

# Equioptic points of a triangle

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## Abstract

The locus of points where two non-concentric circles  $c_1$  and  $c_2$  are seen under equal angles is the equioptic circle  $e$ . The equioptic circles of the excircles of a triangle  $\Delta$  have a common radical axis  $r$ . Therefore the excircles of a triangle share up to two real points, *i.e.*, the equioptic points of  $\Delta$  from which the circles can be seen under equal angles. The line  $r$  carries a lot of known triangle centers. Further we find that any triplet of circles tangent to the sides of  $\Delta$  has up to two real equioptic points. The three radical axis of triplets of circles containing the incircle are concurrent in a new triangle center.

*Key Words:* Triangle, excircle, incircle, equioptic circle, equioptic points, center of similarity, radical axis.

## 1. Introduction

Let there be given a triangle  $\Delta$  with vertices  $A$ ,  $B$ , and  $C$ . The incenter shall be denoted by  $I$ , the incircle by  $\Gamma$ . The excenters are labeled with  $I_1$ ,  $I_2$ , and  $I_3$ . We assume that  $I_1$  is opposite to  $A$ , *i.e.*, it is the center of the excircle  $\Gamma_1$  touching the line  $[B, C]$  from the outside of  $\Delta$ , cf. Fig. 1. Sometimes it is convenient to number vertices as well as sides of  $\Delta$ : The side (lines)  $[B, C]$ ,  $[C, A]$ ,  $[A, B]$  shall be the first, second, third side (line) and  $A$ ,  $B$ ,  $C$  shall be the first, second, third vertex, respectively. According to [1, 2] we denote the centers of  $\Delta$  with  $X_i$ . For example the incenter  $I$  is labeled with  $X_1$ .

The set of points where two curves can be seen under equal angles is called *equioptic curve*, see [3]. It is shown that any pair  $(c_1, c_2)$  of non-concentric circles has a circle  $e$  for its equioptic curve [3]. The circle  $e$  is the Thales circle of the segment bounded by the internal and external centers of similarity of either given circle, *i.e.*, the center of  $e$  is the midpoint of the two centers of similarity, see Fig. 1. In case of two congruent circles  $e$  becomes the bisector of the centers of  $c_1$  and  $c_2$ , provided that  $c_1$  and  $c_2$  are not concentric.

The four circles  $\Gamma$ ,  $\Gamma_i$  (with  $i \in \{1, 2, 3\}$ ) tangent to the sides of a triangle  $\Delta$  can be arranged in six pairs and, thus, they define six equioptic circles. Among them we find four triplets of equioptic circles which have a common radical axis instead of a radical center, *i.e.*, the three circles of such a triplet form a pencil of circles. These shall be the contents of Sec. 2 and Sec. 3.

We use homogeneous trilinear coordinates of points and lines, respectively. The homogeneous triplet of real (complex) numbers  $(x_0 : x_1 : x_2)$  are said to be the homogeneous trilinear coordinates of a point  $X$  if  $x_i$  are the oriented distances of  $X$  with respect to the sides  $[B, C]$ ,  $[C, A]$ , and  $[A, B]$  up to a common non vanishing factor, see. [1]. When we deal with trilinear

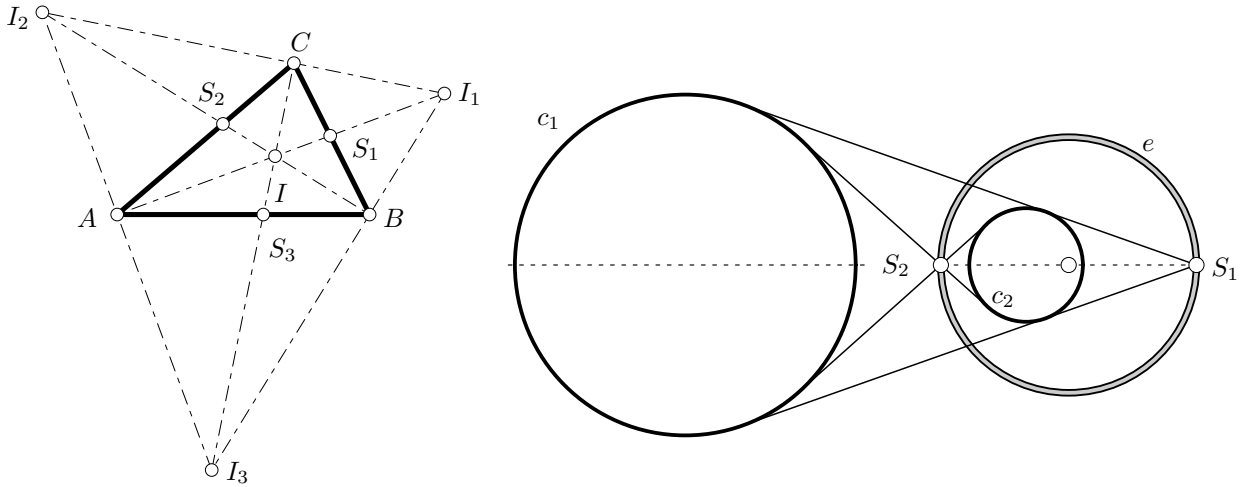


Figure 1: Left: Notations in and around the triangle  $\Delta$ . Right: Equioptic circle of two circles.

coordinates of points expressed in terms of homogeneous polynomials in  $\Delta$ 's side lengths  $a = \overline{BC}$ ,  $b = \overline{CA}$ , and  $c = \overline{AB}$  we use a function  $\zeta$  with the property  $\zeta(f(a, b, c)) = f(b, c, a)$  and further a function  $\sigma$  with  $\sigma(X_0 : x_1 : x_2) = (x_2 : x_0 : x_1)$ .

