Frégier's theorem in three dimensions

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Abstract. We present a three-dimensional analog of Frégier's theorem. This yields Frégier quadrics associated with each quadric. The Frégier quadrics of regular quadrics can be obtained as so-called generalized offsets to quadrics with an offset function depending on the Gaussian curvature. Further, we shall discuss some of the common features and differences of the planar and spatial Frégier transform.

Keywords: Frégier's theorem, quadrics, generalized offset, Gaussian curvature

1 Introduction

1.1 Known results, related work

In its original version, Frégier's theorem (see [3, 4]) relates metric notions (right angles) with objects from Projective Geometry (conics): *The chords of a conic c cut out by the legs of a right angle centered at a point P on c pass through a fixed point F while the right angle rotates about P* (see Fig. 1, left). F is called the *Frégier point* of P with respect to c. Obviously, F lies on c's normal at P. The set of all Frégier points of a conic c is a conic f, called the *Frégier conic* of c. (If c is a circle or an equilateral hyperbola, then f shrinks to the center or becomes the line at infinity.) The conic f is coaxial with c, similar to c, and thus, of the same affine type (see [4]). From the viewpoint of Projective Geometry, the Frégier

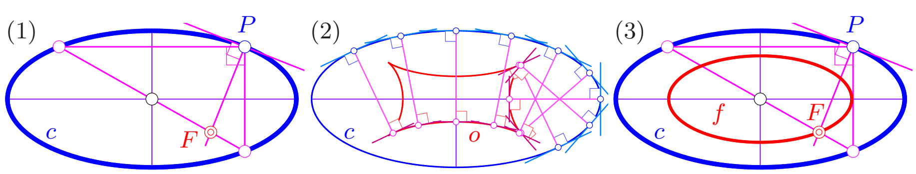


Fig. 1. (1) Frégier point F of $P \in c$, (2) offset o of a conic c, (3) Frégier conic f of c.

point is nothing but the center of an involution acting on the conic obtained by lifting the involution of right angles at P to c (see [4]). Replacing the right angle in Frégier's construction by arbitrary angles, yields 1-parameter families of conics as envelopes of chords, see [11]. These families are always pencils of the third kind. Both, the right angle and the conic can indeed be explained within the setting of Projective Geometry as described in [12] which is the basis for the non-Euclidean versions of Frégier's theorem discussed in [15].

The *ordinary offsets* to (non-circular) conics (see Fig. 1, center), *i.e.*, the set of points at a constant distance of a conic are algebraic curves of degree 8 (cf. [4, p. 118]). A curve g is called a *generalized offset* to a curve c if the distance function d (offset function) applied on the normals of c is not constant and depends on the curve point. The fact that generalized offsets of conics can be conics as well if the offset function is a special multiple of the cube root of the curvature function was first mentioned in [5] and thoroughly studied and clarified in [14]. The relations between the curvature function of a conic and the distance between curve points and Frégier points yields to the finding that the Frégier conics are the only so-called generalized offsets to a given conic which are also conics, see [11].

The algebraic and computational approach towards generalized Frégier conics in [11] delivers as a by-catch poristic triangle families interscribed in between the initial conic and two generalized Frégier conics, all belonging to the same pencil of conics of the third kind. Various generalizations of and variations on Frégier's theorem can be found in [1, 6, 9, 17, 18].

1.2 Contributions of the present paper

In the following, we shall give a 3-dimensional version of Frégier's theorem. The proofs of our results are not only based on computations. Especially the existence of the Frégier point of a point with respect to a quadric is proved by means of analytic geometry and synthetically. This proof is more general than that given in [17] (it covers all regular quadrics) and it is more rigorous than the one given in [2]. There are different types of quadrics within the setting of Euclidean geometry, and thus, their Frégier sets have to be treated separately. The case of the ellipsoid is treated first. The computations are explained in more detail here and rely mainly on the computations in the proof of the existence of the Frégier point. In order to improve the readability of the present note, we shall keep further computations as short as possible.

Sec. 2 discusses the Frégier quadrics of ellipsoids were we also explain the relation between the Gaussian curvature and the offset function between the given quadric and its Frégier quadric. Afterwards, we describe the Frégier sets of hyperboloids (see Sec. 3) and paraboloids (see Sec. 4). In the latter two cases, we see that ruled quadrics have ruled Frégier sets. We shall also see that regular quadrics can have singular Frégier quadrics. The short Sec. 5 is devoted to the minor appealing case of cones and cylinders. Finally in Sec. 6, we shall explain at hand of two examples how the planar (ordinary) Frégier construction applied to 1-parameter families of planar intersections of a given quadric results in Frégier sets different from the Frégier quadrics described in the previous sections.

