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Equioptic Points of a Triangle

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Abstract. The locus of points where two non-concentric circles c_1 and c_2 are seen under equal angles is the equioptic circle e. The equioptic circles of the excircles of a triangle Δ have a common radical axis r. Therefore the excircles of a triangle share up to two real points, i.e., the equioptic points of Δ from which the circles can be seen under equal angles. The line r carries a lot of known triangle centers. Further we find that any triplet of circles tangent to the sides of Δ has up to two real equioptic points. The three radical axes of triplets of circles containing the incircle are concurrent in a new triangle center.

Key Words: Triangle, excircle, incircle, equioptic circle, equioptic points, center of similarity, radical axis.

MSC 2010: 51M04

1. Introduction

Let there be given a triangle Δ with vertices A, B, and C. The incenter shall be denoted by I, the incircle by Γ. The excenters are labeled with I_1, I_2, and I_3. We assume that I_1 is opposite to A, i.e., it is the center of the excircle Γ_1 touching the line $[B, C]$ from the outside of Δ (cf. Fig. 1). Sometimes it is convenient to number vertices as well as sides of Δ: The side (lines) $[B, C]$, $[C, A]$, $[A, B]$ shall be the first, second, third side (line) and A, B, C shall be the first, second, third vertex, respectively. According to [1, 2] we denote the centers of Δ with X_i. For example the incenter I is labeled with X_1.

The set of points where two curves can be seen under equal angles is called *equioptic curve*, see [3]. It is shown that any pair (c_1, c_2) of non-concentric circles has a circle e for its equioptic curve [3]. The circle e is the Thales circle of the segment bounded by the internal and external centers of similarity of either given circle, i.e., the center of e is the midpoint of the two centers of similarity (see Fig. 1). In case of two congruent circles e becomes the bisector of the centers of c_1 and c_2, provided that c_1 and c_2 are not concentric.

The four circles Γ, Γ_i (with $i \in \{1, 2, 3\}$) tangent to the sides of a triangle Δ can be arranged in six pairs and, thus, they define six equioptic circles. Among them we find four triplets of equioptic circles which have a common radical axis instead of a radical center, i.e.,

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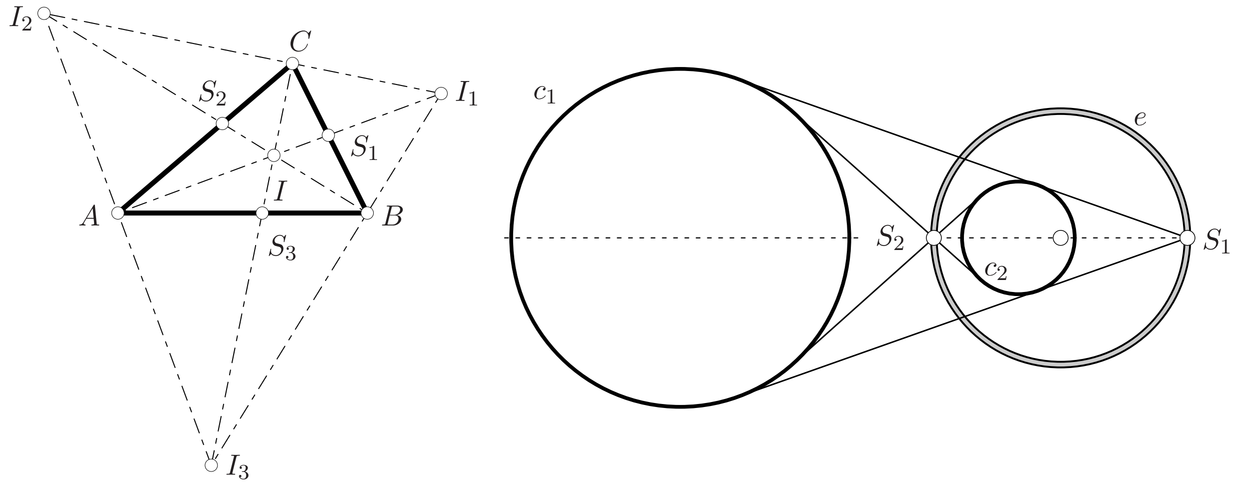

Figure 1: Left: Notations in and around the triangle Δ. Right: Equioptic circle of two circles.

the three circles of such a triplet form a pencil of circles. These shall be the contents of Sections 2 and 3.

We use homogeneous trilinear coordinates of points and lines, respectively. The homogeneous triplet of real (complex) numbers $(x_0 : x_1 : x_2)$ are said to be the homogeneous trilinear coordinates of a point X if x_i are the oriented distances of X with respect to the sides $[B, C]$, $[C, A]$, and $[A, B]$ up to a common non vanishing factor, see [1]. When we deal with trilinear coordinates of points expressed in terms of homogeneous polynomials in Δ's side lengths $a = \overline{BC}$, $b = \overline{CA}$, and $c = \overline{AB}$ we use a function ζ with the property $\zeta(f(a, b, c)) = f(b, c, a)$ and further a function σ with $\sigma(x_0 : x_1 : x_2) = (x_2 : x_0 : x_1)$.