In this paper mappings will be written as superscripts, *e.g.*,  $\sigma \circ \zeta(X) = X^{\sigma \circ \zeta} = X^{\zeta \sigma}$  if applied to points. Note that  $\zeta \circ \sigma = \sigma \circ \zeta$ , provided that  $x_i$  are homogeneous functions in  $a, b, c$ .

## 2. Equioptic circles of the excircles

In order to construct the equioptic circles of a pair of excircles we determine the respective centers of similarity. First we observe that the internal centers of similarity of  $\Gamma_i$  and  $\Gamma_j$  is the  $k$ -th vertex of  $\Delta$ , where  $(i, j, k) \in \mathbb{I}^3 := \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ . Second we have to find the external centers of similarity. For any pair  $(\Gamma_i, \Gamma_j)$  the  $k$ -th side of  $\Delta$  is an exterior common tangent of both  $\Gamma_i$  and  $\Gamma_j$ , respectively, and thus  $[I_i, I_j]$  and the  $k$ -th side of  $\Delta$  meet in the external center of similarity  $S_{ij}$  of  $\Gamma_i$  and  $\Gamma_j$ . Now we are able to show a first result:

### Corollary 2.1.

*The external centers of similarity  $S_{ij}$  of the excircles  $\Gamma_i$  and  $\Gamma_j$  of a triangle  $\Delta$  are collinear. The line carrying these points is the polar of  $X_1$  with respect to  $\Delta$  and the polar line with respect to the excentral triangle of  $\Delta$  at the same time.*

*Proof.* We construct the polar line of the incenter  $X_1$  with respect to  $\Delta$ . For that purpose we project  $I$  from  $C$  to the line  $[A, B]$ . This gives  $S_3 := [A, B] \cap [I, C]$ . Then we determine a fourth point  $C'$  on  $[A, B]$  such that  $(A, B, S_3, C')$  is a harmonic quadrupel. The four lines  $[C, A]$ ,  $[C, B]$ ,  $[C, I]$ , and  $[I_1, I_2]$  obviously form a harmonic quadrupel and thus any line (which is not passing through  $C$ ) meets these four lines in four points of a harmonic quadrupel. So we have  $C' = [I_1, I_2] \cap [A, B]$  and obviously  $C' = S_{12}$ . Cyclically reordering labels of points and numbers we find  $S_{23}$  and  $S_{31}$  which are collinear with  $S_{12}$  and gather on the polar of  $X_1$ . On the other hand we have the harmonic quadruples  $(I_1, I_2, C, S_{12})$  (cyclic) which shows that  $[S_{12}, S_{23}]$  is the polar of  $X_1$  with respect to the excentral triangle.  $\square$

The centers  $T_{ij}$  of the equioptic circles  $e_{ij}$  are the midpoints of the line segments bounded by  $S_{ij}$  and the  $k$ -th vertex of  $\Delta$  with  $(i, j, k) \in \mathbb{I}^3$ . In terms of homogeneous trilinear coordinates

we have

$$S_{12} = (-1 : 1 : 0), \quad S_{i+1,j+1} = S_{ij}^\sigma.$$

The centers  $T_{ij}$  are thus

$$T_{12} = (c : -c : a - b), \quad T_{i+1,j+1} = T_{ij}^{\sigma\zeta} \quad (1)$$

and we can easily prove:

**Corollary 2.2.**

*The centers  $T_{ij}$  of the equioptic circles  $e_{ij}$  of any pair  $(\Gamma_i, \Gamma_j)$  of excircles of  $\Delta$  are collinear.*

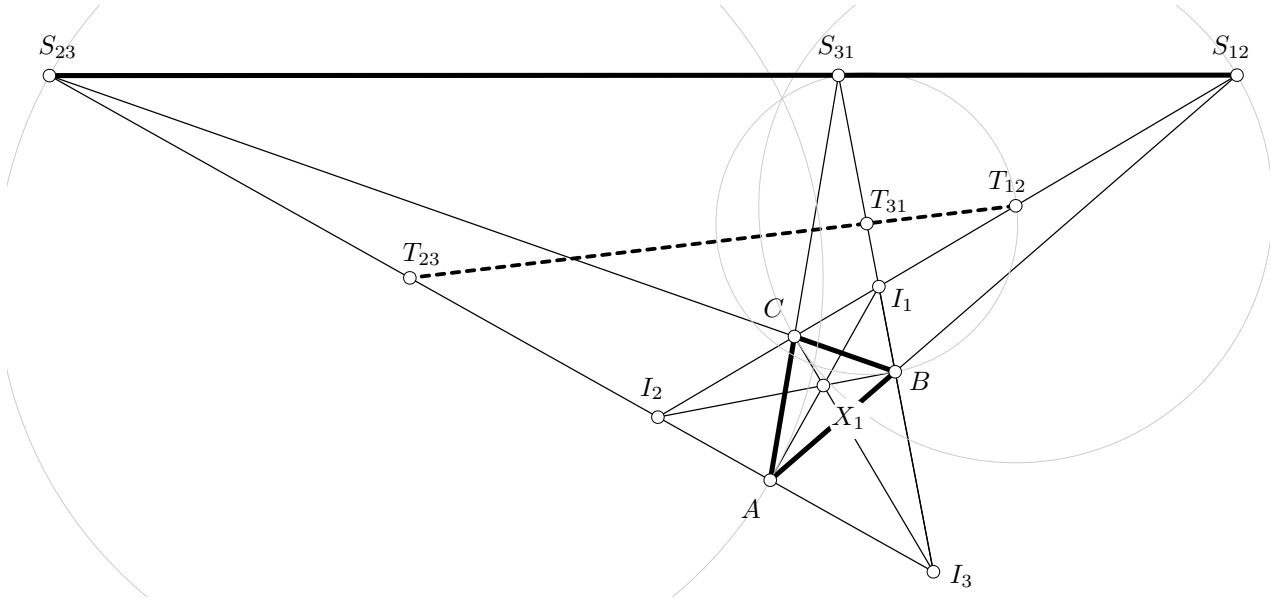


Figure 2: Centers of similarity  $S_{ij}$ , the centers of the equioptic circles  $T_{ij}$ , and harmonic quadruples.

