Pencils of Frégier conics

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Abstract

For each point P on a conic c, the involution of right angles at P induces an elliptic involution on c whose center F is called the Frégier point of P. Replacing the right angles at P between assigned pairs of lines with an arbitrary angle ϕ yields a projective mapping of lines in the pencil about P, and thus, on c. The lines joining corresponding points on c do no longer pass through a single point and envelop a conic f which can be seen as the generalization of the Frégier point and shall be called a generalized Frégier conic. By varying the angle, we obtain a pencil of generalized Frégier conics which is a pencil of the third kind. We shall study the thus defined conics and discover, among other objects, general Poncelet triangle families.

Keywords: conic, angle, projective mapping, Frégier point, Frégier conic, Poncelet porism, envelope.

MSC (AMS 2020): 51M04, 51N15, 14H50

1 Introduction

1.1 Known results, contributions of the present paper

FRÉGIER's theorem in its original form says that *the chords of a conic c which are seen from a point $P \in c$ under a right angle pass through one point F* (cf. [1, 6, 7] and see Fig. 1). The point F is usually called the *Frégier point* of P. If P moves along c, then

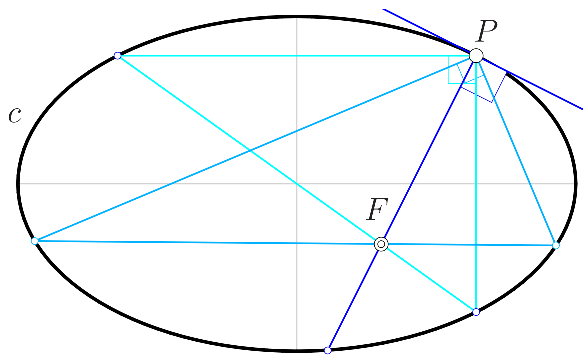

Figure 1: The Frégier point F is the center of the involution on c that is induced by the involution of right angles at P.

F traces a conic f (see Fig. 2) homothetic to c with similarity factor $(a^2 - b^2)/(a^2 + b^2)$ (in the case of a non-circular ellipse, i.e., $a \neq b$) or $(a^2 + b^2)/(a^2 - b^2)$ (in the case of a non-equilateral hyperbola, i.e., $a \neq b$), where a and b are the semi-major and semi-minor axes lengths. For a parabola c, the conic f is even congruent to c. The conic f is sometimes called *Frégier conic* (see [7, 14]). However, the Frégier conic f and c are always of the same affine type.

According to [8, 13], a conic-shaped *generalized offset* to a conic c with center (ellipse or hyperbola) can only be found by applying a multiple of the cube root of the curvature radius ρ at P on c's normal at P in order to find the corresponding point P' of the generalized offset. In [13] it is shown that the distance function

$$k\sqrt[3]{\rho(t)}$$

is unique up to a constant $k \in \mathbb{R}$. The case of a parabola differs slightly, i.e., the distance function is no longer unique. Surprising enough, until now it is obviously not recognized what is illustrated in Fig. 2:

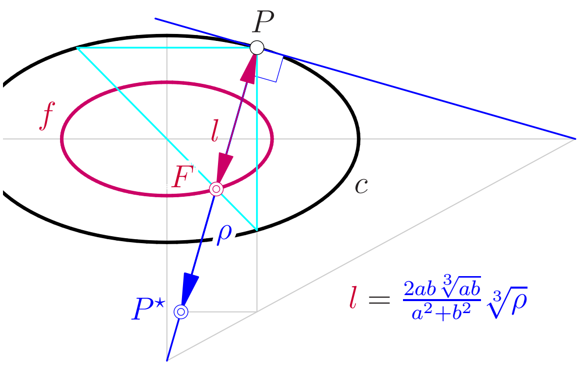

Figure 2: The Frégier conic f is a generalized offset of the conic c.

Theorem 1.1. *Frégier conics are conic-shaped generalized offsets (in the sense of* [8] *and* [13]*).*

Proof. We first recall that the Frégier point of a point $P \in c$ lies on c's normal at P. Let ρ denote the radius of curvature of c at P and let l denote the distance between P and its Frégier point, then it is elementary to verify that ρ and l are bound to

$$8a^4b^4\rho = (a^2 \pm b^2)^3 l^3,$$

where the plus stands for the ellipse and the minus for the hyperbola. Hence, the offset distance equals a multiple of the cube root of the curvature radius ρ in both cases. For the parabola $x^2 = 2qy$ ($q \neq 0$) we find

$$8q^2\rho = l^3. \qquad \square$$

FRÉGIER's theorem can be considered a result of Euclidean geometry, for it involves right angles, or a result of projective geometry, since the Frégier point F of a point P on a conic c is the center of the involution of right angles in the pencil about P projected onto c, see [7].

Variants of FRÉGIER's theorem in higher dimensions do exist (see, e.g., [9, 17]). Further, connections to linear 2-parameter and 3-parameter families of conics are studied in [11]. FRÉGIER's theorem is also studied in relation to quadratic mappings recently in [17] and even earlier in [15].

In [16], the authors define a *Frégier involution* using right angles in Euclidean and non-Euclidean sense which gives rise to a possible generalization of FRÉGIER's theorem also in higher dimensions, but completely different from the approach made in [9]. Conics in non-Euclidean planes with singular Frégier conics are studied in [14]. Many relations of the Frégier point and FRÉGIER's theorem in Euclidean geometry to various construction tasks in connections with conics were disclosed, see [2, 3, 5, 12], to name just a few.

In this article, we replace the right angle which is usually the main ingredient of FRÉGIER's theorem by a different Euclidean angle $\phi \neq 0, \frac{\pi}{2}$ and study the chords cut out of c by the legs of the rotating rigid angle (with vertex P on c). Since the mapping that assigns to each line g the rotated copy g' with the fixed angle $\phi = \sphericalangle\, g, g'$ is a projectivity, we first show that the chords of c that join pairs (Q, Q') of assigned points envelope a conic. This does not depend on the affine type of c. These envelopes are then called *generalized Frégier conics*.

Although, we have this rather general result, the equations of the generalized Frégier conics of the three different affine types of base conics c have to be elaborated separately. We will find that the generalized Frégier conics of a point $P \in c$ belong to a pencil of conics (of the third kind) which also contains the initial conic c and the Frégier point F as a limiting case. Further, the algebraic proofs of the results yield computational artifacts that allow for a geometric interpretation and give rise to general Poncelet porisms as described in [4].

The remainder of this section is dedicated to the technical details we use in the computational proofs and in the derivation of the equations of the generalized Frégier conics. Section 2 provides some general results and it is shown that the generalized Frégier conics form a pencil of the third kind. The proofs in Section 2 use synthetic reasoning. In Section 3, we shall derive the equations of the generalized Frégier conics. This enables us to show some more results on the variety of generalized Frégier conics. Along the way, we will discover some Poncelet families of triangles. Although we have to treat the different affine types of conics separately, we will lay down the computations in detail only for the case of the ellipse. This is done in order to make the presentation of results clear. In all other cases, we just point out what the differences are.