In this paper mappings will be written as superscripts, *e.g.*, $\sigma \circ \zeta(X) = X^{\sigma\circ\zeta} = X^{\zeta\sigma}$ if applied to points. Note that $\zeta \circ \sigma = \sigma \circ \zeta$, provided that x_i are homogeneous functions in a, b, c.

2. Equioptic circles of the excircles

In order to construct the equioptic circles of a pair of excircles we determine the respective centers of similarity. First we observe that the internal centers of similarity of Γ_i and Γ_j is the k-th vertex of Δ, where $(i, j, k) \in \mathbb{I}^3 := \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Second we have to find the external centers of similarity. For any pair (Γ_i, Γ_j) the k-th side of Δ is an exterior common tangent of both Γ_i and Γ_j, respectively, and thus $[I_i, I_j]$ and the k-th side of Δ meet in the external center of similarity S_{ij} of Γ_i and Γ_j. Now we are able to show a first result:

Corollary 1. *The external centers of similarity S_{ij} of the excircles Γ_i and Γ_j of a triangle Δ are collinear. The line carrying these points is the polar of X_1 with respect to Δ and the polar line with respect to the excentral triangle of Δ at the same time.*

Proof: We construct the polar line of the incenter X_1 with respect to Δ. For that purpose we project I from C to the line $[A, B]$. This gives $S_3 := [A, B] \cap [I, C]$. Then we determine a fourth point C' on $[A, B]$ such that (A, B, S_3, C') is a harmonic quadrupel. The four lines $[C, A]$, $[C, B]$, $[C, I]$, and $[I_1, I_2]$ obviously form a harmonic quadrupel and thus any line (which is not passing through C) meets these four lines in four points of a harmonic quadrupel. So we have $C' = [I_1, I_2] \cap [A, B]$ and obviously $C' = S_{12}$. By cyclical reordering of labels of points and numbers we find S_{23} and S_{31} which are collinear with S_{12} and gather on the polar of X_1. On the other hand we have the harmonic quadruples (I_1, I_2, C, S_{12}) (cyclic) which shows that $[S_{12}, S_{23}]$ is the polar of X_1 with respect to the excentral triangle. □

The centers T_{ij} of the equioptic circles e_{ij} are the midpoints of the line segments bounded by S_{ij} and the k-th vertex of Δ with $(i,j,k) \in \mathbb{I}^3$. In terms of homogeneous trilinear coordinates we have

$$S_{12} = (-1 : 1 : 0), \quad S_{i+1j+1} = S_{ij}^{\sigma}.$$

The centers T_{ij} are thus

$$T_{12} = (c : -c : a-b), \quad T_{i+1,j+1} = T_{ij}^{\sigma\zeta} \tag{1}$$

and we can easily prove:

Corollary 2. *The centers T_{ij} of the equioptic circles e_{ij} of any pair (Γ_i, Γ_j) of excircles of Δ are collinear.*

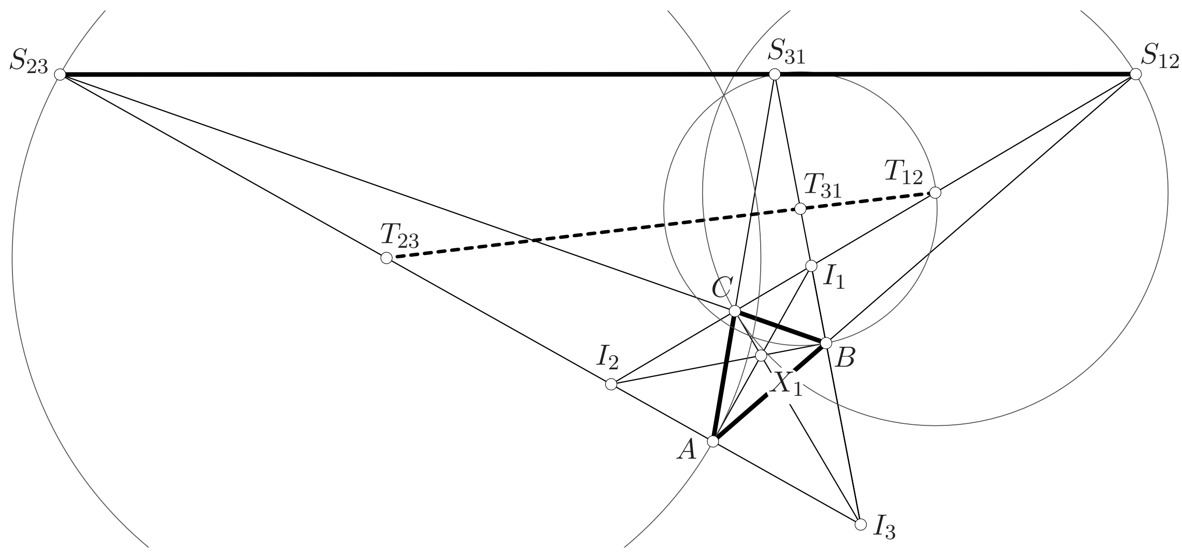

Figure 2: Centers of similarity S_{ij}, the centers of the equioptic circles T_{ij}, and harmonic quadruples.