*Proof.* The coordinate vectors of  $T_{ij}$  given in (1) are linearly dependent.  $\square$

**Remark 1.** *The line  $t$  connecting any  $T_{ij}$  with any  $T_{jk}$  (with  $(i, j, k) \in \mathbb{I}^3$ ) has trilinear coordinates  $[\lambda_0 : \lambda_1 : \lambda_2]$ , where  $\lambda_0 = a(-a + b + c)$  and  $\lambda_i = \zeta^i(\lambda_0)$  with  $i \in \{0, 1, 2\}$ . Obviously  $t$  is the polar of  $X_{55}$  with respect to  $\Delta$ . The center  $X_{55}$  is the center of homothety of the tangential triangle, the intangent triangle, and the extangent triangle, see [1]. Further it is the internal center of similarity of the incircle and the circumcircle of  $\Delta$ .*

We use the formula for the distance of two points given by their *actual trilinear coordinates* given in [1, p. 31] and compute the radii  $\rho_{ij}$  of the equioptic circles  $c_{ij}$  and find

$$\rho_{12} = \text{dist}(C, T_{12}) = \text{dist}(S_{12}, T_{12}) = \frac{ab}{a-b} \sin \frac{C}{2}, \quad (2)$$

$$\rho_{i+1,j+1} = \zeta(\rho_{ij}).$$

By means of the distance formula from [1, p. 31] or equivalently by means of the more complicated equation for a circle given by center and radius from [1, p. 223] we write down the equations of the equioptic circles

$$\begin{aligned} e_{12} : -c_A x_0^2 + c_B x_1^2 + (1 - c_C)x_2(x_1 - x_0) + (c_B - c_A)x_0x_1 &= 0, \\ e_{i+1j+1} &= \zeta(e_{ij}), \end{aligned} \quad (3)$$

where  $c_A$ ,  $c_B$ , and  $c_C$  are shorthand for  $\cos A$ ,  $\cos B$ , and  $\cos C$ , respectively. Now it is easy to verify that the following holds true:

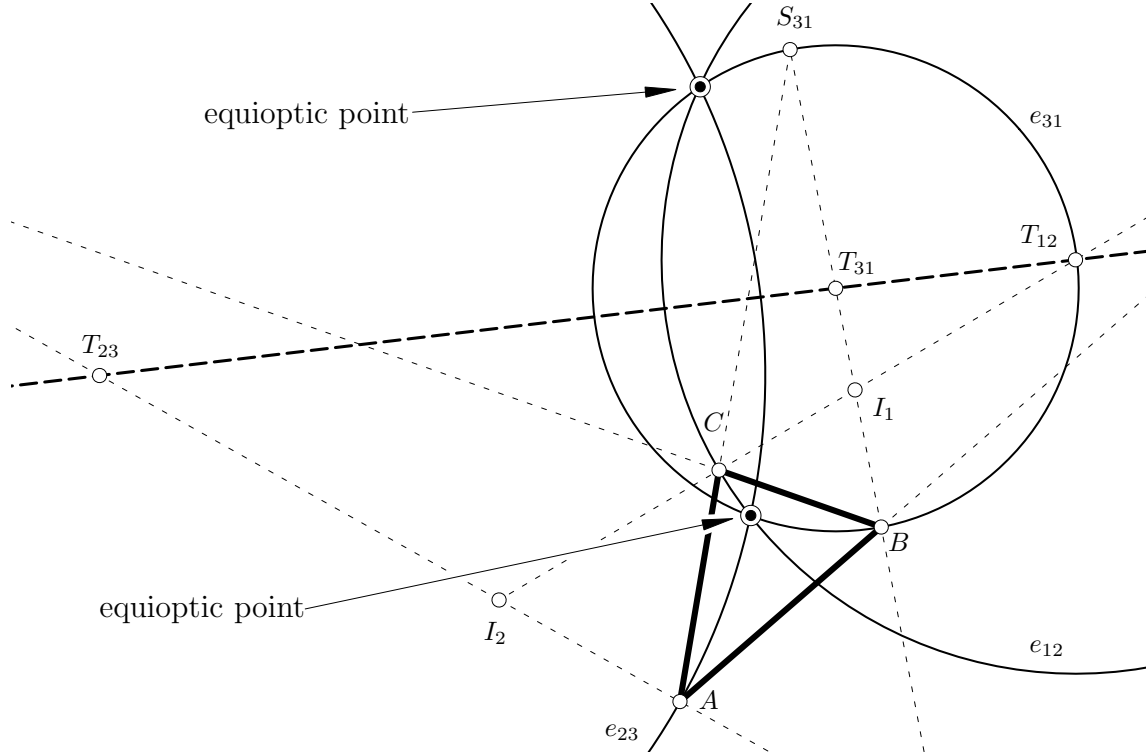


Figure 3: The equioptic circles and equioptic points of a triangle.