2 Frégier points of quadrics with center

2.1 Moving tripods and the Frégier point

Assume that a quadric $\mathcal{Q}$ with center is given by its equation in terms of Cartesian coordinates as

$$\mathcal{Q}: \; q(\mathbf{x},\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Q}\,\mathbf{x} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1 \tag{1}$$

with $a, b, c \in \mathbb{R}^+$ and $\mathbf{Q} = \operatorname{diag}(a^{-2}, b^{-2}, \pm c^{-2})$. For the moment, we exclude quadrics with rotational symmetry, *i.e.*, $a \neq b \neq c \neq a$. Now, we show:

Theorem 1. *(Frégier's theorem in three-space.) Let $\mathcal{Q}$ be a regular quadric with center, let $P \in \mathcal{Q}$ be some point, and let $P_1, P_2, P_3 \neq P$ be three further points on $\mathcal{Q}$ such that the lines $[P, P_i]$ form a triple of pairwise orthogonal lines (Cartesian tripod). Then, the normal n of $\mathcal{Q}$ at P intersects the plane $\varepsilon := [P_1, P_2, P_3]$ in a point F independent of the pose of the tripod.*

Proof. We assume that the initial pose of the tripod placed at $P = (u, v, w) \in \mathcal{Q}$. Since $\mathcal{Q}$ is symmetric with respect to the planes $x = 0$, $y = 0$, $z = 0$, the points $P_1 = (-u, v, w)$, $P_2 = (u, -v, w)$, $P_3 = (u, v, -w)$ are also located on $\mathcal{Q}$. Hence, the legs of the initial tripod can be parametrized by $\mathbf{p}_k = \mathbf{p} + t \cdot \mathbf{b}_k$ (with $t \in \mathbb{R}$, $k \in \{1, 2, 3\}$, and the canonical basis vectors $\mathbf{b}_k$). The plane ε through the points P_k can be given by the equation $xu^{-1} + yv^{-1} + zw^{-1} = 1$. $\mathcal{Q}$'s normal n at P allows for the parametrization $n : \mathbf{x}(\lambda) = \mathbf{p} + \lambda\mathbf{n}$ with $\lambda \in \mathbb{R}$ and the normal vector $\mathbf{n} = \frac{1}{2}\operatorname{grad}\mathcal{Q}(\mathbf{p}) = (ua^{-2}, vb^{-2}, wc^{-2}) = \mathbf{Qp}$. Thus, the coordinate vector of $F = \varepsilon \cap n$ equals

$$\mathbf{f} = \mathbf{p} + \lambda_F \mathbf{Qp} \quad \text{with} \quad \lambda_F = -2a^2b^2c^2(a^2b^2 + b^2c^2 + c^2a^2)^{-1}. \tag{2}$$

(Note that the sign of c^2 has to be altered in the case of a hyperboloid.) The initial tripod centered at P can be moved to any other pose by simply applying a rotation about some axis through P. For that purpose, we assume that $\mathbf{D} \in \mathrm{SO}_3$ with entries d_{ij} is an orthogonal matrix (*i.e.*, $\mathbf{DD}^{\mathrm{T}} = \mathbf{I}_3$) with $\det \mathbf{D} = +1$. The legs of the tripod in the new pose can be parametrized by $\mathbf{p}'_k = \mathbf{p} + \lambda_k \mathbf{Db}_k$, where λ_k have to be determined such that $\mathbf{p}'_k \in \mathcal{Q}$. This means $q(\mathbf{p}'_k, \mathbf{p}'_k) = q(\mathbf{p}, \mathbf{p}) + 2\lambda_k q(\mathbf{p}, \mathbf{Db}_k) + \lambda_k^2 q(\mathbf{Db}_k, \mathbf{Db}_k) = 1$ which yields

$$\lambda_k = -2q(\mathbf{p}, \mathbf{Db}_k)/q(\mathbf{Db}_k, \mathbf{Db}_k). \tag{3}$$

From that, we find the equation of the plane ε' joining the endpoints $\mathbf{p}'_k$ as $\sum\limits_{k=1}^{3} \frac{1}{\lambda_k} \langle \mathbf{Db}_k, \mathbf{x} - \mathbf{p} \rangle = 1$. Finally, we have to verify that the coordinate vector of the Frégier point F (2) satisfies the equation of ε, *i.e.*, we have to show that $\sum\limits_{k=1}^{3} \frac{1}{\lambda_k} \langle \mathbf{Db}_k, \lambda_F \mathbf{Qp} \rangle = 1$. First, we see $\langle \mathbf{Db}_k, \mathbf{Qp} \rangle = \mathbf{p}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Db}_k = q(\mathbf{p}, \mathbf{Db}_k)$. This, together with (3), yields $\sum\limits_{k=1}^{3} \frac{1}{\lambda_k} \langle \mathbf{Db}_k, \lambda_F \mathbf{Qp} \rangle = -\frac{\lambda_F}{2} \sum\limits_{k=1}^{3} q(\mathbf{Db}_k, \mathbf{Db}_k)$. Second, we find $q(\mathbf{Db}_k, \mathbf{Db}_k) = \mathbf{b}_k{}^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{QDb}_k = d_{1k}^2 a^{-2} + d_{2k}^2 b^{-2} + d_{3k}^2 c^{-2}$. Since