Proof: The coordinate vectors of T_{ij} given in (1) are linearly dependent. □

Remark 1. The line t connecting any T_{ij} with any T_{jk} (with $(i,j,k) \in \mathbb{I}^3$) has trilinear coordinates $[\lambda_0 : \lambda_1 : \lambda_2]$, where $\lambda_0 = a(-a+b+c)$ and $\lambda_i = \zeta^i(\lambda_0)$ with $i \in \{0,1,2\}$. Obviously t is the polar of X_{55} with respect to Δ. The center X_{55} is the center of homothety of the tangential triangle, the intangent triangle, and the extangent triangle, see [1]. Further it is the internal center of similarity of the incircle and the circumcircle of Δ.

We use the formula for the distance of two points given by their *actual trilinear coordinates* given in [1, p. 31] and compute the radii ρ_{ij} of the equioptic circles c_{ij} and find

$$\rho_{12} = \operatorname{dist}(C, T_{12}) = \operatorname{dist}(S_{12}, T_{12}) = \frac{ab}{a-b} \sin\frac{C}{2}, \qquad \rho_{i+1j+1} = \zeta(\rho_{ij}). \tag{2}$$

By means of the distance formula from [1, p. 31] or equivalently by means of the more complicated equation for a circle given by center and radius from [1, p. 223] we write down the equations of the equioptic circles

$$e_{12}: \quad -\,\mathrm{c_A}\,x_0^2 + \mathrm{c_B}\,x_1^2 + (1-\mathrm{c_C})x_2(x_1-x_0) + (\mathrm{c_B}-\mathrm{c_A})x_0x_1 = 0, \quad e_{i+1j+1} = \zeta(e_{ij}), \tag{3}$$

where $\mathrm{c_A}$, $\mathrm{c_B}$, and $\mathrm{c_C}$ are shorthand for $\cos A$, $\cos B$, and $\cos C$, respectively. Now it is easy to verify that the following holds true:

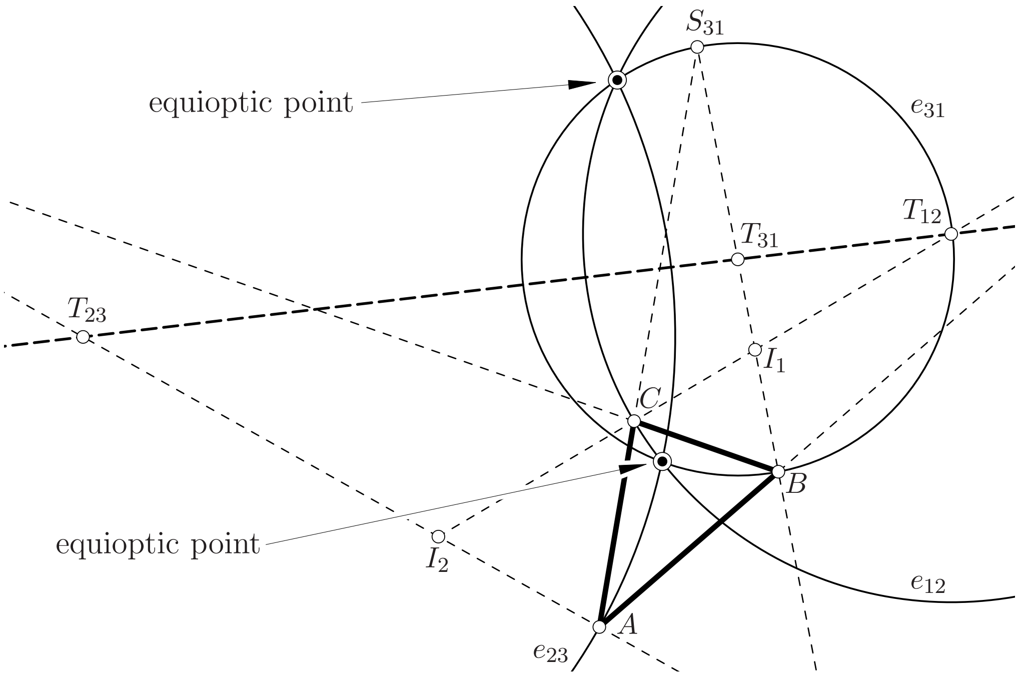

Figure 3: The equioptic circles and equioptic points of a triangle.

Theorem 3.

1. *The three equioptic circles of the excircles of a generic triangle* Δ *have a common radical axis* r *and thus they have up to two common real points, i.e., the equioptic points of the excircles from which the excircles can be seen under equal angles.*
2. *The radical axis* r *contains* X_4 *(ortho center),* X_9 *(Mittenpunkt),* X_{10} *(Spieker center), and further* X_i *with*

$$\begin{aligned} i \in \{&19, 40, 71, 169, 242, 281, 516, 573, 966, 1276, 1277, 1512,\\ &1542, 1544, 1753, 1766, 1826, 1839, 1842, 1855, 1861, 1869,\\ &1890, 2183, 2270, 2333, 2345, 2354, 2550, 2551, 3496, 3501\}. \end{aligned} \tag{4}$$

Proof:

1. Let $P_1 := \mu e_{12} + \nu e_{23}$, $P_2 := \mu e_{23} + \nu e_{31}$, and $P_3 := \mu\, e_{31} + \nu e_{12}$ be the equations of the conic sections in the pencils spanned by any pair of equioptic circles e_{ij}. We compute the singular conic sections in the pencils and find that for all pencils the real singular conic sections consist of the ideal line $\omega:\ ax_0 + bx_1 + cx_2$ and the line

$$(b-c)\,\mathrm{c_A}\,x_0 + (c-a)\,\mathrm{c_B}\,x_1 + (a-b)\,\mathrm{c_C}\,x_2 = 0, \tag{5}$$

which is the radical axis of these three equioptic circles.