1.2 General setup and technical preliminaries

In order to describe points, we use inhomogeneous Cartesian coordinates (x, y) in the Euclidean plane as well homogeneous coordinates $x_0 : x_1 : x_2$. These are linked by $x = x_1 x_0^{-1}$ and $y = x_2 x_0^{-1}$, provided that $x_0 \neq 0$, i.e., the point $x_0 : x_1 : x_2$ is not a point at infinity, and thus, it allows for a representation as (x, y). The points at infinity (ideal points) lie on the line with the homogeneous equation $x_0 = 0$. Sometimes, we make use of the complex extension of the Euclidean plane. This leads to the finding that all Euclidean circles pass through the *absolute points* I and $J = \overline{I}$ of Euclidean geometry with homogeneous coordinates

$$I = 0 : 1 : \mathrm{i} \quad \text{and} \quad J = 0 : 1 : -\mathrm{i}.$$

Conversely, any conic through I and J is a Euclidean circle. The tangents from any circle's center M to the circle are so-called *isotropic lines*, i.e., the joins $[M, I]$ and $[M, J]$ with the absolute points. Any two concentric circles touch each other at I and J, and thus, they span a pencil of conics of the third kind.

We describe the three affine types of conics by their equations

$$\mathcal{E}, \mathcal{H} : \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \ \mathcal{P} : x^2 - 2qy = 0, \quad (1)$$

with respect to the standard frame assuming $a \neq b$, $a, b \in \mathbb{R}^+$, and $q \in \mathbb{R} \setminus \{0\}$. The computational proofs make use of their rational parametrizations

$$\begin{aligned}\mathbf{e}(t) &= \left(a\tfrac{1-t^2}{1+t^2}, b\tfrac{2t}{1+t^2}\right), t \in \mathbb{R},\\ \mathbf{h}(t) &= \left(a\tfrac{1+t^2}{1-t^2}, b\tfrac{2t}{1-t^2}\right), t \in \mathbb{R}\setminus\{-1,1\}, \\ \mathbf{p}(t) &= \left(2qt, 2qt^2\right), \quad t \in \mathbb{R}.\end{aligned} \tag{2}$$

At this point, we shall recall that for any conic there exists a huge variety of equivalent rational parametrizations. For example, the reparametrization $t \to \frac{a_{00}+a_{01}t}{a_{10}+a_{11}t}$ turns (2) in to an equivalent parametrizations and describe a projective mapping acting on the conic (provided that $a_{00}a_{11} - a_{10}a_{01} \neq 0$). In the computations, we should see that some geometric objects will then be described in a different way.

Later, we also need (Euclidean) rotation matrices. With the substitution

$$\cos\xi = \frac{1-x^2}{1+x^2} \quad \text{and} \quad \sin\xi = \frac{2x}{1+x^2} \tag{3}$$

the rotation matrices $\mathbf{R}(\phi)$ can be given with rational entries as

$$\mathbf{R}(\phi) = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix} = \begin{pmatrix}\frac{1-f^2}{1+f^2} & \frac{-2f}{1+f^2}\\ \frac{2f}{1+f^2} & \frac{1-f^2}{1+f^2}\end{pmatrix}. \tag{4}$$

In the following, we assume that $\phi \neq 0, \pm\frac{\pi}{2}$ since we are interested in generalized Frégier conics different from the Frégier point. Further, $f \neq 0, \pm 1, \pm \mathrm{i}$ since these values correspond to $\phi \neq 0, \pm\frac{\pi}{2}$ and $\pm\mathrm{i}$ are the poles of the rational equivalents of sine and cosine, i.e., the poles of the arctangent. We will not repeatedly and explicitly write these assumptions any further.

2 Projective mappings on a conic

In this section, we shall have a closer look at projective mappings acting on conics. This will lead to a general and unifying result. In [7], we can find some results on projective mappings on conics and how to treat projective mappings on conics (especially involutive ones).

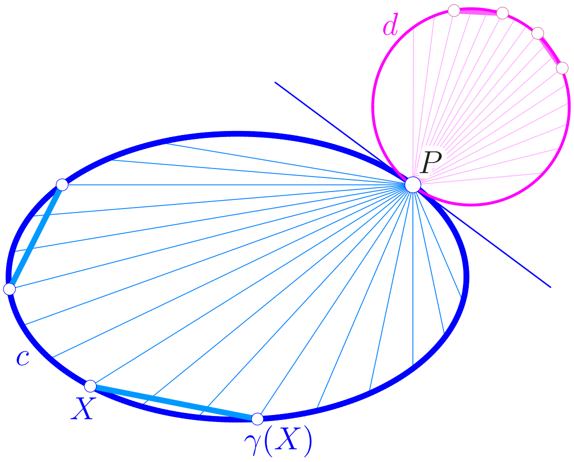

Figure 3: The perspectivity $c \to d$ can be extended to a collineation $c \to d$.

However, we need the following (apparently new) result:

Theorem 2.1. *Let c be a conic in a projective plane and P be some point on c. Further, assume that $\gamma : c \to c$ is the non-involutive projective mapping acting on c induced by the Euclidean rotation through a fixed angle $\phi \neq 0, \frac{\pi}{2}, \pi$ about P. Then, the chords $s = [X, \gamma(X)]$ of c that join a each point $X \in c$ with its γ-image $\gamma(X) \in c$ envelope a conic f.*

Proof. We use a result from [7, p. 247]: The projective mapping on a line or in a pencil of lines can be transferred via a perspectivity

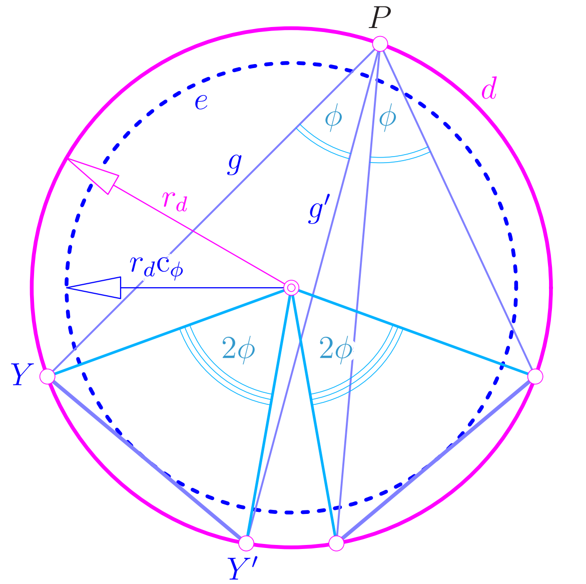

Figure 4: The Frégier conic e of $P \in d$ of the circle d to the angle ϕ is a concentric circle.

onto a conic c, and *vice versa*. For that purpose the center P of the perspectivity has to lie on the conic c in order to guarantee for a one-to-one correspondence (between line/pencil and conic). Thus, a projective mapping on a conic c can be transferred to any other conic d, for example, onto a circle d (of radius r_d) that touches c at P (as illustrated in Fig. 3).

Now, the rotation about P sends each line g through P to a line g' through P with $\sphericalangle g, g' = \phi$. Consequently, the projective mapping on c is transferred to the projective mapping on d. From $P \in c, d$, each segment spanned by a point Y and its image point Y' is seen under the constant angle ϕ, and thus, it is seen from the center of d under the angle 2ϕ (see Fig. 4). Therefore, the chords joining corresponding points envelop a circle e concentric with d and of radius $r_d \cos\phi$.