$\mathbf{D}$ is an orthogonal matrix, its columns as well as its rows are unit vectors. Therefore, $\sum_{k=1}^{3} q(\mathbf{D}\mathbf{b}_k, \mathbf{D}\mathbf{b}_k) = a^{-2} + b^{-2} + c^{-2}$, and by virtue of (2), we have $\sum_{k=1}^{3} \frac{1}{\lambda_k} \langle \mathbf{D}\mathbf{b}_k, \lambda_F \mathbf{Q}\mathbf{p}\rangle = 1$.

In the case of a paraboloid, the point P_3 is to be replaced with the ideal point of the z-axis and $\mathcal{Q}$ is not diagonal, but still regular. □

Proof. (Synthetic proof of Thm. 1.) We use the tripod defined in the analytic proof. Assume that one leg, say $[P, P_3]$, is fixed and the tripod is rotated around it. The right angle hook with legs $[P, P_1]$, $[P, P_2]$ then rotates around P – P_1, P_2 move on the conic $e_3 = [P, P_1, P_2] \cap \mathcal{Q}$ – and, according to the planar Frégier theorem, the chord $[P_1, P_2]$ always passes through the (ordinary) Frégier point F_3 (see Fig. 2). Thus, during the rotation of the tripod, the line $[P_3, F_3]$ remains fixed, to be more precise, it remains the same for the plane ε is just changing, not rotating. The quadric's normal n at P lies coplanar with $[P, F_3]$ (normal to the planar curve e_3), hence F, P, and F_3 are collinear. Since any spatial rotation (around P) can be decomposed into rotations about the three legs of the initial tripod, at least one of the lines $[P, F_i]$ remains unchanged, and thus, it passes through the point F. Hence, the plane joining the endpoints of any tripod (centered at P) passes through the point F. □

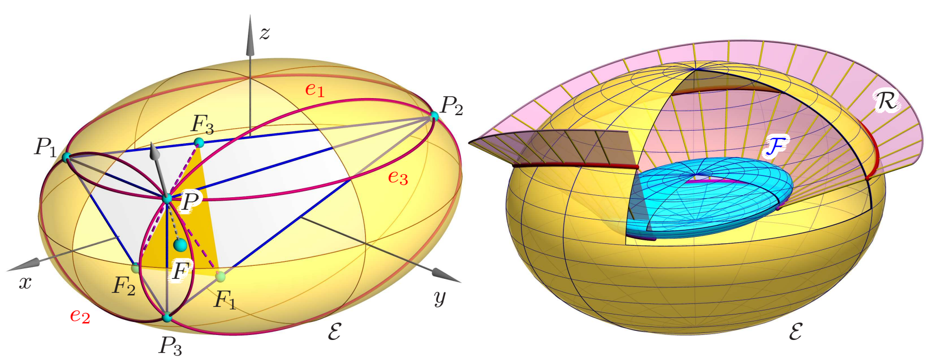


Fig. 2. Left: The Frégier point F of P with respect to $\mathcal{E}$ and the triple of ordinary Frégier points in the tripod's faces. Right: Frégier ellipsoid $\mathcal{F}$ of an ellipsoid $\mathcal{E}$ with a part of the equally wide strip of the ruled surface $\mathcal{R}$ joining the points of equal Gaussian curvature on $\mathcal{E}$ with the corresponding points on $\mathcal{F}$.

The linear mapping $\varphi: P \mapsto F$ that assigns to each point P the Frégier point F with respect to some quadric $\mathcal{Q}$ shall be called the *Frégier transformation*. The analytic representation given in (2) can be converted into matrix form: $\mathbf{f} = (\mathbf{I}_3 + \lambda_F \mathbf{Q})\mathbf{p}$, where λ_F depends on the quadric $\mathcal{Q}$. Compared with the planar case, φ is not necessarily a similarity.