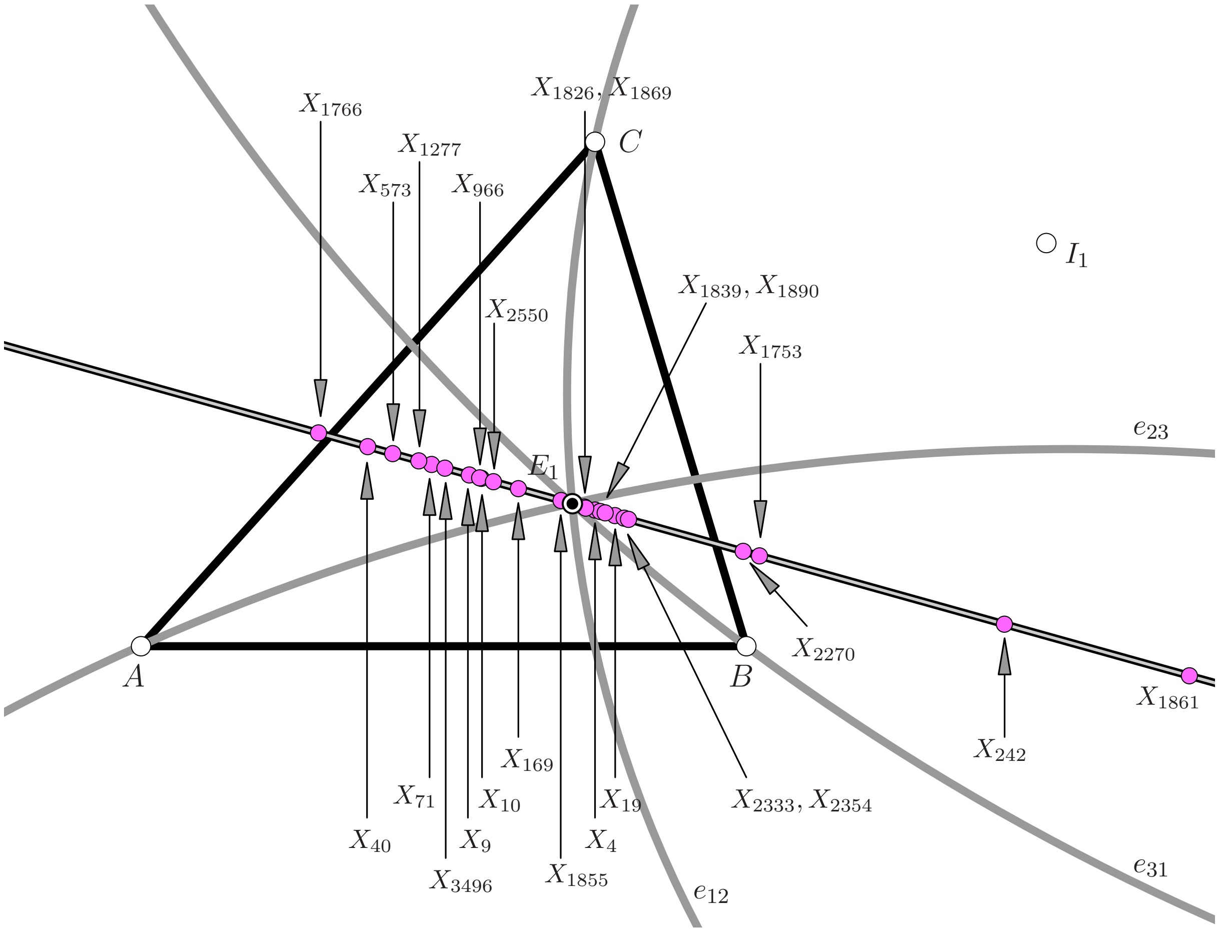

Figure 4: The line $[X_4, X_9]$ and a bunch of triangle centers on it. Some of the centers mentioned in Theorem 3 are not depicted for they are far out and especially X_{516} is the ideal point of the line $[X_4, X_9]$. The base triangle is acute.

2. In [1, pp. 64 ff.] we find $X_4 = [\operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C]$ and $X_9 = [b + c - a, c + a - b, a + b - c]$ and obviously these coordinate vectors annihilate Eq. (5). By inserting the trilinears of the other points mentioned in the theorem we proof the incidence. The trilinears of points X_i with $i \leq 360$ can be found in [1] whereas the trilinears for $i > 360$ can be found in [2].

□

In Fig. 3 the equioptic circles as well as the equioptic points of an acute triangle are depicted. Figure 4 shows some of the centers mentioned in Theorem 3 located on r. Here, the base triangle is acute. Figure 5 shows the centers on the line $r = [X_4, X_9]$ for an obtuse base triangle.