**Theorem 2.1.**

1. *The three equioptic circles of the excircles of a generic triangle  $\Delta$  have a common radical axis  $r$  and thus they have up to two common real points, i.e., the equioptic points of the excircles from which the excircles can be seen under equal angles.*
2. *The radical axis  $r$  contains  $X_4$  (ortho center),  $X_9$  (Mittenpunkt),  $X_{10}$  (Spieker center), and further  $X_i$  with*

$$\begin{aligned} i \in \{19, 40, 71, 169, 242, 281, 516, 573, 966, 1276, 1277, 1512, \\ 1542, 1544, 1753, 1766, 1826, 1839, 1842, 1855, 1861, 1869, \\ 1890, 2183, 2270, 2333, 2345, 2354, 2550, 2551, 3496, 3501\}. \end{aligned} \quad (4)$$

*Proof.* 1. Let  $P_1 := \mu e_{12} + \nu e_{23}$ ,  $P_2 := \mu e_{23} + \nu e_{31}$ , and  $P_3 := \mu e_{31} + \nu e_{12}$  be the equations of the conic sections in the pencils spanned by any pair of equioptic circles  $e_{ij}$ . We

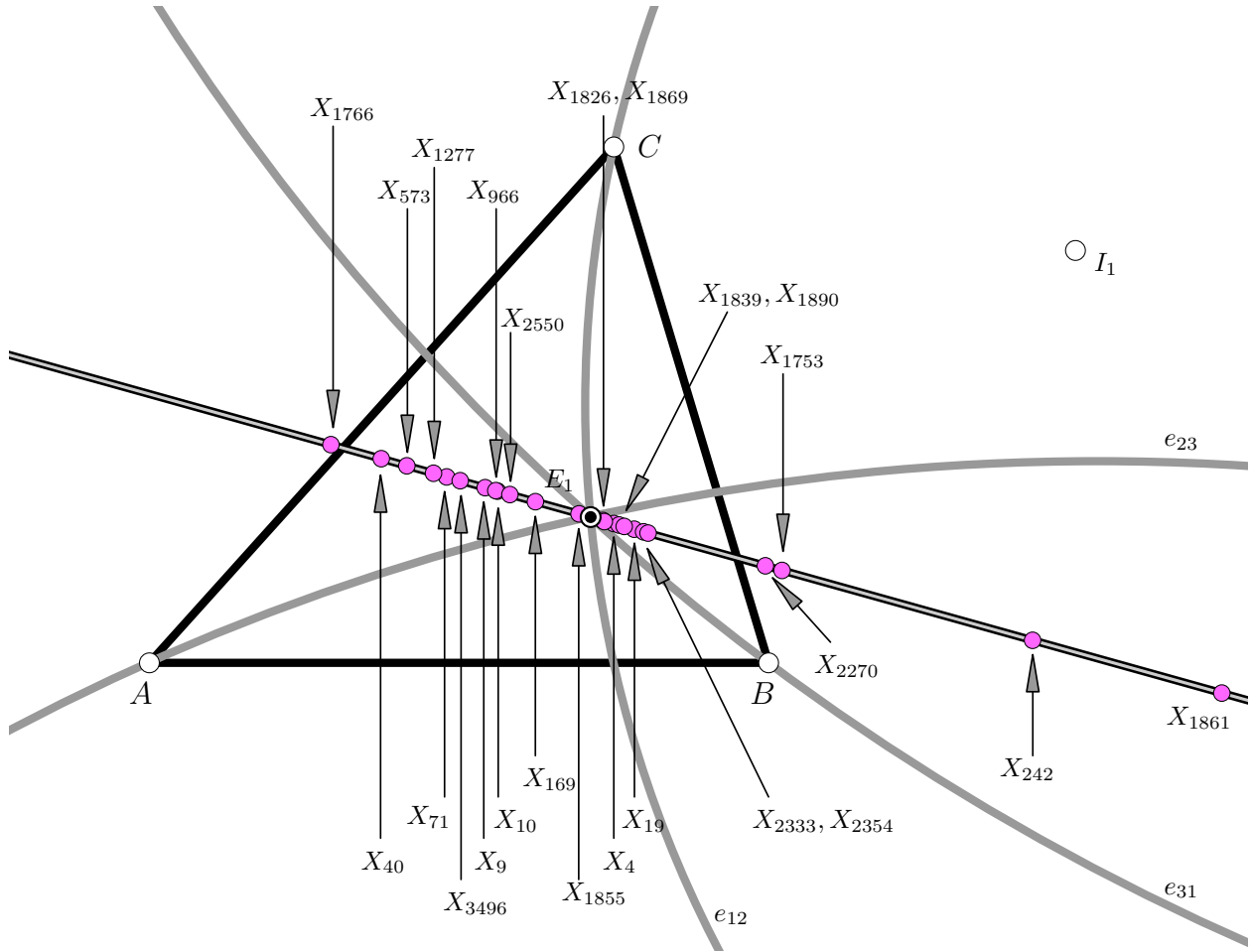


Figure 4: The line  $[X_4, X_9]$  and a bunch of triangle centers on it. Some of the centers mentioned in Th. 2.1 are not depicted for they are far out and especially  $X_{516}$  is the ideal point of the line  $[X_4, X_9]$ . The base triangle is acute.

compute the singular conic sections in the pencils and find that for all pencils the real singular conic sections consist of the ideal line  $\omega : ax_0 + bx_1 + cx_2$  and the line

$$(b - c)c_A x_0 + (c - a)c_B x_1 + (a - b)c_C x_2 = 0, \quad (5)$$

which is the radical axis of these three equioptic circles.

2. In [1, pp. 64 ff.] we find  $X_4 = [\operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C]$  and  $X_9 = [b + c - a, c + a - b, a + b - c]$  and obviously these coordinate vectors annihilate Eq. (5). By inserting the trilinears of the other points mentioned in the theorem we proof the incidence. The trilinears of points  $X_i$  with  $i \leq 360$  can be found in [1] whereas the trilinears for  $i > 360$  can be found in [2].

