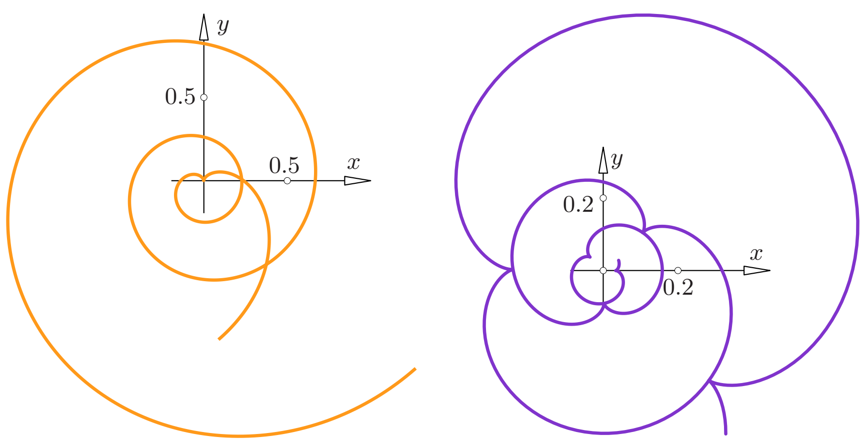

Fig. 10. The polynomial approximations imitate the shape of the curves found by interpolation in the space of elementary exponential solutions up to a certain extent depending on the degree of the power series.

6 Conclusion

The case of time dependent delays is not treated here, since ODDEs with time dependent delays will in general have no closed solutions, *i.e.*, solutions defined by elementary analytic functions. Exact solutions may exist only in very rare cases, if at all.

References

1. L. Euler: *Investigatio curvarum quae evolutae sui similes producunt.* Commentarii academiae scientiarum Petropolitanae **12** (1750), 3–52.
2. L. Euler: *Investigatio curvarum quae similes sint suis evolutis vel primis vel secundis vel tertiis vel adeo ordinis cuiuscunque.* Nova Acta Academiae Scientarum Imperialis Petropolitanae **1** (1787), 75–116.
3. G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics.* From the ancient Greeks to 21^{st} century developments. Springer-Spektrum, Springer-Verlag, Heidelberg, 2016.
4. J.K. Hale, S.M. Verduyn Lunel: *Introduction to Functional Differential Equations.* Applied Mathematical Sciences 99. Springer, 1993.
5. J.M. Heffernan, R.M. Corless: *Solving some delay differential equations with computer algebra.* The Mathematical Scientist **31**/1 (2006), 21–34.
6. R. Kunkli, I. Papp, M. Hoffmann: *Isoptics of Bézier curves.* Comp. Aided Geom. Design **30**/1 (2013), 78–84.
7. G. Loria: *Spezielle algebraische und transzendente ebene Kurven I, II.* B.G. Teubner, Leipzig-Berlin, 1910.
8. B. Odehnal: *Equioptic curves of conic sections.* J. Geom. Graph. **14**/1 (2010), 29–43.
9. R.F. Puerta, M.A. Gómez-Elvira Gonzáles, Á. Miguel, C. San Antonio Gómez, E. Martín, M. Conejo: *The logarithmic spiral, autoisoptic curve.* Anales de Ingeniería Gráfica **21**/2009, 41–44.
10. A.F. Timan: *Theory of Approximation of Functions of a Real Variable.* Dover Publications, New York, 1994.
11. H. Wieleitner: *Spezielle ebene Kurven.* G.J. Göschen'sche Verlagshandlung, Leipzig, 1908.
12. E. Wölffing: *Ueber Pseudotrochoiden.* Zeitschr. Math. Phys. **44** (1897), 139–166.
13. W. Wunderlich: *Autoevoluten.* Elem. Math. **XVII**/6 (1962), 121–128.