Remark 2. The circles in the pencil of circles spanned by the equioptic circles can share two real points, one real point with multiplicity two, or no real points. Thus a triangle has either two real equioptic points (cf. Fig. 4 or Fig. 8), or a single equioptic point (cf. Fig. 7), or no real equioptic point as is the case in Fig. 5.

In case of an equilateral triangle Δ there is only one equioptic point E that coincides with the center of Δ. The three equioptic circles become straight lines: e_{ij} is the k-th interior angle bisector and the k-th altitude of Δ. Figure 6 illustrates this case. From E any excircle

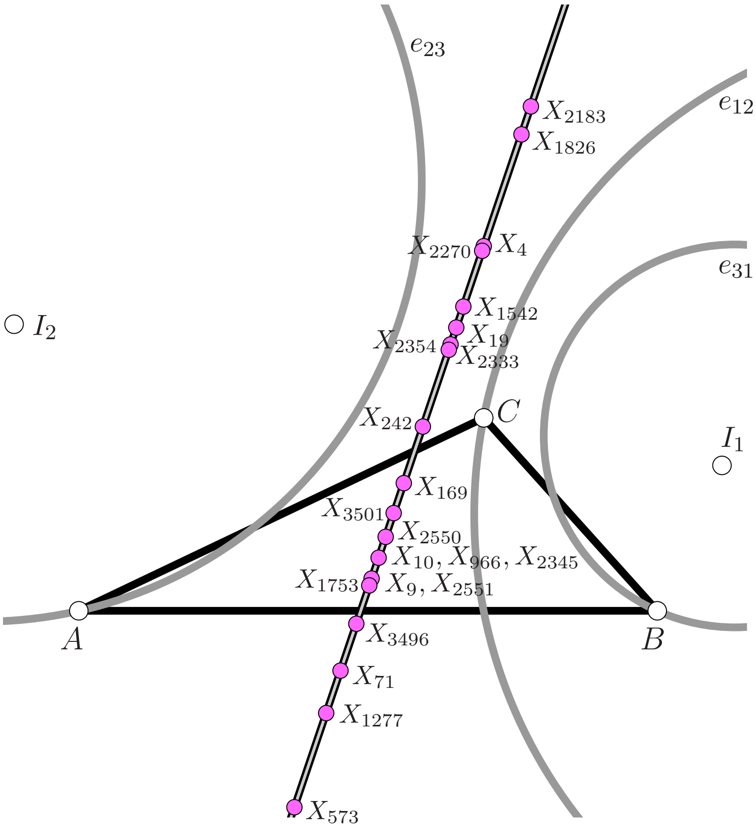

Figure 5: The line $[X_4, X_9]$ and a bunch of triangle centers on it. Some of the centers mentioned in Theorem 3 are not depicted for they are far out and especially X_{516} is the ideal point of the line $[X_4, X_9]$. The base triangle is obtuse.

i_j can be seen under the angle $\arccos\left(-\frac{1}{8}\right) \approx 97.180756°$.

The case of a single equioptic point is illustrated in Fig. 7 at hand of an isosceles triangle. It is easily shown that in this case one has to choose $\angle ACB = 2\arcsin(\sqrt{3}-1) \approx 94.11719432°$ in order to have a unique equioptic point. Thus the triangle is obtuse. The unique (real) equioptic angle now equals $2\arcsin\left(\frac{3-\sqrt{3}}{2}\right) \approx 78.68794716°$.

Here and in the following we use the abbreviations $\widehat{a} = b + c$, $\widehat{b} = c + a$, and $\widehat{c} = a + b$. We can show:

Theorem 4. *A generic triangle Δ has a unique equioptic point if and only if*

$$\sum_{cyclic} \left(a^6 + 2\widehat{a}a^5 - a^4(\widehat{a}^2 + 4bc) - 4a^3b(\widehat{a}^2 - 3bc) - 2a^2b^2c^2\right) = 0. \tag{6}$$

Proof: We have already shown that the three equioptic circles have a common radical axis r. Furthermore, if one of these three circles touches the common radical axis at a point E, then any other circle touches precisely at E. This is caused by the fact that E has zero power with respect to the circle touching r and, as a point of the common radical axis r, it has to have the same power with respect to the other circles. Thus it is sufficient to derive a contact condition of r and one of the equioptic circles. Therefore, we compute the resultant of the equation of one circle as given in Eq. (3) and the radical axis from Eq. (5) with respect to one variable, say x_2. This yields a quadratic form $q(x_0, x_1)$. Now the condition on a, b, c in

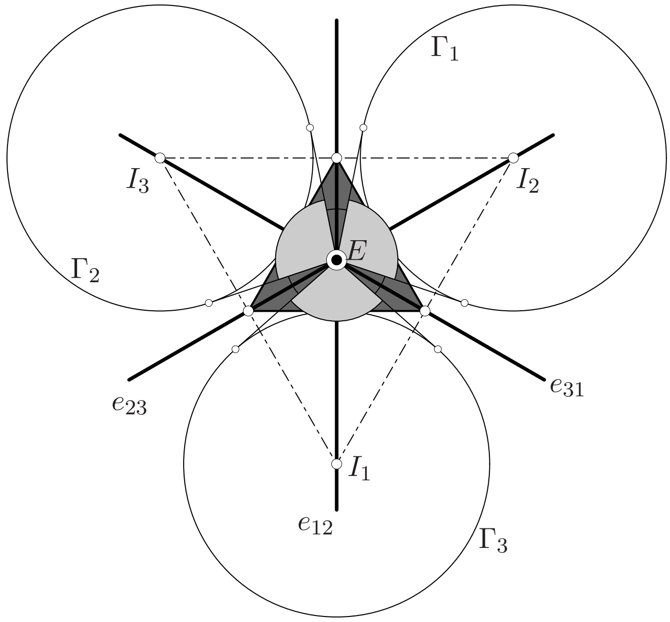

Figure 6: The only equioptic point of an equilateral triangle.