The perspectivity from $c \to d$ can be extended to a perspective collineation κ with center P that sends the envelope e to a conic f, i.e., the generalized Frégier conic that touches c' chords of assigned points. $\square$

We have excluded the case of involutive projectivities, because then the envelope of the chords is the center of the involution on the conic (cf. [7, p. 251]). With small

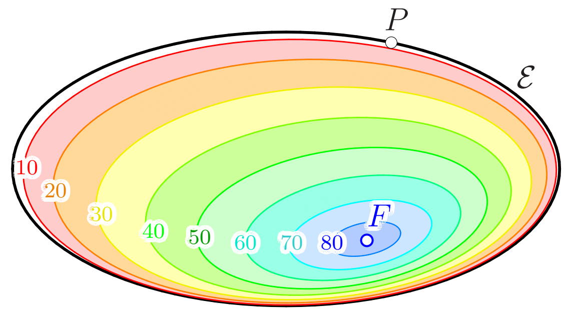

Figure 5: Some generalized Frégier conics of $P \in \mathcal{E}$ ($\phi = 10°, \ldots, 80°$): For $\phi \to \frac{\pi}{2}$ the conics shrink to the Frégier point F of P.

modifications, Thm. 2.1 is valid for any projective mapping acting on c. The projective mapping c mentioned in Thm. 2.1 is elliptic. However, the above result is true for elliptic, parabolic, and hyperbolic projectivities. There is something more important that we can deduce from Thm. 2.1:

Theorem 2.2. *The generalized Frégier conics (for variable ϕ) of a point P on a conic c form a pencil of conics of the third kind.*

Proof. We recall that the Frégier conics e of the circle d which is a collinear image of the initial conic c form a pencil of concentric circles. This pencil consist of all conics that pass through the absolute points of Euclidean geometry sharing the isotropic tangents through the common center, and therefore, they form a pencil of conics of the third kind. The (perspective) collineation κ (defined in the proof of Thm. 2.1) that sends d back to c maps all circles concentric with d to the conics of a pencil of the third kind. $\square$

It is clear that the initial conic c is also a member of the pencil of generalized Frégier conics. Further, the Frégier point F considered as the real intersection of a pair of complex conjugate lines is also a (singular) member of the pencil.

Fig. 5 shows some generalized Frégier conics of a point P on an ellipse $\mathcal{E}$. The smaller the angle ϕ, the shorter the chords of assigned points are, and therefore, the generalized Frégier conics come closer to the ellipse $\mathcal{E}$. If $\phi \to \frac{\pi}{2}$, then the conics shrink to the Frégier point F of P.

3 Equations of Frégier conics

In this section, we compute the equations of the generalized Frégier conics. Unfortunately, we have to treat the three different affine types of conics separately. However, the generalized Frégier conics of all types of conics have some properties in common and we can simplify the description by leaving some things aside. The computational approach yields some results that could not be shown in a purely synthetic way.

3.1 Frégier conics of ellipses

Let an ellipse $\mathcal{E}$ be given by the equation (1). It means no restriction to assume that $a > b$ holds. The generic point P on the ellipse $\mathcal{E}$ can be described by means of a real parameter T as $P = \mathbf{e}(T)$ in (2).

The lines g in the pencil shall be determined by choosing a second point $Q \in \mathcal{E}$ is given as $\mathbf{e}(U)$ with $U \neq T$ in (2). Hence, we obtain the equation of the chord $g := [P, Q]$ of $\mathcal{E}$ as

$$g : b(1-TU)\mathbf{x}+a(T+U)\mathbf{y}=b(1+TU). \quad (5)$$

If we rotate the normal vector

$$\mathbf{n} = (b(TU-1), -a(T+U))$$

through the angle $\phi \in (0, \frac{\pi}{2})$ either clockwise or counter clockwise, we obtain the normal vectors of those lines g^+, g^- enclosing the angles $\pm\phi$ with g. The rotation is described by the multiplication of $\mathbf{n}$ with either of the matrices $\mathbf{R}(\phi)$ or $\mathbf{R}(-\phi)$ from (4).

Now, the lines g^+ and g^- have the equations

$$\begin{aligned} g^+ : (1+T^2)\Big(\big(b(1-f^2)UT+2fa(T+U)+\\ +b(f^2-1)\big)x+\big(2TUbf + a(f^2-1)(T+U)-\\ -2fb\big)y\Big)+ab(1-f^2)(1+T^2)(1+TU)+\\ +2a^2f(T+U)(T^2-1)-4b^2f(1+TU)T=0, \end{aligned}$$

and

$$\begin{aligned} g^- : (1+T^2)\Big(\big(b(1-f^2)UT-2fa(T+U)+\\ +bf^2-b\big)x+\big(-2TUbf+a(f^2-1)(T+U)+\\ +2fb\big)y\Big)+ab(1-f^2)(1+T^2)(1+TU)-\\ -2a^2f(T+U)(1+T^2)+4b^2f(TU-1)T=0. \end{aligned}$$

The chords' endpoints $Q^+ = g^+ \cap \mathcal{E}$ and

$Q^- = g^- \cap \mathcal{E}$ are

$$\begin{aligned}
Q^+ &= b^2(a^2(f^2-1)^2+4b^2f^2)(1+T^2U^2)+\\
&+4abf(f^2-1)(a^2-b^2)(T+U)(1-TU)+\\
&+a^2(b^2(f^2-1)^2+4a^2f^2)(T^2+U^2)+\\
&+8f^2(a^2-b^2)(a^2+b^2)TU :\\
&:\Big(ab(1-U)(1+T^2)(1-f^2)-\big(2b^2(1+T^2)+\\
&+2a^2(T^2+U)+2(a^2-b^2)T(1+U)\big)f\Big)\cdot\\
&\cdot\Big(ab(1+T^2)(1+U)(f^2-1)-\big(2b^2(1-T^2)+\\
&+2a^2(T^2-U)-2(a^2-b^2)T(1-U)\big)f\Big):\\
&:\Big(abU(1+T^2)(f^2-1)-\big(2a^2T^2+\\
&+2(a^2-b^2)UT+2b^2\big)f\Big)\cdot\Big(ab(1+T^2)\cdot\\
&\cdot(f^2-1)+\big(2b^2T^2U+2(a^2-b^2)T+2a^2U\big)f\Big)
\end{aligned}$$

and Q^- admits a similar representation.

Now, we can state and prove:

Theorem 3.1. *The lines $s^+ := [Q, Q^+]$ and $s^- := [Q, Q^-]$ envelop the same ellipse $\mathcal{F}_\mathcal{E}$. The centers of all these ellipses trace an ellipse $\mathcal{M}$ homothetic to $\mathcal{E}$.*

Proof. The parametrizations of Q^+ and Q^- enable us to derive the equations of the lines $s^+ = [Q, Q^+]$ and $s^- = [Q, Q^-]$. The computation of the envelopes is now straight forward: We eliminate U from the equations of s^- and s^+ and we can immediately see that both families of lines envelop the same curve with the equation

$$\begin{aligned}
\mathcal{F}_\mathcal{E} : & \, b^2\big(\mathrm{s}_\phi{}^2\mathrm{c}_\tau{}^2\epsilon^2-4a^2b^2\big)\mathbf{x}^2+\\
&+a^2\big((a^2+b^2)^2-\mathrm{s}_\tau{}^2\mathrm{s}_\phi{}^2\epsilon^2\big)\mathbf{y}^2+\\
&-2abf^2\mathrm{s}_\tau\mathrm{c}_\tau(1+\mathrm{c}_\phi)^2\epsilon^2\mathbf{xy}+\\
&-2ab(a^4-b^4)\mathrm{s}_\phi{}^2\big(b\mathrm{c}_\tau\mathbf{x}-a\mathrm{s}_\tau\mathbf{y}\big)+\\
&-a^2b^2\big(\mathrm{c}_\phi(a^2+b^2)^2-\epsilon^2\big)\big)=0,
\end{aligned} \tag{6}$$

where we changed back to the trigonometric representation. For the sake of simplicity, we have set

$$\begin{aligned}
\mathrm{s}_\phi &:= \sin\phi, \ \mathrm{c}_\phi := \cos\phi,\\
\mathrm{s}_\tau &:= \sin\tau, \ \mathrm{c}_\tau := \cos\tau,
\end{aligned}$$

and $\epsilon^2 := a^2 - b^2$ is the square of the linear excentricity of the ellipse $\mathcal{E}$. In order to show that the curves $\mathcal{F}_\mathcal{E}$ are ellipses, we find their centers as

$$\mathbf{m}(T) = \frac{f^2(a^4-b^4)}{(a^2f^2+b^2)(b^2f^2+a^2)}\,\mathbf{e}(-T)$$