2.2 The Frégier points of an ellipsoid

With the results from the analytic proof of Thm. 1, and by virtue of (1) and (2), we are able to describe the set of Frégier points of an ellipsoid. It means no restriction to assume that the ellipsoid's normal n at P can be parametrized by

$$N(t) = \mathbf{p} + \lambda \cdot \mathbf{n}, \quad t \in \mathbb{R}, \tag{4}$$

where $\mathbf{n} := W^{-1}(\frac{u}{a^2}, \frac{v}{b^2}, \frac{w}{c^2})$ with $W := \sqrt{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{w^2}{c^4}}$ is the ellipsoid's unit normal vector at P (which will be of importance later). This yields the coordinate representation of the Frégier point $F = \pi \cap n_P$ of P with respect to $\mathcal{E}$ as

$$F = \mathbf{p} + \lambda_F\, \mathbf{n}, \quad \text{with} \quad \lambda_F = -2a^2b^2c^2W\tau_{\mathcal{E}}^{-1}, \tag{5}$$

where $\tau_{\mathcal{E}} := a^2b^2 + b^2c^2 + c^2a^2$. Note that λ_F is the Euclidean distance between P and F. From (4), we can immediately infer:

Theorem 2. *The Frégier points with respect to an ellipsoid $\mathcal{E}$ form an ellipsoid $\mathcal{F}$, the* Frégier ellipsoid *of $\mathcal{E}$. $\mathcal{F}$ is co-axial with $\mathcal{E}$, and if the semiaxes lengths of $\mathcal{E}$ are $a, b, c \in \mathbb{R}^+$, then the semiaxes lengths a', b', c' of $\mathcal{F}$ are*

$$a' = a(\tau_{\mathcal{E}} - 2b^2c^2)\tau_{\mathcal{E}}^{-1}, \quad b' = b(\tau_{\mathcal{E}} - 2c^2a^2)\tau_{\mathcal{E}}^{-1}, \quad c' = c(\tau_{\mathcal{E}} - 2a^2b^2)\tau_{\mathcal{E}}^{-1}. \tag{6}$$

Proof. The Frégier transform (5) $\varphi : P \mapsto F$ fixes the center of $\mathcal{E}$ and its coordinate matrix equals diag$\,(a'a^{-1}, b'b^{-1}, c'c^{-1})$. Hence, the ellipsoid $\mathcal{E}$ is mapped to the ellipsoid mentioned in the theorem with the given semiaxis lengths. □

The Frégier sets of Ellipsoids whose semiaxis lengths are proportional to

$$(A^2 - B^2)(A^2 + B^2) : 2AB(A^2 \mp B^2) : 2AB(A^2 \pm B^2),$$

with $A{:}B \neq 0{:}0$ and $A, B \in \mathbb{R}$ are singular (planes of multiplicity two).

In planar geometry, Frégier conics (to non-circular Euclidean or pseudo-Euclidean conics) can only be singular in non-Euclidean planes, see [15].

There is another difference between planar and spatial Frégier sets: While the Frégier conic of a circle is its center, the Frégier set of a sphere is a sphere. Provided that $a = b = c = R$, (6) yields $a' = b' = c' = \frac{R}{3}$ implies: *The Frégier quadric of a sphere of radius R is a concentric sphere with radius $\frac{R}{3}$* (cf. [17]). In general, ellipsoids of revolution have Frégier sets with rotational symmetry.

2.3 Support function and Gaussian curvature of an ellipsoid

The ellipsoid (1) can be described by its support function $d :\ \mathrm{S}^2 \to \mathbb{R}$ which is the oriented distance function of $\mathcal{E}$'s tangent planes to the origin of the coordinate frame, *i.e.*, the center of $\mathcal{E}$. From the Hessian normal form of the tangent planes $\frac{1}{W}\left(\frac{xu}{a^2} + \frac{yv}{b^2} + \frac{zw}{c^2} - 1\right) = 0$, where (u, v, w) are the coordinates of a point P on $\mathcal{E}$, we obtain the support function by setting $x = y = z = 0$ and find

$$d = -W^{-1}. \tag{7}$$

According to [10, 13, 19], the support function d and the Gaussian curvature function K of the ellipsoid are related via

$$K = d^4 (abc)^{-2}. \tag{8}$$

By virtue of (5) and (7), we have $\lambda_F = 2a^2b^2c^2(d\,\tau_{\mathcal{E}})^{-1} = 2(abc)^{\frac{3}{2}} / \big(\tau_{\mathcal{E}} \sqrt[4]{K}\big)$.

This allows us to formulate the following result:

Theorem 3. *The Frégier set $\mathcal{F}$ of an ellipsoid $\mathcal{E}$ is a generalized offset of $\mathcal{E}$ in the shape of an ellipsoid with a multiple of the fourth root of the Gaussian curvature as offset function.*

We recall that the ordinary offset of an ellipsoid, *i.e.*, the set of all points at constant distance from the ellipsoid is not an ellipsoid (cf. [13, p. 429]). Thm. 3 is the 3-dimensional version of the result for conics given in [11, Thm. 1.1, p. 2].

3 Hyperboloids and their Frégier sets

3.1 Ruled and oval hyperboloids

There are two different types of hyperboloids: one-sheeted and two-sheeted hyperboloids. These hyperboloids can be given by their equations as

$$\mathcal{H}_1 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{and} \quad \mathcal{H}_2 : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \tag{9}$$

where $a, b, c \in \mathbb{R}^+$ are the semiaxis lengths.