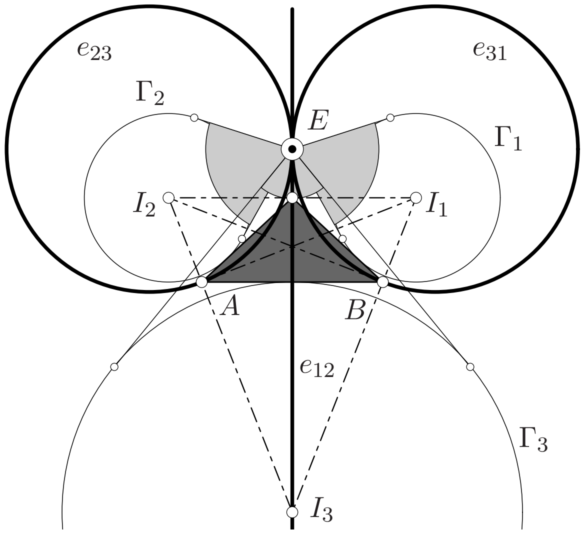

Figure 7: An iscosceles triangle with a unique equioptic point.

order to make r a tangent of any equioptic circle is that the quadractic form q factors which is equivalent to $\det(H_q) = 0$ where H_q is the Hessian of q. We find that H_q is the sextic form given in Eq. (6). □

Remark 3. The case of an equilateral triangle is not covered by Eq. (6). In the case of an equilateral triangle the points T_{ij} are ideal points and thus there are no points S_{ij}. The equioptic circles of the excircles of an equilateral triangle degenerate and become the altitudes of the triangle and the one and only equioptic point is the one and only center of the equilateral triangle as can seen in Fig. 6.

3. Equioptic circles of the incircle and an excircle

We recall that the equioptic circles of a pair (Γ, Γ_i) is the Thales circle of the line segment bounded by the internal and external center of similarity of the incircle Γ and the i-th excircle Γ_i. We observe that the i-th vertex of Δ is the external center of similarity of the above given pair of circles. The internal center is the intersection of a common internal tangent, i.e, Δ's i-th side and the line $[I, I_i]$ connecting the respective centers. Consequently the internal centers of similarity are the points S_i ($i \in \{1,2,3\}$). We have

$$S_1 = (0:1:1), \quad S_{i+1} = S_i^\sigma.$$

As a consequence of Corollary 1 we have:

Corollary 5. *The two internal centers S_i, S_j of similarity of Γ and Γ_i, Γ_j are collinear with the external center S_{ij} of similarity of Γ_i and Γ_j for $(i,j) \in \{(1,2),(2,3),(3,1)\}$.*

Proof: The collinearity is easily checked by showing the linear dependency of the respective coordinate vectors. □

The centers T_i of equioptic circles e_i of Γ and Γ_i are the midpoints of Δ's i-th vertex and S_i. Thus we have

$$T_1 = (b+c:a:a), \quad T_{i+1} = T_i^{\sigma\zeta}.$$

Now we observe the following phenomenon:

Corollary 6. *The two centers T_i and T_j of equioptic circles of Γ and the i-th and j-th excircle are collinear with the center T_{ij} of the equioptic circle e_{ij} of Γ_i and Γ_j.*

Proof: This is easily veryfied using the trilinear representation of all the involved points. □

Again we use the formulae given in [1, p. 223] in order to compute the equations of the equioptic circles e_i of the incircle Γ and the i-th excircle Γ_i and arrive at

$$e_1\colon\ x^{\mathrm{T}} \cdot \begin{bmatrix} 2a^2s(a-s) & 2abs(a-s) & 2acs(a-s) \\ 2abs(a-s) & (\star\star) & (\star) \\ 2acs(a-s) & (\star) & (\star\star\star) \end{bmatrix} \cdot x = 0, \qquad e_{i+1} = \zeta(e_i), \tag{7}$$

where

$$\begin{aligned} (\star) &= 2b^2c^2 + 3b^3c + 3bc^3 - 3a^2bc + 2b^4 + 2c^4 - 2a^2b^2 - 2a^2c^2, \\ (\star\star) &= b(\widehat{a}^2(4c-b) + a^2b - 8b^2c), \\ (\star\star\star) &= c(2bc^2 + 7b^2c - c^3 + a^2c + 4b^3). \end{aligned}$$

Here $s = (a+b+c)/2$ is the semiperimeter of Δ.