(with $\mathbf{e}$ from (2)) which parametrizes the ellipse $\mathcal{M}$ mentioned above. Obviously, $\mathcal{M}$ is homothetic to $\mathcal{E}$ and its semi-axes lengths are

$$\begin{aligned}
\text{major} &= \frac{af^2(a^4-b^4)}{(a^2f^2+b^2)(b^2f^2+a^2)},\\
\text{minor} &= \frac{bf^2(a^4-b^4)}{(a^2f^2+b^2)(b^2f^2+a^2)},
\end{aligned}$$

provided $b < a$. Their ratio equals $a : b$ and they never vanish as long as $a \neq b$. □

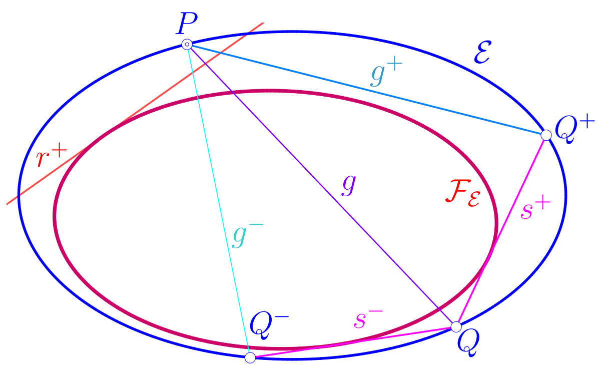

Figure 6: Both chords s^+ and s^- envelop the same conic $\mathcal{F}_\mathcal{E}$.

The fact that the generalized Frégier conics of an ellipse are always ellipses can also be deduced from the construction used in the proof of Thm. 2.1: For real rotation angles ϕ, the envelopes of the chords are in the interior of the auxiliary circle d. The collineation $d \to c$ with center P maps these interior circles to conics in the interior of the ellipse c (or $\mathcal{E}$, respectively). Hence, the generalized Frégier conics of an ellipse can only be ellipses.

Only if we allow the rotation angle ϕ to be a pure imaginary number, the radii of the envelopes of d's chords can become arbitrarily large:

$$r_d \cos(\mathrm{i}\phi) = r_d \cosh\phi \geq r_d,$$

and thus, there exist outer generalized Frégier conics of any affine type but not corresponding to real angles.

The Frégier ellipses (6) constitute a pencil of conics of the third kind (cf. Thm. 2.2). All conics in this pencil touch each other in a pair of complex conjugate points

$$B_1 = \left(a\frac{(a^2+b^2)(1-T^2)+4ab\mathrm{i}T}{(1+T^2)(a^2-b^2)}, \right.$$
$$\left. 2b\frac{ab\mathrm{i}(1-T^2)-(a^2+b^2)T}{(1+T^2)(a^2-b^2))} \right),$$

and $B_2 = \overline{B_1}$. These points are the collinear images of the absolute points of Euclidean geometry common to all circles concentric with the auxiliary circle d used in the proof of Thms. 2.1 and 2.2. Since, the points B_1 and B_2 are each others complex conjugates they span a real line

$$\begin{aligned} p:\ & \epsilon^2(b(1-T^2)x - 2aTy) = \\ & = ab(a^2+b^2)(1+T^2) \end{aligned} \tag{7}$$

which is the polar line of the Frégier point

$$F = \frac{a^2-b^2}{a^2+b^2}\left(\frac{a(1-T^2)}{1+T^2}, \frac{-2bT}{1+T^2}\right). \tag{8}$$

with pivot point $P \in c$. The line p given by (7) is sometimes called the *Frégier* line of P with respect to c (cf. [15]). The Frégier line (with multiplicity two) is a singular conic in the pencil of generalized Frégier conics.

The following can also be shown:

Theorem 3.2. *For variable point $P \in \mathcal{E}$, the Frégier ellipses $\mathcal{F}_{\mathcal{E}}$ envelop two ellipses $\mathcal{E}_i$, $\mathcal{E}_e$ which are homothetic to $\mathcal{E}$.*

Proof. The elimination of the parameter T from the equation (6) of $\mathcal{F}_{\mathcal{E}}$ and its derivative with respect to T yields

$$\begin{aligned} \mathcal{E}_o:\ & \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(b^2f^2-a^2)^2}{(b^2f^2+a^2)^2}, \\ \mathcal{E}_i:\ & \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(a^2f^2-b^2)^2}{(a^2f^2+b^2)^2}. \end{aligned}$$

Obviously, $\mathcal{E}_o$ and $\mathcal{E}_i$ are concentric with $\mathcal{E}$, there axes are parallel to those of $\mathcal{E}$, and since the semi-axes lengths of the latter ellipses are

$$a_i = a\frac{a^2-b^2f^2}{b^2f^2+a^2},\quad b_i = b\frac{a^2-b^2f^2}{b^2f^2+a^2}$$
and
$$a_o = a\frac{a^2f^2-b^2}{a^2f^2+b^2},\quad b_o = b\frac{a^2f^2-b^2}{a^2f^2+b^2}.$$

The ratio of both pairs of semi-axes lengths equals a/b. □

The outer and inner envelope $\mathcal{E}_o$ and $\mathcal{E}_i$ coincide if $f = \pm 1$ and become the ordinary Frégier conic being the trace (8) of the Frégier points of $\mathcal{E}$.

Fig. 7 shows the two ellipses $\mathcal{E}_o$ and $\mathcal{E}_i$ comprising the envelope of the Frégier ellipses of $\mathcal{E}$.

The sketch of the computational proof of Thm. 3.1 hides a detail: The resultant of

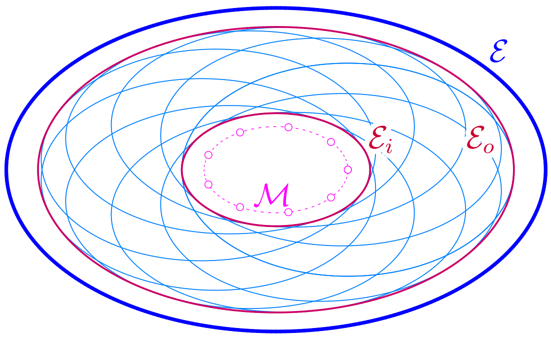

Figure 7: The envelope of the generalized Frégier conics of the ellipse $\mathcal{E}$ consists of an outer ellipse $\mathcal{E}_o$ and an inner ellipse $\mathcal{E}_i$.