For the moment, we exclude the case $T := \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} = 0$ and state:

Theorem 4. *The set of all Frégier points of a one- / two-sheeted hyperboloid with $T \neq 0$ is a one- / two-sheeted hyperboloid. If the semiaxis lengths of both are $a, b, c \in \mathbb{R}^+$, the semiaxis lengths of the Frégier quadrics $\mathcal{F}_1$ and $\mathcal{F}_2$ are*

$$a' = a\left(\tau_{\mathcal{H}} + 2b^2c^2\right)\tau_{\mathcal{H}}^{-1}, \quad b' = b\left(\tau_{\mathcal{H}} + 2c^2a^2\right)\tau_{\mathcal{H}}^{-1}, \quad c' = c\left(\tau_{\mathcal{H}} - 2a^2b^2\right)\tau_{\mathcal{H}}^{-1},$$

where $\tau_{\mathcal{H}} := a^2b^2 - b^2c^2 - c^2a^2$.

Proof. The proof follows nearly the same tracks as the proof of Thm. 2. $\square$

The Frégier quadrics of hyperboloids of revolution are also hyperboloids of revolution. In Thm. 4, we have excluded those quadrics whose equations involve a quadratic form with vanishing trace. Such hyperboloids can carry the regulus of the altitudes of a tetrahedron (see [7, 8]). The vanishing trace $T = 0$ is equivalent to $\tau_{\mathcal{H}} = 0$, and thus, the Frégier quadrics degenerate and become the plane at infinity. This case could be viewed as the spatial analog to the case of an equilateral hyperbola.

For both quadrics in (2), the choice of principal axes lengths in the ratios

$$\begin{aligned}
&(A^2 - B^2)(A^2 + B^2) : 2AB(A^2 - B^2) : 2AB(A^2 + B^2),\\
&2AB(A^2 - B^2) : (A^2 - B^2)(A^2 + B^2) : 2AB(A^2 + B^2)
\end{aligned}$$

yield singular Frégier quadrics which are planes with multiplicity two.

A parametrization of the Frégier set of a one-sheeted hyperboloid reads

$$F = \mathbf{p} + t_F \mathbf{n} \quad \text{with} \quad t_F = 2a^2b^2c^2W\tau_{\mathcal{H}}^{-1} \tag{10}$$

(which is analog to (5)) and with the Frégier transform φ, the following is clear:

Theorem 5. *The Frégier points F of all points on a ruling of a one-sheeted hyperboloid $\mathcal{H}_1$ form a ruling of the Frégier hyperboloid $\mathcal{F}$.*

Thm. 5 immediately implies the following result:

Theorem 6. *The Frégier quadric $\mathcal{F}$ of a one-sheeted hyperboloid $\mathcal{H}_1$ is always a ruled hyperboloid, and thus, one-sheeted.*

The contents of Thms. 5 and 6 are illustrated in Fig. 3 and are natural consequences of the Frégier transformation.

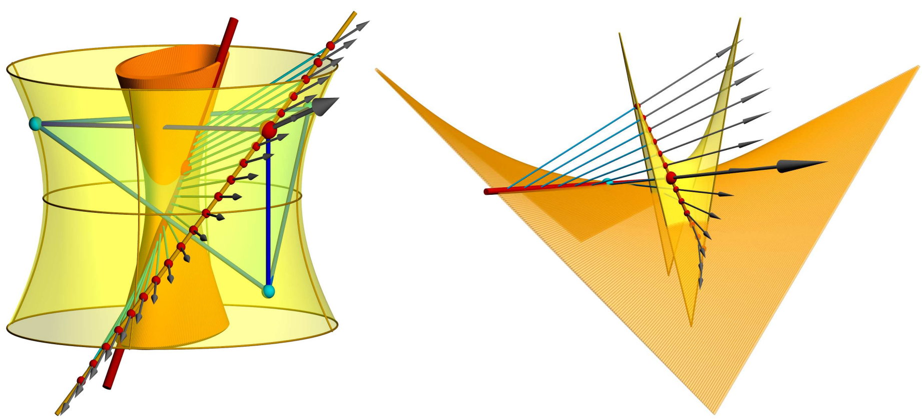


Fig. 3. The Frégier quadric of a ruled quadric is also ruled: one-sheeted hyperboloid (left), hyperbolic paraboloid (right).