Now we can show:

Theorem 7. *The equioptic circles e_i, e_j, and e_{ij} defined by the incircle Γ and the excircles Γ_i, Γ_j have a common radical axis r_k (with $(i,j,k) \in \mathbb{I}^3$) and thus Γ, Γ_i, and Γ_j have up to two real equioptic points.*

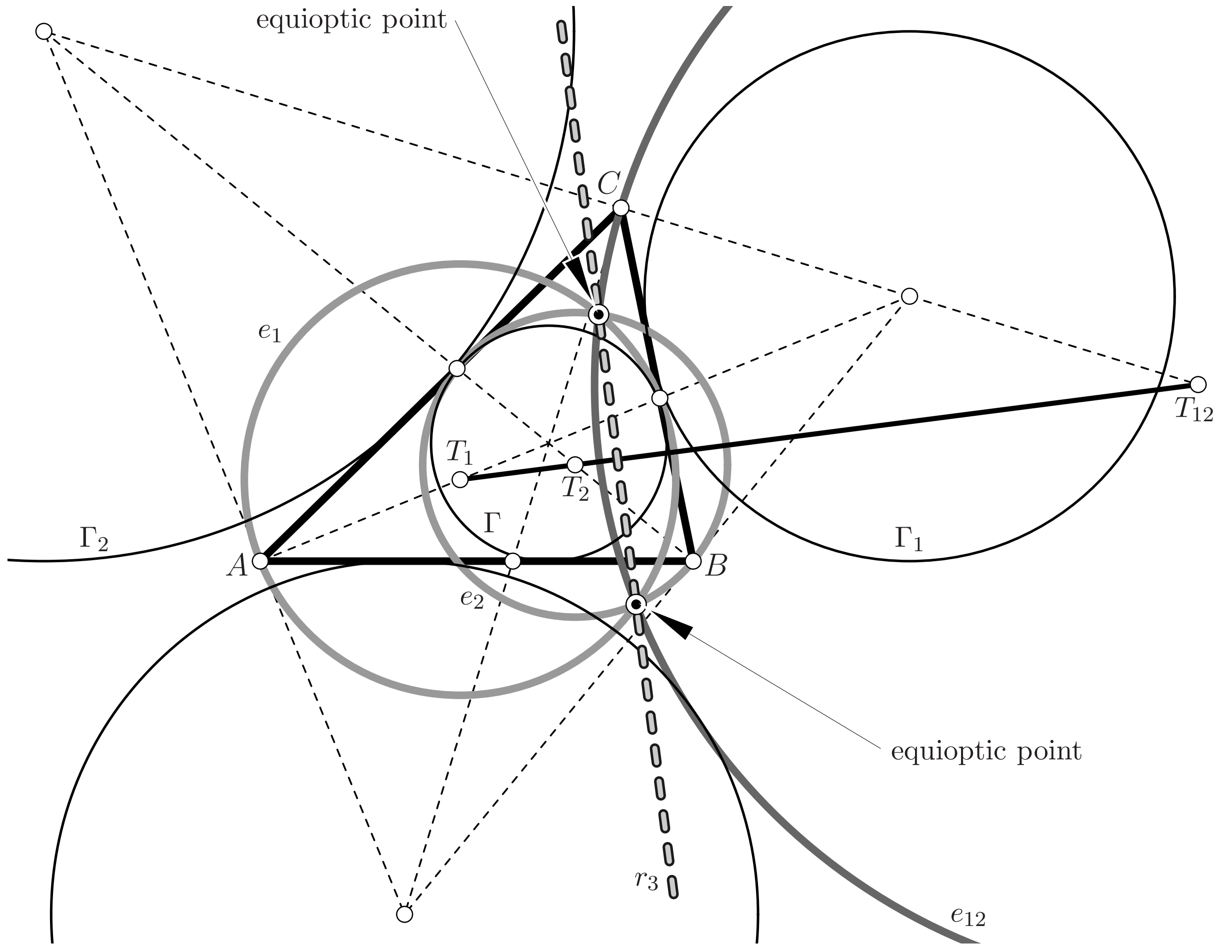

Figure 8: Equioptic circles and points of Γ, Γ_1, and Γ_2 of the incircle and the three excircles.

Proof: With Eq. (3) and (7) we compute the radical axis r_k of Γ, Γ_i, and Γ_j (where $(i,j,k) \in \mathbb{I}^3$) as the singular conic sections in the pencil of conics spanned by either two circles, cf. the proof of Theorem 3. The radical axis r_3 is given by

$$\begin{aligned} r_3 = [&-ba^5 - (\widehat{a}^2 + 2bc)a^4 + (\widehat{a}^2 b + c(2\widehat{a}^2 - bc))a^3 + \widehat{a}^2(\widehat{a}^2 + 6c^2)a^2 + c^2\widehat{a}^2(b+4c) : \\ &: ab^5 + 2(\widehat{b}^2 + 2ac)b^4 - (a\widehat{b}^2 + c(2\widehat{b}^2 - ac))b^3 - \widehat{b}^2(\widehat{b}^2 + 6c^2)b^2 - c^2\widehat{b}^2(a+4c)\widehat{b} : \\ &: (b-a)c^5 + 4(a-b)\widehat{c}c^4 + (a-b)(7\widehat{c}^3 - 3ab)c^3 + 4(a-b)\widehat{c}(\widehat{c}^2 + ab)c^2 \\ &+7ab(a-b)\widehat{c}^2 c + 4a^2b^2(a-b)\widehat{c})]. \end{aligned} \tag{8}$$

Finally we have $r_1 = r_3^{\sigma\zeta}$ and $r_2 = r_1^{\sigma\zeta}$. □

A certain triplet of equioptic circles is shown in Fig. 8. Now we are able to state and prove:

Theorem 8. *The three radical axes r_k (cf. Theorem 7) are concurrent in a triangle center.*

Proof: The homogeneous coordinate vectors of the lines r_i given in (8) are linearly dependent. This proves the concurrency.