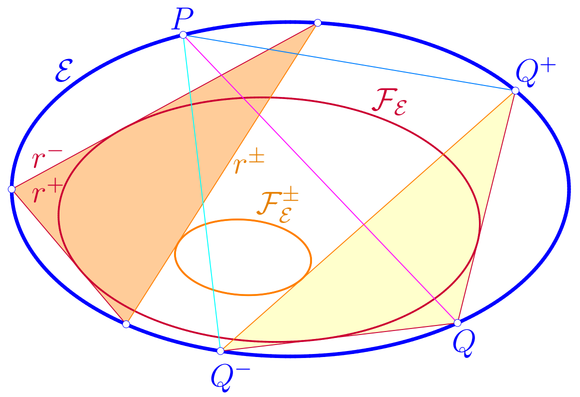

Figure 8: The triangle built by r^+, r^-, $r^\pm$ already indicates the existence of a poristic family of triangles interscribed between $\mathcal{E}$ and the Frégier ellipses $\mathcal{F}_\mathcal{E}$ and $\mathcal{F}_\mathcal{E}^\pm$.

the equation of s^+ and its derivative with respect to U turns out to be the product of a polynomial of degree one (equation of a line r^+) and a polynomial of degree two (equation of $\mathcal{F}_\mathcal{E}$). This is also the case with s^- (yielding the equation of a line $r^- \neq r^+$ and the equation of $\mathcal{F}_\mathcal{E}$). However, the two resultants share the quadratic factor describing $\mathcal{F}_\mathcal{E}$ and differ in the linear parts. The lines r^+ and r^- belong to the pencil about $(-a, 0)$ (the left principal vertex of $\mathcal{E}$ which corresponds to the parameter value $T = \infty$) and their equations are

$$\begin{gathered} r^+, r^- :\ b\big(\epsilon^2 \mathrm{s}_\tau \sin\phi \pm 2abc_\phi\big)\mathbf{x} + \\ + a \sin\phi\big(\epsilon^2 \mathrm{c}_\tau + a^2 + b^2\big)\mathbf{y} + \\ + ab\big(\epsilon^2 \mathrm{s}_\tau \sin\phi \pm 2abc_\phi\big) = 0 \end{gathered}$$

From the computational point of view, the lines r^+ and r^- do not have any further meaning. It is quite the opposite from the geometric point of view as we shall see soon.

The vertex of the pencil depends on the parametrization (2) and can be replaced with any other point on $\mathcal{E}$ (simply by substituting any linear rational function for T).

It is by no means surprising that the lines $s^\pm := [Q^-, Q^+]$ also envelop a conic $\mathcal{F}_\mathcal{E}^\pm$ since $\sphericalangle g^- g = \sphericalangle g g^+ = \frac{1}{2} \sphericalangle g^- g^+$. Further, a computational proof of the latter fact (comparable to that of Thm. 3.1) would also produce the equation of a line $r^\pm$ which is tangent to the Frégier ellipse $\mathcal{F}_\mathcal{E}^\pm$.

At this point, we emphasize that the respective coefficient matrices of the conics satisfy

$$\mathcal{F}_\mathcal{E}^\pm = \mathcal{F}_\mathcal{E} \mathcal{E}^{-1} \mathcal{F}_\mathcal{E},$$

which identifies $\mathcal{F}_\mathcal{E}^\pm$ as the *conjugate conic of* $\mathcal{E}$ with respect to $\mathcal{F}_\mathcal{E}$ in the sense of [10]. Also in that sense, the conics $\mathcal{E}$ and $\mathcal{F}_\mathcal{E}$ span an *exponential pencil* of conics which also contains $\mathcal{F}_\mathcal{E}^\pm$. Because of the nestedness of $\mathcal{E}$, $\mathcal{F}_\mathcal{E}$, and $\mathcal{F}_\mathcal{E}^\pm$, the exponential pencil has a point shaped limit which equals the Frégier point F given by (8). This holds in the like manner for the generalized Frégier conics of hyperbolae and parabolae.

There exists a triple $(r^+, r^-, r^\pm)$ of lines which are the sides of a triangle interscribed between $\mathcal{E}$, $\mathcal{F}_\mathcal{E}$, and $\mathcal{F}_\mathcal{E}^\pm$ independent of the choice of $P \in \mathcal{E}$. This gives rise to the following:

Theorem 3.3. *The triangles bounded by the lines r^+, r^-, $r^\pm$ form a one-parameter family of triangles interscribed between the conic $\mathcal{E}$ and the Frégier ellipses $\mathcal{F}_\mathcal{E}$ and $\mathcal{F}_\mathcal{E}^\pm$. The triangles form a Poncelet family.*

Proof. We only have to show that the conics $\mathcal{E}$, $\mathcal{F}_\mathcal{E}$, and $\mathcal{F}_\mathcal{E}^\pm$ belong to a linear pencil (cf. [7, p. 259]) in order to meet the requirements of a general Poncelet porism (cf. [4]).

This can either be done by referring to Thm. 2.2 according to which the two Frégier conics to angles ϕ and 2ϕ belong to a pencil of conics (of the third kind) or by means of computation:

For that purpose, we homogenize the equations of $\mathcal{E}$, $\mathcal{F}_\mathcal{E}$, and $\mathcal{F}_\mathcal{E}^\pm$, extract the coefficient matrices, and find that they are linearly dependent since

$$\begin{aligned}&\Big(4(1-f^2)^2\,\mathcal{F}_\mathcal{E}-\mathcal{F}_\mathcal{E}^\pm\Big)(1+f^2)^{-1}=\\&=a^4b^4(3f^2-1)(f^2-3)^2(1+T^2)^2\,\mathcal{E},\end{aligned}\tag{9}$$

provided that $f\neq\pm1$. In the cases $f=\pm\sqrt{3},\pm1/\sqrt{3}$, i.e., $\phi\neq\pm\frac{\pi}{6}$, the Frégier ellipses $\mathcal{F}_\mathcal{E}$ and $\mathcal{F}_\mathcal{E}^\pm$ coincide. □

We have shown that $\mathcal{F}_\mathcal{E}$, $\mathcal{F}_\mathcal{E}^\pm$, $\mathcal{E}$ belong to one pencil of conics. This is a pencil of the third kind that contains the two singular conics. The first of which is a line with multiplicity two:

$$\epsilon^2\big(b(T^2-1)\mathbf{x}+2aT\mathbf{y}\big)=ab(a^2+b^2)(1+T^2).$$

The second one is a pair of complex conjugate lines concurring in the (real) Frégier point (8) with directions

$$\frac{x}{y}=\pm\frac{\mathrm{ib}}{2\mathrm{a}}\,\frac{(a^2+b^2)(1-T^2)+4ab\mathrm{i}T}{ab(T^2-1)-\mathrm{i}(a^2+b^2)T}.$$

This pair of complex conjugate lines is the image of the isotropic lines through the center of d under the perspective collineation $d\to c$ used in the proof of Thm. 2.1.

3.2 Frégier conics of hyperbolae

In analogy to the previous section, we assume that a hyperbola $\mathcal{H}$ is given by the middle equation of (1) with real semi-axes a,b. The vertex of the rotating angle(s) is now $P=\mathbf{h}(T)$ with $\mathbf{h}$ from (2) where $T\in\mathbb{R}\setminus\{-1,1\}$.