□

In Fig. 3 the equioptic circles as well as the equioptic points of an acute triangle are depicted. Fig. 4 shows some of the centers mentioned in Th. 2.1 located on  $r$ . Here, the base triangle is acute. Fig. 5 shows the centers on the line  $r = [X_4, X_9]$  for an obtuse base triangle.

**Remark 2.** *The circles in the pencil of circles spanned by the equioptic circles can share two real points, one real point with multiplicity two, or no real points. Thus a triangle has either*

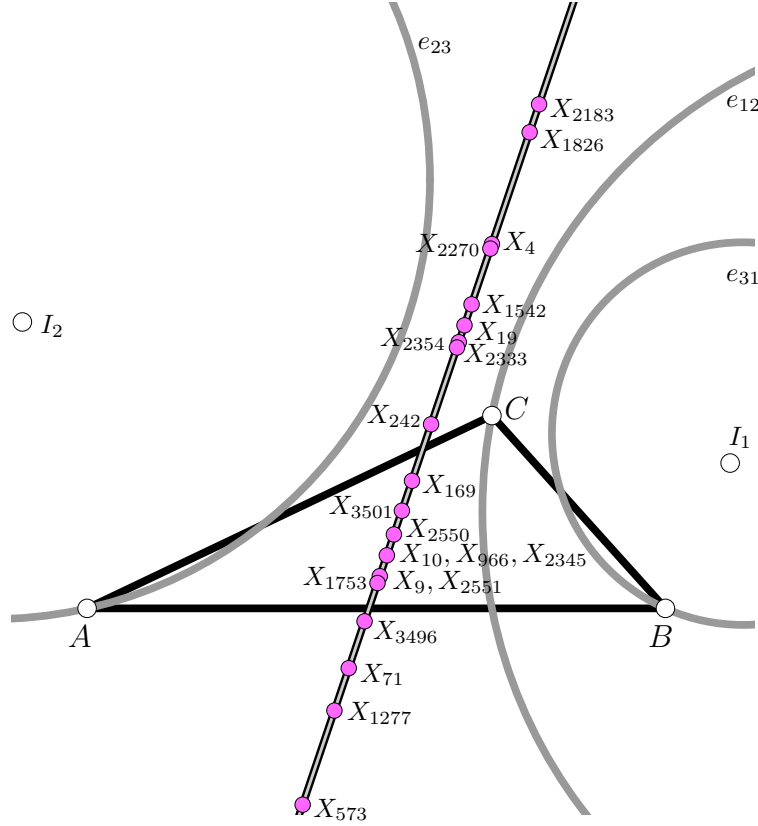


Figure 5: The line  $[X_4, X_9]$  and a bunch of triangle centers on it. Some of the centers mentioned in Th. 2.1 are not depicted for they are far out and especially  $X_{516}$  is the ideal point of the line  $[X_4, X_9]$ . The base triangle is obtuse.

two real equioptic points (cf. Fig. 4 or Fig. 8), or a single equioptic point (cf. Fig. 7), or no real equioptic point as is the case in Fig. 5.

In case of an equilateral triangle  $\Delta$  there is only one equioptic point  $E$  that coincides with the center of  $\Delta$ . The three equioptic circles become straight lines:  $e_{ij}$  is the  $k$ -th interior angle bisector and the  $k$ -th altitude of  $\Delta$ . Fig. 6 illustrates this case. From  $E$  any excircle  $i_j$  can be seen under the angle  $\arccos(-\frac{1}{8}) \approx 97.180756^\circ$ .

The case of a single equioptic point is illustrated in Fig. 7 at hand of an isosceles triangle. It is easily shown that in this case one has to choose  $\angle ACB = 2 \arcsin(\sqrt{3} - 1) \approx 94.11719432^\circ$  in order to have a unique equioptic point. Thus the triangle is obtuse. The unique (real) equioptic angle now equals  $2 \arcsin\left(\frac{3-\sqrt{3}}{2}\right) \approx 78.68794716^\circ$ .

Here and in the following we use the abbreviations  $\hat{a} = b + c$ ,  $\hat{b} = c + a$ , and  $\hat{c} = a + b$ . We can show:

**Theorem 2.2.** *A generic triangle  $\Delta$  has a unique equioptic point if and only if*

$$\sum_{cyclic} (a^6 + 2\hat{a}a^5 - a^4(\hat{a}^2 + 4bc) + 4a^3b(\hat{c}a - b^2) - 2a^2b^2c^2) = 0. \quad (6)$$

*Proof.* We have already shown that the three equioptic circles have a common radical axis  $r$ . Furthermore, if one of these three circles touches the common radical axis at a point  $E$ , then

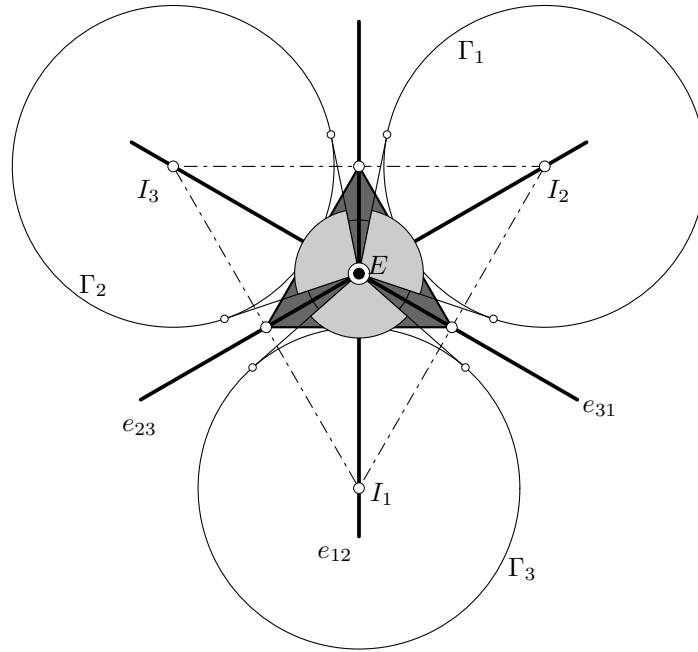


Figure 6: The only equioptic point of an equilateral triangle.