3.2 Support function and Gaussian curvature of a hyperboloid

The Gaussian curvature function of the two-sheeted hyperboloid $\mathcal{H}_2$ equals that of the ellipsoid given in (8). For the one-sheeted hyperboloid $\mathcal{H}_1$, the Gaussian curvature is given by

$$K = -d^4(abc)^{-2}. \tag{11}$$

The support function of the two-sheeted hyperboloid equals

$$d = W^{-1}. \tag{12}$$

In analogy to Thm. 3, we can formulate:

Theorem 7. *The Frégier quadrics of one- or two-sheeted hyperboloids are their generalized offsets which are also one- or two-sheeted hyperboloids.*

Proof. Use the expression relating the Gaussian curvature K and the support function d in order to express the parameter t_F as a function of d on the unit surface normal. □

4 Frégier sets of paraboloids

4.1 Ruled and oval paraboloids

We have to distinguish between elliptic and hyperbolic paraboloids. While the elliptic paraboloid $\mathcal{P}_e$ is oval, the hyperbolic paraboloid $\mathcal{P}_h$ is ruled. With the very properties of the Frégier transformation, we can say:

Theorem 8. *The set of Frégier points of all points on elliptic or hyperbolic paraboloids are elliptic or hyperbolic paraboloids. The Frégier points of the points on a ruling of a hyperbolic paraboloid form a ruling of the Frégier paraboloid.*

If $\mathcal{P}_e$ and $\mathcal{P}_h$ are given by their equations (in normal form, cf. [13])

$$\mathcal{P}_e: \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=2z, \quad \mathcal{P}_h: \ \frac{x^2}{a^2}-\frac{y^2}{b^2}=2z, \tag{13}$$

then the respective Frégier quadrics are

$$\begin{aligned}\mathcal{F}_e &: (a^2+b^2)^3\left(b^2x^2+a^2y^2\right)-2a^2b^2(a^2+b^2)(a^2-b^2)^2z+4a^4b^4(a^2-b^2)^2=0,\\ \mathcal{F}_h &: (a^2-b^2)^3\left(b^2x^2-a^2y^2\right)-2a^2b^2(a^2-b^2)(a^2+b^2)^2z-4a^4b^4(a^2+b^2)^2=0.\end{aligned} \tag{14}$$

Hence, the Frégier set of a paraboloid of revolution ($\mathcal{P}_e$ with $a=b$) is a pair of complex conjugate planes through the axis of $\mathcal{P}_e$. The Frégier set of an orthogonal hyperbolic paraboloid $\mathcal{P}_h$ with $a=b$ is the ideal plane with multiplicity two.

4.2 Generalized offsets to paraboloids

The Gaussian curvature functions of paraboloids $K_\star$ do not solely depend on the support functions d, they also involve the coordinate function in the direction of the paraboloids' axis (*i.e.*, in our setting $K_{e,h}=\pm d^4(ab)^{-2}z^{-4}$). The Gaussian curvature functions K_e, K_h of $\mathcal{P}_e$ and $\mathcal{P}_h$ from (13) can be expressed as

$$K_{e,h}=\pm(ab)^{-2}W^{-4},$$

where $W:=\sqrt{1+\frac{x^2}{a^4}+\frac{y^2}{b^4}}$, while the distances between the surface points $(x,y,\star)$ and the corresponding Frégier points are given by

$$\lambda_{F_{e,h}}=\pm 2a^2b^2\,W\,(a^2\pm b^2)^{-1},$$

where we have to choose either the upper or the lower sign. Hence, the distance $\lambda_{F_{e,h}}$ and the Gaussian curvature are related via

$$K_e=16a4b^4f_e^{-4}(a^2+b^2)^{-4}, \quad K_h=16a^4b^4f_h^{-4}(a^2-b^2)^{-4}.$$

Thus, in analogy to Thm. 3 and Thm. 7, we can say:

Theorem 9. *The Frégier paraboloids of elliptic/hyperbolic paraboloids are the generalized offsets which are again elliptic/hyperbolic paraboloids.*

5 Quadratic cones and cylinders

A quadratic cone without rotational symmetries can be given by its equation as

$$\mathcal{K}:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0 \tag{15}$$

with $a,b,c\in\mathbb{R}^+$. It is no restriction to assume that $a>b$ holds, while c can take almost any value in $\mathbb{R}^+$. The computation of the Frégier point of a point $P=(u,v,w)\in\mathcal{K}$ can be done as outlined in Sec. 2.2. So, we find

$$F=\Big(u\left(\tau_{\mathcal{K}}+2b^2c^2)\tau_{\mathcal{K}}^{-1}\right),v\left(\tau_{\mathcal{K}}+2c^2a^2)\tau_{\mathcal{K}}^{-1}\right),w\left(\tau_{\mathcal{K}}-2a^2b^2)\tau_{\mathcal{K}}^{-1}\right)\Big),$$

where $\tau_{\mathcal{K}}:=a^2b^2-b^2c^2-c^2a^2$. This can be formulated as:

Theorem 10. *The set of Frégier points of a quadratic cone* (15) *is a quadratic cone with the equation*

$$\mathcal{F}:\ \frac{x^2}{a^2(\tau_{\mathcal{K}}+2b^2c^2)^2}+\frac{y^2}{b^2(\tau_{\mathcal{K}}+2c^2a^2)^2}-\frac{z^2}{c^2(\tau_{\mathcal{K}}+2a^2b^2)^2}=0$$

which is singular if, and only if, $c=\frac{ab}{\sqrt{a^2-b^2}}$.