Again, the point Q is obtained by assuming $Q=\mathbf{h}(U)$ with $T\neq U$ and the chords $g:=[P,Q]$ of $\mathcal{H}$ have an equation similar to that of $\mathcal{E}$ in (5). Now, the chords' normal vectors are proportional to

$$\mathbf{n}=(b(1+TU),-a(T+U)).$$

The normal vectors of the legs g^+ and g^- of the moving angles attached to g are found by applying the linear mappings induced by the matrices $\mathbf{R}(\phi)$ and $\mathbf{R}(-\phi)$ from (4).

This allows us to write down the equations of g^+ and g^-, compute the points Q^+, Q^-, and furthermore, to determine the envelopes of the lines $s^+:=[Q,Q^+]$, $s^-:=[Q,Q^-]$, and $s^\pm:=[Q^-,Q^+]$, and we find $\mathcal{F}_{\mathcal{H}}^{+}=\mathcal{F}_{\mathcal{H}}^{-}=\mathcal{F}_{\mathcal{H}}$ with the equation

$$
\begin{aligned}
\mathcal{F}_{\mathcal{H}}: \Big((a^2-b^2f^2)(a^2f^2-b^2)\big(1+T^4\big)+\\
+2\big(a^2b^2(1+f^2)^2+\varepsilon^2f^2\big)T^2\Big)\mathbf{x}^2+\\
+\Big(a^2b^2(1+f^2)^2\big(1+T^4\big)+\\
+2\big(2\varepsilon^2f^2-a^2b^2(1+f^2)^2\big)T^2\Big)\mathbf{y}^2+\\
+4a^2b\varepsilon^2f^2T(1+T^2)\mathbf{xy}+\\
-2ab^2(a^2-b^2)\varepsilon f^2(1-T^4)\mathbf{x}+\\
-4a^2b(a^2-b^2)\varepsilon f^2T(1-T^2)\mathbf{y}+\\
+a^2b^2(a^2f^2+b^2)\cdot\\
\cdot(b^2f^2+a^2)(1-T^2)^2=0,
\end{aligned}
\tag{10}
$$

where $\varepsilon^2=a^2+b^2$ is the square of the linear excentricity of the hyperbola $\mathcal{H}$.

Analogously to Thm. 3.1, we can formulate

Theorem 3.4. *The lines s^+ and s^- envelop the same conic $\mathcal{F}_{\mathcal{H}}$, the generalized Frégier conic of the hyperbola $\mathcal{H}$.*
The generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ of a hyperbola $\mathcal{H}$ can be conics of any affine type.

Proof. Since the determinant of the quadratic term of (10) equals

$$
\begin{aligned}
D_{12}:=4a^4b^4(1+f^2)^2(a^2-b^2f^2)^3\cdot\\
\cdot(a^2f^2-b^2)^3(1-T^2)^4
\end{aligned}
$$

and vanishes exactly if $f=\pm\frac{a}{b},\pm\frac{b}{a}$, the generalized Frégier conics in these particular four cases coincide and the equation of $\mathcal{F}_{\mathcal{H}}$ simplifies to

$$
\begin{aligned}
(a^2+b^2)^2\big(2bT\mathbf{x}+a(1+T^2)\mathbf{y}\big)^2+\\
-2ab(1-T^2)\big(b(a^4-b^4)\big(1+T^2\big)\mathbf{x}+\\
2aT(a^4-b^4)\mathbf{y}-ab(a^4+b^4)(1-T^2)\big)=0.
\end{aligned}
$$

The latter equation describes a parabola with ideal point

$$0:-a(1+T^2):2bT$$

(for all four values of f). For proper choices of f, D_{12} can be positive as well as negative, and therefore, the generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ of $\mathcal{H}$ can also be ellipses and hyperbolae. Using the collineation applied in the proof of Thm. 2.1, we can also argue that all affine types of conics can show up here as generalized Frégier conics.

The Frégier conics of hyperbolae are regular since the determinant of the homogeneous equation equals

$$D_{012}:=-8a^{10}b^{10}(1-f^2)^2(1+f^2)^4(1-T^2)^6$$

which vanishes only if $f=\pm1$ (right angle, Frégier point) or if $T=\pm1$ (which can be avoided by reparametrizing $\mathcal{H}$). □

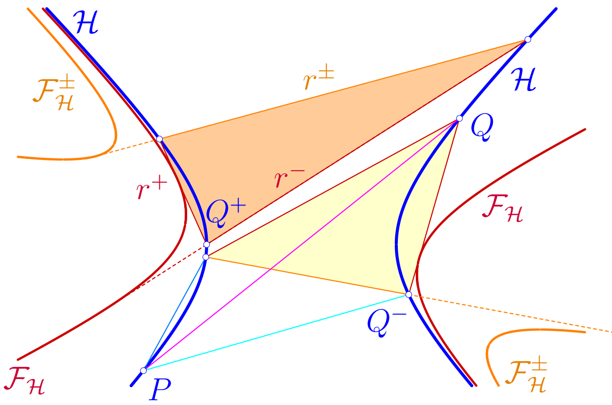

Figure 9: Two triangles from the Poncelet family interscribed between $\mathcal{H}$, $\mathcal{F}_{\mathcal{H}}$, and $\mathcal{F}_{\mathcal{H}}^{\pm}$.

The one-parameter family of generalized Frégier conics of a hyperbola shows a behaviour similar to that of an ellipse. Comparable to Thm. 3.2, we can show:

Theorem 3.5. *For variable point $P\in\mathcal{H}$, the generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ envelop two hyperbolae $\mathcal{H}_i$, $\mathcal{H}_o$ which are homothetic to $\mathcal{H}$.*

Proof. We eliminate the parameter T from the equation (10) of the generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ and of the derivatives of (10) with respect to T. This elimination yields besides the equations $ay \mp bx = 0$ of $\mathcal{H}$'s asymptotes, the hyperbola $\mathcal{H}$, and a further hyperbola $\mathcal{H}'$ that does not contribute to the envelope.

The two components of the envelope are two hyperbolae

$$\mathcal{H}_i : \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(b^2f^2+a^2)^2}{(b^2f^2-a^2)^2},$$
$$\mathcal{H}_o : \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{(a^2f^2+b^2)^2}{(a^2f^2-b^2)^2}.$$

It is obvious that $\mathcal{H}_i$ and $\mathcal{H}_o$ are homothetic to $\mathcal{H}$. Their semi-axes are

$$a_o = a\frac{b^2f^2+a^2}{a^2-b^2f^2}, \quad b_o = b\frac{b^2f^2+a^2}{a^2-b^2f^2}$$
and
$$a_i = a\frac{a^2f^2+b^2}{a^2f^2-b^2}, \quad b_i = b\frac{a^2f^2+b^2}{a^2f^2-b^2}$$

(provided that $f \neq \pm\frac{a}{b}, \pm\frac{b}{a}$) and the axes ratio equals a/b. □

It is a rather simple task to show that the centers of the generalized Frégier conics move on a hyperbola $\mathcal{M}$ homothetic to $\mathcal{H}$ with semi-axes

$$\text{principal} = \frac{af^2(a^4+b^4)}{(a^2-b^2f^2)(a^2f^2-b^2)},$$
$$\text{auxiliary} = \frac{bf^2(a^4+b^4)}{(a^2-b^2f^2)(a^2f^2-b^2)},$$

provided that $f \neq \pm\frac{a}{b}, \pm\frac{b}{a}$.

In the previous section, we have seen that the computation of the generalized Frégier conics as the envelopes of chords of a conic produced straight lines as some by-product.