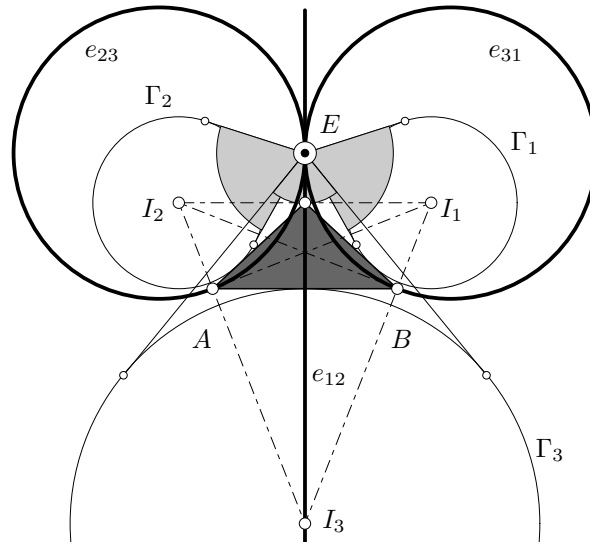


Figure 7: An isosceles triangle with a unique equioptic point.

any other circle touches precisely at  $E$ . This is caused by the fact that  $E$  has zero power with respect to the circle touching  $r$  and, as a point of the common radical axis  $r$ , it has to have the same power with respect to the other circles. Thus it is sufficient to derive a contact condition of  $r$  and one of the equioptic circles. Therefore, we compute the resultant of the equation of one circle as given in Eq. (3) and the radical axis from Eq. (5) with respect to one variable, say  $x_2$ . This yields a quadractic form  $q(x_0, x_1)$ . Now the condition on  $a, b, c$  in

order to make  $r$  a tangent of any equioptic circle is that the quadratic form  $q$  factors which is equivalent to  $\det(H_q) = 0$  where  $H_q$  is the Hessian of  $q$ . We find that  $H_q$  is the sextic form given in Eq. (6).  $\square$

**Remark 3.** *The case of an equilateral triangle is not covered by Eq. (6). In the case of an equilateral triangle the points  $T_{ij}$  are ideal points and thus there are no points  $S_{ij}$ . The equioptic circles of the excircles of an equilateral triangle degenerate and become the altitudes of the triangle and the one and only equioptic point is the one and only center of the equilateral triangle as can be seen in Fig. 6.*

### 3. Equioptic circles of the incircle and an excircle

We recall that the equioptic circles of a pair  $(\Gamma, \Gamma_i)$  is the Thales circle of the line segment bounded by the internal and external center of similarity of the incircle  $\Gamma$  and the  $i$ -th excircle  $\Gamma_i$ . We observe that the  $i$ -th vertex of  $\Delta$  is the external center of similarity of the above given pair of circles. The internal center is the intersection of a common internal tangent, i.e.  $\Delta$ 's  $i$ -th side and the line  $[I, I_i]$  connecting the respective centers. Consequently the internal centers of similarity are the points  $S_i$  ( $i \in \{1, 2, 3\}$ ). We have

$$S_1 = (0 : 1 : 1), \quad S_{i+1} = S_i^\sigma.$$

As a consequence of Cor. 2.1 we have:

**Corollary 3.1.**

*The two internal centers  $S_i, S_j$  of similarity of  $\Gamma$  and  $\Gamma_i, \Gamma_j$  are collinear with the external center  $S_{ij}$  of similarity of  $\Gamma_i$  and  $\Gamma_j$  for  $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$ .*

*Proof.* The collinearity is easily checked by showing the linear dependency of the respective coordinate vectors.  $\square$

The centers  $T_i$  of equioptic circles  $e_i$  of  $\Gamma$  and  $\Gamma_i$  are the midpoints of  $\Delta$ 's  $i$ -th vertex and  $S_i$ . Thus we have

$$T_1 = (b + c : a : a), \quad T_{i+1} = T_i^{\sigma\zeta}.$$

Now we observe the following phenomenon:

**Corollary 3.2.**

*The two centers  $T_i$  and  $T_j$  of equioptic circles of  $\Gamma$  and the  $i$ -th and  $j$ -th excircle are collinear with the center  $T_{ij}$  of the equioptic circle  $e_{ij}$  of  $\Gamma_i$  and  $\Gamma_j$ .*

*Proof.* This is easily verified using the trilinear representation of all the involved points.  $\square$

Again we use the formulae given in [1, p. 223] in order to compute the equations of the equioptic circles  $e_i$  of the incircle  $\Gamma$  and the  $i$ -th excircle  $\Gamma_i$  and arrive at

$$e_1 : x^\top \cdot \begin{bmatrix} 2a^2s(a-s) & 2abs(a-s) & 2acs(a-s) \\ 2abs(a-s) & (\star\star) & (\star) \\ 2acs(a-s) & (\star) & (\star\star\star) \end{bmatrix} \cdot x = 0, \quad (7)$$

$$e_{i+1} = \zeta(e_i),$$



where

$$\begin{aligned} (\star) &= 2b^2c^2 + 3b^3c + 3bc^3 - 3a^2bc + 2b^4 + 2c^4 - 2a^2b^2 - 2a^2c^2, \\ (\star\star) &= b(\widehat{a}^2(4c - b) + a^2b - 8b^2c), \\ (\star\star\star) &= c(2bc^2 + 7b^2c - c^3 + a^2c + 4b^3). \end{aligned}$$

Here  $s = (a + b + c)/2$  is the semiperimeter of  $\Delta$ .