The Frégier cone $\mathcal{F}$ of the cone of revolution $\mathcal{K}:k^2(x^2+y^2)=z^2$ (with $k\in]0,\frac{\pi}{2}[$, *i.e.*, with angle of aperture equal to $\omega=2\operatorname{arccot}k$) is the cone of revolution with the equation $\mathcal{F}:(2k^2+1)^2(x^2+y^2)=z^2$. The angle of aperture of $\mathcal{F}$ equals $\omega_{\mathcal{F}}=2\operatorname{arccot}(2k^2+1)$.

Thm. 10 describes the Frégier conics in the elliptic plane. Obviously, singular Frégier conics to regular conics can occur (see also [15]).

For quadratic cylinders we can state:

Theorem 11. *The Frégier quadrics of quadratic (elliptic — non-circular, parabolic, hyperbolic — not equilateral) cylinders are cylinders of the same affine type erected above the Frégier conic of a planar cross section orthogonal to the generators.*

The Frégier set of a cylinder of revolution is its axis. The Frégier set of an equilateral hyperbolic cylinder is the plane at infinity.

Thm. 11 can be verified by means of the Frégier transformation or by direct computation with a rational representation of the rotation matrix $\mathbf{D}$. Finally, we note that the Frégier quadric of a cylinder (except those two cases excluded in Thm. 11) is always regular.

For both types of singular quadrics, the Frégier sets are *not generalized offsets* of the initial quadric in the sense of Thms. 3, 7, 9 since their Gaussian curvature equals zero (all over the surface).

6 The planar version of Frégier's theorem in connection with quadrics

We can apply the planar Frégier transform to any planar intersection of a quadric $\mathcal{Q}$. Since there is a 3-parameter family of planes in 3-dimensional space, we find a 3-parameter family of intersection conics, and thus, a 3-parameter family of Frégier conics associated with the latter.

On the other hand, we can look at partial flags $F = (P, \varepsilon)$, *i.e.*, a plane ε with an incident point P. Thus, there exists a unique Frégier point F assigned to P with respect to the intersection $e = \varepsilon \cap \mathcal{Q}$. If the plane component of F traverses a pencil of planes, we obtain a 1-parameter family of conics sweeping a surface.

In the following, we shall choose two relatively simple subsets of planes.

6.1 Normal sections along principal tangents

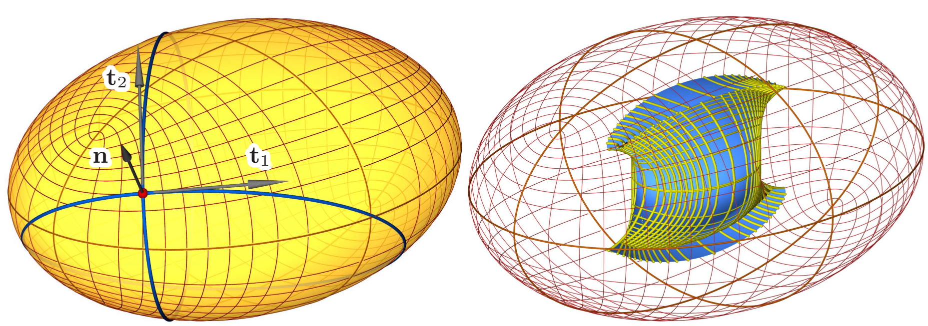


Fig. 4. The Frégier points of a point $P \in \mathcal{E}$ with respect to the two normal sections of $\mathcal{E}$ in the directions of the principal tangents (left) are located on a single surface.

The (ordinary) Frégier points of the normal sections of the ellipsoid $\mathcal{E}$ given in (1) in the direction of the principal curvature lines are

$$F_1 = \begin{pmatrix} e_1 \frac{a^2b^2c^2 + (\tau_E - 2b^2c^2)v}{a^2b^2c^2 + \tau_E v} \\ e_2 \frac{a^2b^2c^2 + (\tau_E - 2c^2a^2)v}{a^2b^2c^2 + \tau_E v} \\ e_3 \frac{a^2b^2c^2 + (\tau_E - 2a^2b^2)v}{a^2b^2c^2 + \tau_E v} \end{pmatrix}, F_2 = \begin{pmatrix} e_1 \frac{a^2b^2c^2 + (\tau_E - 2b^2c^2)u}{a^2b^2c^2 + \tau_E u} \\ e_2 \frac{a^2b^2c^2 + (\tau_E - 2c^2a^2)u}{a^2b^2c^2 + \tau_E u} \\ e_3 \frac{a^2b^2c^2 + (\tau_E - 2a^2b^2)u}{a^2b^2c^2 + \tau_E u} \end{pmatrix}, \quad (16)$$

where e_i are the coordinate functions of the principal parametrization of $\mathcal{E}$:

$$\mathbf{e} = \left(a\sqrt{\frac{(a^2+u)(a^2+v)}{(a^2-b^2)(a^2-c^2)}}, b\sqrt{\frac{(b^2+u)(b^2+v)}{(b^2-c^2)(b^2-a^2)}}, c\sqrt{\frac{(c^2+u)(c^2+v)}{(c^2-a^2)(c^2-b^2)}} \right).$$