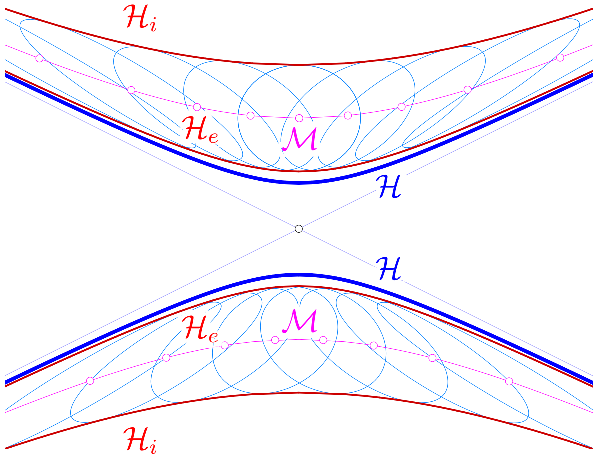

Figure 10: Frégier conics of a hyperbola can be ellipses. In any case, the Frégier conics of a hyperbola $\mathcal{H}$ envelop two hyperbolae $\mathcal{H}_i$ and $\mathcal{H}_e$ (homothetic to $\mathcal{H}$) and with their centers on a further homothetic hyperbola $\mathcal{M}$.

These lines depend on the parametrization of the initial conic, but nevertheless, they allow us to conclude that there exist general Poncelet families of triangles interscribed between $\mathcal{H}$ and the generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ and $\mathcal{F}_{\mathcal{H}}^{\pm}$.

Therefore, and without repeating the similar computations, and in analogy to Thm. 3.3, we can state:

Theorem 3.6. *The hyperbola $\mathcal{H}$ and the pair of generalized Frégier conics $\mathcal{F}_{\mathcal{H}}$ and $\mathcal{F}_{\mathcal{H}}^{\pm}$ admit an interscribed one-parameter family of triangles,* i.e., *a one-parameter family of billiards with two caustics.*

According to Thm. 2.2 and because of

$$\mathcal{F}_{\mathcal{H}}^{\pm} - 4(1-f^2)^3\mathcal{F}_{\mathcal{H}} =$$
$$= (3f^2-1)(f^2-3)(1+f^2)^2(1-T^2)^2a^4b^4\mathcal{H}$$

the conics $\mathcal{H}$, $\mathcal{F}_{\mathcal{H}}$, and $\mathcal{F}_{\mathcal{H}}^{\pm}$ belong to a pencil of conics.

3.3 Frégier conics of parabolae

Finally, we assume that the parabola $\mathcal{P}$ is given by the third equation in (1). Now, we let $P=\mathbf{p}(T)$ and $Q=\mathbf{p}(U)$ with $T,U\in\mathbb{R}$ and $T\neq U$ be two points on $\mathcal{P}$ spanning the line $g=[P,Q]$ rotating about P through ϕ. With (4) applied to the normal vector

$$\mathbf{n}=(T+U,-1),$$

we find the line g^+ with the equation

$$\begin{aligned} g^+ : &\big((T+U)(1-f^2)+2f\big)\mathbf{x}+\\ &+\big(2(T+U)f+f^2-1\big)\mathbf{y}=\\ =&\,2pT\big(U(1-f^2)-2fT(T+U)+2f\big)\end{aligned} \tag{11}$$

and the line g^- admits a similar representation. The lines g^+ and g^- intersect $\mathcal{P}$ in the points $Q^+,Q^-\neq P$ where

$$Q^+=2p\left(\frac{U(f^2-1)-2f(T+U)T-2f}{2f(T+U)+f^2-1},\right.\\ \left.\frac{(2f(T+U)T+U(1-f^2)+2f)^2}{(2f(T+U)+f^2-1)^2}\right). \tag{12}$$

The point Q^- admits a similar coordinate representation. This yields the equations of the chords $s^-:=[Q,Q^-]$, $s^+:=[Q,Q^-]$, and $s^\pm:=[Q^-,Q^+]$, where

$$\begin{aligned} s^+ :\ &2(f(T^2-U^2)+U(1-f^2)+f)\mathbf{x}+\\ &+(2f(T+U)+f^2-1)\mathbf{y}\\ =&\,2pU(2f(T+U)T+U(1-f^2)+2f),\end{aligned} \tag{13}$$

and the equations of the other chords s^- and $s^\pm$ can be given in a similar form.

We compute their envelopes and find again that the Frégier conics $\mathcal{F}_\mathcal{P}^-=\mathcal{F}_\mathcal{P}^+=:\mathcal{F}_\mathcal{P}$ are identic. An equation of the parabola's generalized Frégier conics can be given as

$$\begin{aligned}\mathcal{F}_\mathcal{P} :\ &(4T^2f^2+(1+f^2)^2)\mathbf{x^2}+\\ &+4f^2T\mathbf{xy}+f^2\mathbf{y^2}+\\ &+8qf^2T(1+T^2)\mathbf{x}-\\ &-2q(f^4-2T^2f^2+1)\mathbf{y}+\\ &+4q^2f^2(1+T^2)^2=0.\end{aligned} \tag{14}$$

Now, we can state (comparable to Thm. 3.1 and Thm. 3.4):

Theorem 3.7. *The chords s^+ and s^- cut out of a parabola $\mathcal{P}$ by congruent angles centered at a point $P\in\mathcal{P}$ envelope the same conic $\mathcal{F}_\mathcal{P}$ with the equation* (14).
The generalized Frégier conics $\mathcal{F}_\mathcal{P}$ of a parabola $\mathcal{P}$ are ellipses if $\phi\in\mathbb{R}$.

Proof. The chords' envelope is already given in (14). Since the determinant of the homogeneous equation of $\mathcal{F}_\mathcal{P}$ equals

$$D_{012}=-8q^2(1-f^2)^2(1+f^2)^4,$$

it never vanishes (for, by assumption $q\neq 0$, $f\neq 0,\pm1,\pm\mathrm{i}$). Hence, the generalized Frégier conics of the parabola are always regular. The determinant of the quadratic term in the inhomogeneous equation (14) of $\mathcal{F}_\mathcal{P}$ equals

$$D_{12}=4f^2(1+f^2)^2$$

and is always positive (provided $f\neq 0,\pm\mathrm{i}$, which is excluded from the very beginning). Hence, (14) describes ellipses independent of the choice of f and T (since $p\neq 0$, $f\neq\pm1,\pm\mathrm{i}$). In order to verify the second part of the theorem, just discuss the quadratic part of (14). □

The centers of the ellipses (14) showing up as generalized Frégier conics are located on a parabola with the parametrization

$$(-2qT, f^{-2}q(2T^2f^2 + f^4 + 1))$$

and the equation

$$\mathcal{P}_c: \ \mathbf{x}^2 - 2q\mathbf{y} = -2f^{-2}q^2(1 + f^4).$$

Obviously, this parabola is congruent to $\mathcal{P}$. This parabola is also shown in Fig. 11

Like in the case with the ellipse $\mathcal{E}$, the elimination process delivers two lines r^+ and r^-, which are parallel and tangent to $\mathcal{F}$ and have the equations

$$r^+, r^- : \ 2fpT \mp p(1-f^2) + f\mathbf{x} = 0. \quad (15)$$

The parallelity of r^+ and r^- depends on the parametrization (2) of $\mathcal{P}$ since $t = \infty$ in the third equation of (2) yields the point $0 : 0 : 1 = r^+ \cap r^-$. A suitable linear rational reparametrization of the parabola (2) can move the point $r^+ \cap r^-$ to any other point on $\mathcal{P}$.