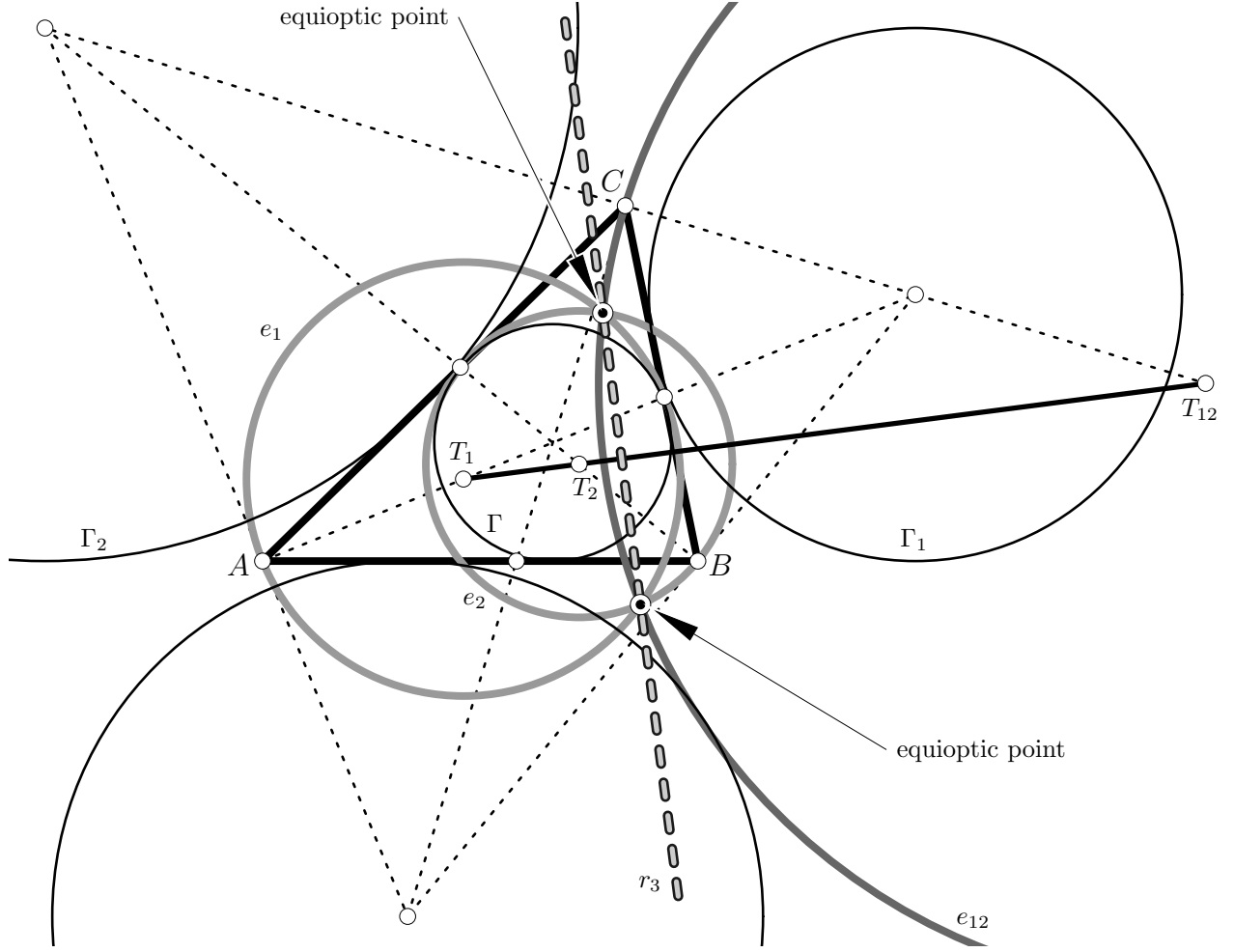


Figure 8: Equioptic circles and points of  $\Gamma$ ,  $\Gamma_1$ , and  $\Gamma_2$  of the incircle and the three excircles.

Now we can show:

**Theorem 3.1.**

*The equioptic circles  $e_i$ ,  $e_j$ , and  $e_{ij}$  defined by the incircle  $\Gamma$  and the excircles  $\Gamma_i$ ,  $\Gamma_j$  have a common radical axis  $r_k$  (with  $(i, j, k) \in \mathbb{I}^3$ ) and thus  $\Gamma$ ,  $\Gamma_i$ , and  $\Gamma_j$  have up to two real equioptic points.*