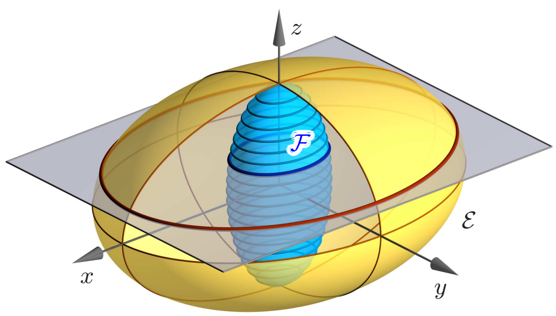
Both parametrizations in (16) fulfill the equation of a single surface $\mathcal{F}$ of degree 12 that passes three times through the intersection of $\mathcal{E}$ with the ideal plane.

6.2 Pencils of parallel planes

Among the many subsets in the manifold of planes we choose the following special example and show:

Theorem 12. *The 1-parameter family of Frégier ellipses f of the planar intersections e of an ellipsoid $\mathcal{E}$ with the planes of a pencil of parallel planes form an ellipsoid $\mathcal{F}$ which touches $\mathcal{E}$ in a pair (T_1, T_2) of opposite points. At the points T_1, T_2 the common tangent planes of $\mathcal{E}$ and $\mathcal{F}$ are parallel to the planes of the pencil,* i.e., *$[T_1, T_2]$ is $\mathcal{E}$'s (and also) $\mathcal{F}$'s diameter conjugate to the (ideal) axis of the pencil of planes.*

Proof. It is sufficient to verify the theorem for the special case of the planes $z = w =$ const. and the ellipsoid (1). The generic case can be obtained by applying a shear transformation in the direction of the horizontal planes $z = w$ with $|w| \leq c$.



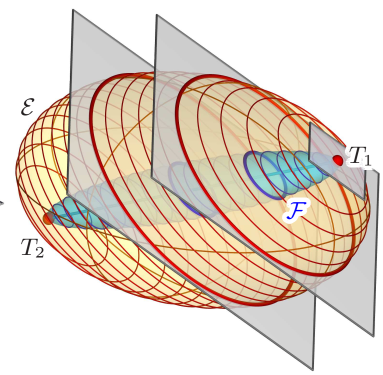


Fig. 5. Left: The Frégier ellipsoid $\mathcal{F}$ is the union of the Frégier conics of the horizontal sections of $\mathcal{E}$. Right: The Frégier ellipsoid $\mathcal{F}$ of arbitrary parallel sections.

In the special case, the horizontal sections (intersections of $\mathcal{E}$ with the planes of the pencil) are given by the equations $x^2a^{-2} + y^2b^{-2} = 1 - w^2c^{-2}$. Hence, its semiaxis lengths are equal to $a/c\sqrt{c^2 - w^2}$ and $b/c\sqrt{c^2 - w^2}$. According to [4, p. 257], the semiaxis lengths of the Frégier conics are $a' = a/\sqrt{c^2 - w^2}(a^2 - b^2)/(a^2 + b^2)$ and $b' = b/\sqrt{c^2 - w^2}(a^2 - b^2)/(a^2 + b^2)$, Thus, in each plane $z = w$, the equations of the Frégier conics can be given as $\frac{x^2}{\frac{a^2}{c^2}(c^2 - w^2)\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} + \frac{y^2}{\frac{b^2}{c^2}(c^2 - w^2)\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} = 1$. After substituting $z = w$ and reshaping, we find the equation of $\mathcal{F}$

$$\mathcal{F}: \ \frac{x^2}{\frac{a^2}{c^4}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} + \frac{y^2}{\frac{b^2}{c^4}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2} + \frac{z^2}{c^2} = 1.$$

Now, it is obvious that $\mathcal{F}$ is an ellipsoid and touches $\mathcal{E}$ in the pair of points $(0, 0, \pm c)$ which span the diameter conjugate to the horizontal plane. $\square$

A computational proof for the *generic case*, *i.e.*, a diameter that is not orthogonal to the planes in the pencil is best done with a curvature line parametrization of $\mathcal{E}$. Thm. 12 is also valid for one- and two-sheeted hyperboloids.

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