We compute the intersection $G = (g_0 : g_1 : g_2)$ of any pair (r_i, r_j) of radical axes and find

$$\begin{aligned} g_0 = {}& bc\widehat{a}^5(b-c)^2 + 2bc\widehat{a}^2(2\widehat{a}^4 - 10\widehat{a}^2bc + 5b^2c^2)a \\ &+\widehat{a}^3(\widehat{a}^4 - 8\widehat{a}^2bc + 4b^2c^2)a^2 - 2(\widehat{a}^6 + 3bc\widehat{a}^4 - 10b^2c^2\widehat{a}^2 + b^3c^3)a \\ &-\widehat{a}(8\widehat{a}^4 - 23bc\widehat{a}^2 + 4b^2c^2)a^4 - 2(\widehat{a}^4 - 8bc\widehat{a}^2 + 5b^2c^2)a^5 + \widehat{a}(7\widehat{a}^2 - 4bc)a^6 + 2(2\widehat{a}^2 - bc)a^7. \end{aligned} \tag{9}$$

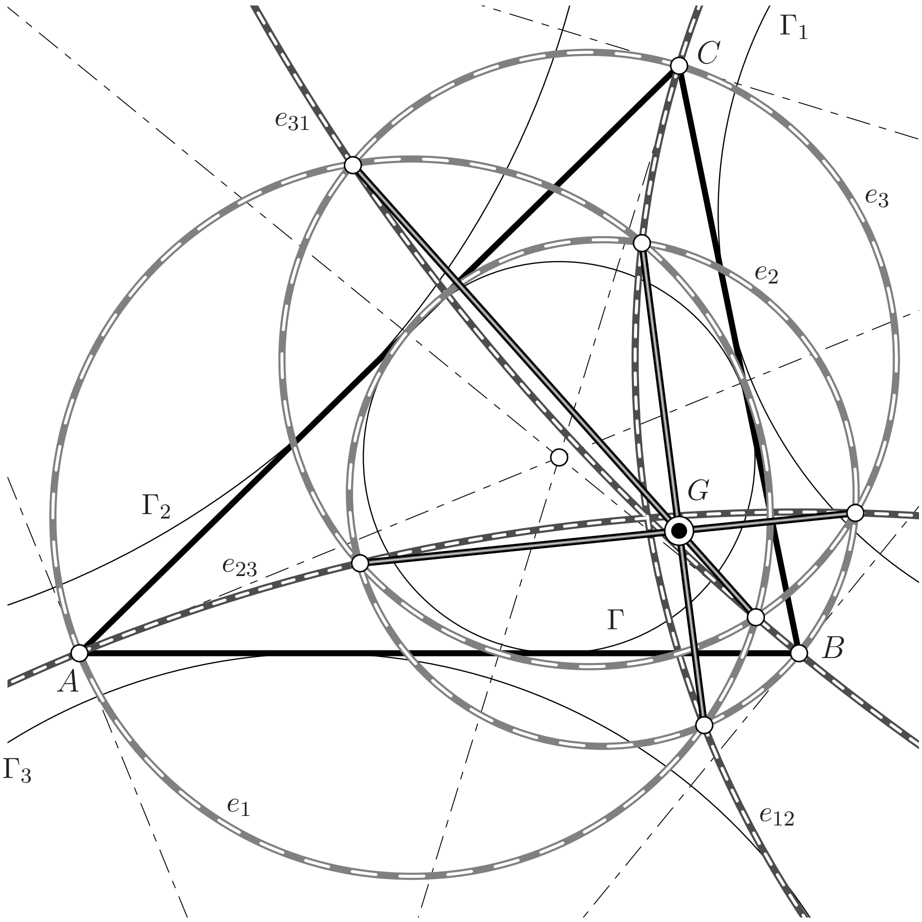

Figure 9: The six equioptic circles of the incircle and the excircle, the three concurrent radical axes, and the center G from Theorem 8.

Since $g_1{=}\zeta(g_0)$ and $g_2{=}\zeta(g_1)$ we find that G is a center of Δ which is not mentioned in [2]. □

References

[1] C. Kimberling: *Triangle Centers and Central Triangles*. Congressus Numerantium, 129, Winnipeg/Canada 1998.

[2] C. Kimberling: *Encylcopedia of Triangle Centers and Central Triangles*. Available at `http://faculty.evansville.edu/ck6/encyclopedia/ECT.html`.

[3] B. Odehnal: *Equioptic curves of conic sections*. J. Geometry Graphics **14**/1, 29–43 (2010).

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