The double angle Frégier conic $\mathcal{F}^{\pm}_{\mathcal{P}}$ has the equation

$$\begin{aligned}\mathcal{F}^{\pm}_{\mathcal{P}} : &\big(16f^2(1-f^2)^2T^2 + (1+f^2)^4\big)\mathbf{x}^2 + \\ &+16f^2(1-f^2)^2T\mathbf{xy} + 4f^2(1-f^2)^2\mathbf{y}^2 + \\ &+32pf^2(1-f^2)^2T(1+T^2)\mathbf{x} + \\ &+\big(16pf^2(1-f^2)^2T^2 - 2p(f^8 - 4f^6 + \\ &+22f^4 - 4f^2 + 1)\big)\mathbf{y} + \\ &+16p^2f^2(1-f^2)^2(1+T^2)^2 = 0.\end{aligned} \quad (16)$$

which is regular as long as $f \neq \pm 1 \pm \sqrt{2}$ and consists of the given parabola $\mathcal{P}$ and the line

$$2pT^2 + 2T\mathbf{x} + \mathbf{y} = 0$$

if $f = \pm 1$. The additional line that comes along with the equation (16) of $\mathcal{F}^{\pm}_{\mathcal{P}}$ has the equation

$$\begin{aligned}r^{\pm} : 2f^2(2T\mathbf{x} + \mathbf{y}) &= \\ = p((1-f^2)^2 - 4f^2T^2).\end{aligned} \quad (17)$$

The three conics $\mathcal{P}$, $\mathcal{F}_{\mathcal{P}}$, and $\mathcal{F}^{\pm}_{\mathcal{P}}$ belong to the same pencil since the respective equations (1), (14), and (16) satisfy

$$\begin{aligned}(3f^2-1)(3-f^2)(1+f^2)^2\mathcal{P} &= \\ = 4(1-f^2)^2\mathcal{F}_{\mathcal{P}} + \mathcal{F}^{\pm}_{\mathcal{P}}.\end{aligned} \quad (18)$$

The comparison of (9) and (18) shows that the latter does neither contain the parameter q nor the curve parameter T, while (9) depends on the semi-axes of $\mathcal{E}$ and on the point P.

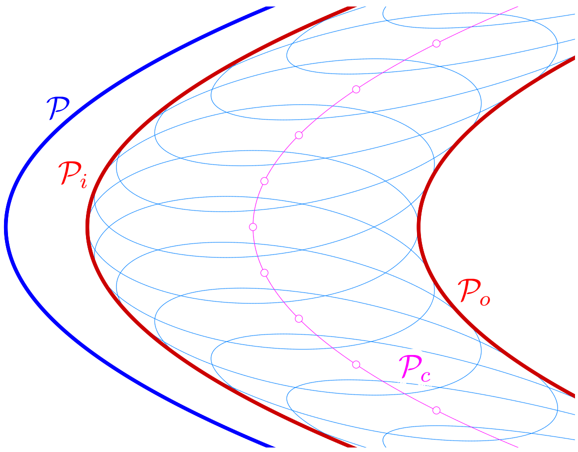

Figure 11: The generalized Frégier conics of a parabola $\mathcal{P}$ envelop two parabolae $\mathcal{P}_i$ and $\mathcal{P}_o$ which are congruent to $\mathcal{P}$.

Comparable to Thms. 3.2 and 3.5, we can show what is illustrated in Fig. 11:

Theorem 3.8. *For variable point $P \in \mathcal{P}$, the generalized Frégier conics $\mathcal{F}_{\mathcal{P}}$ of a* *parabola (1) envelop a pair of congruent parabolas with the equations*

$$\begin{aligned}\mathcal{P}_o:\ & \mathbf{x}^2 + 4f^{-2}q^2 = 2q\mathbf{y},\\ \mathcal{P}_i:\ & \mathbf{x^2} + 4f^2q^2 \quad = 2q\mathbf{y}\end{aligned}$$

which are also congruent to $\mathcal{P}$.

Proof. The computation of these two envelopes is straight forward. Since their quadratic part is a multiple of $x^2 - 2qy$ (as is the case with $\mathcal{P}$), they are congruent to each other and $\mathcal{P}$ as well. $\square$

Because of the existence of one interscribed triangle bounded by the lines (15) and (17) between the conics $\mathcal{P}$, $\mathcal{F}_{\mathcal{P}}$, and $\mathcal{F}^{\pm}_{\mathcal{P}}$ (which belong to a pencil according to Thm. 2.2 and because of (18)), we have (cf. Thm. 3.3 and Thm. 3.6):

Theorem 3.9. *The conics $\mathcal{P}$, $\mathcal{F}_{\mathcal{P}}$, and $\mathcal{F}^{\pm}_{\mathcal{P}}$ allow for a one-parameter family of interscribed triangles.*

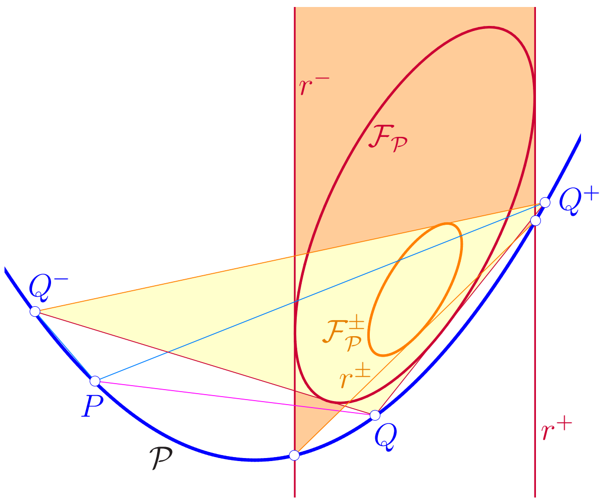

Figure 12: Frégier conics $\mathcal{F}_{\mathcal{P}}$ and $\mathcal{F}^{\pm}_{\mathcal{P}}$ related to a parabola $\mathcal{P}$.

Fig. 12 illustrates that among the triangles in the Poncelet family described in Thm. 3.9 there are degenerate triangles with one vertex at infinity. It is more than one degenerate triangle since each vertex of the triangle can reach one of the positions of $r^+ \cap \mathcal{P}$ or $r^- \cap \mathcal{P}$.

4 Concluding remarks

The generalized Frégier conics can be seen as a blow-up of the ordinary Frégier point just by replacing the right angle between assigned pairs of lines in the projective mapping at some point P on a conic c. This blow-up "enlarges" or blows up the ordinary Frégier conic (the trace of the Frégier point if its pivot $P \in c$ is moving along c) to the two envelopes $\mathcal{E}_o$, $\mathcal{E}_i$ ($\mathcal{H}_o$, $\mathcal{H}_o$ or $\mathcal{P}_o$, $\mathcal{P}_i$). Of course, there are other ways to generalize or adapt Frégier's theorem. We shall postpone this to a future article.

The Poncelet families (one-parameter families of triangles interscribed in between some conics from a pencil) were found just occasionally since the lines bounding these triangles are by-products in the computation. The initial parametrizations (2) lead to just one *initial* triangle in the family. Any other (projectively equivalent) parametrization of the conics would have resulted in another triangle. However, one is enough since it was possible to show that the involved triple of conics $(\mathcal{E}, \mathcal{F}_{\mathcal{E}}, \mathcal{F}^{\pm}_{\mathcal{E}})$ (and also those related to the hyperbola and the parabola) belong to the same pencil.

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