*Proof.* With Eq. (3) and (7) we compute the radical axis  $r_k$  of  $\Gamma$ ,  $\Gamma_i$ , and  $\Gamma_j$  (where  $(i, j, k) \in \mathbb{I}^3$ ) as the singular conic sections in the pencil of conics spanned by either two circles, cf. the proof of Th. 2.1. The radical axis  $r_3$  is given by

$$\begin{aligned} r_3 = & [-ba^5 - (\widehat{a}^2 + 2bc)a^4 + (\widehat{a}^2b + c(2\widehat{a}^2 - bc))a^3 + \widehat{a}^2(\widehat{a}^2 + 6c^2)a^2 + c^2\widehat{a}^2(b + 4c) : \\ & : ab^5 + 2(\widehat{b}^2 + 2ac)b^4 - (\widehat{a}\widehat{b}^2 + c(2\widehat{b}^2 - ac))b^3 - \widehat{b}^2(\widehat{b}^2 + 6c^2)b^2 - c^2\widehat{b}^2(a + 4c)\widehat{b} : \\ & : (b - a)c^5 + 4(a - b)\widehat{c}c^4 + (a - b)(7\widehat{c}^3 - 3ab)c^3 + 4(a - b)\widehat{c}(\widehat{c}^2 + ab)c^2 \\ & \quad + 7ab(a - b)\widehat{c}^2c + 4a^2b^2(a - b)\widehat{c}]. \end{aligned} \quad (8)$$

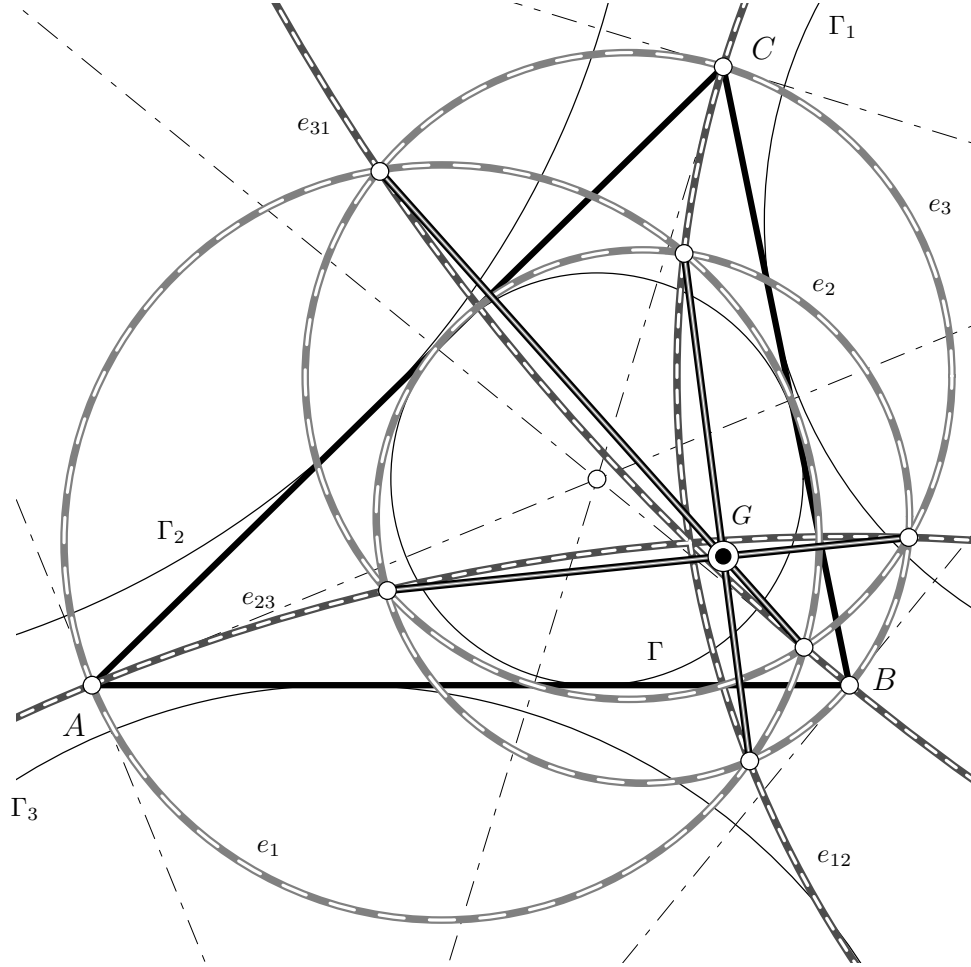


Figure 9: The six equioptic circles of the incircle and the excircle, the three concurrent radical axes, and the center  $G$  from Th. 3.2.

Finally we have  $r_1 = r_3^{\sigma\zeta}$  and  $r_2 = r_1^{\sigma\zeta}$ . □

A certain triplet of equioptic circles is shown in Fig. 8. Now we are able to state and prove:

**Theorem 3.2.**

*The three radical axes  $r_k$  (cf. Th. 3.1) are concurrent in a triangle center.*

*Proof.* The homogeneous coordinate vectors of the lines  $r_i$  given in (8) are linearly dependent. This proves the concurrency.

We compute the intersection  $G = (g_0 : g_1 : g_2)$  of any pair  $(r_i, r_j)$  of radical axes and find

$$\begin{aligned}
 g_0 &= bc\hat{a}^5(b-c)^2 + 2bc\hat{a}^2(2\hat{a}^4 - 10\hat{a}^2bc + 5b^2c^2)a \\
 &+ \hat{a}^3(\hat{a}^4 - 8\hat{a}^2bc + 4b^2c^2)a^2 - 2(\hat{a}^6 + 3bc\hat{a}^4 - 10b^2c^2\hat{a}^2 + b^3c^3)a \\
 &- \hat{a}(8\hat{a}^4 - 23bc\hat{a}^2 + 4b^2c^2)a^4 - 2(\hat{a}^4 - 8bc\hat{a}^2 + 5b^2c^2)a^5 + \hat{a}(7\hat{a}^2 - 4bc)a^6 + 2(2\hat{a}^2 - bc)a^7.
 \end{aligned} \tag{9}$$

Since  $g_1 = \zeta(g_0)$  and  $g_2 = \zeta(g_1)$  we find that  $G$  is a center of  $\Delta$  which is not mentioned in [2]. □

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