On the geometry of spherical trochoids

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Abstract
We provide a synthetic study of the top-views of spherical trochoids. These projections turn out to be higher trochoids, *i.e.*, curves generated by the superposition of more than two rotations. Special shapes of these trochoids show up for special choices of the spherical radii of the rolling circles. A relation to closed algebraic curves of constant width is shown. These curves allow for a kinematic generation.

Key words: spherical trochoid, rolling, evolute, involute, curve of constant width

MSC 2020: 53A17, 51N05 (primary), 14H45 (secondary)

Sačetak
Nudimo sintetičku studiju pogleda odozgo sfernih trohoida. Ispostavilo se da su te projekcije više trohoide, tj. krivulje generirane superpozicijom više od dvije rotacije. Posebni oblici ovih trohoida pokazuju se posebnim odabirom sfernih polumjera kotrljajući krugovi. Prikazan je odnos prema zatvorenim algebarskim krivuljama konstantne širine. Ove krivulje omogućuju kinematičku generaciju.

Ključne riječi: spherical trochoid, rolling, evolute, involute, curve of constant width

1 Introduction

1.1 Motivation, prior work, and contributions of the present paper

This paper is devoted to the memory of Walther Jank (1939–2016). An unpublished and hand written manuscript of a talk given by W. Jank at the *Geometrietagung* in Vorau (Austria) in June 2004 was the basis of this article. It deals with the *geometric* deduction of results on the shapes of the top-views of spherical trochoids. Since W. Jank was a dedicated follower of Walter Wunderlich's work of merit on kinematics (cf. [20]) and especially on trochoids and higher trochoids (see [19]), he applied some of these results to spherical trochoids which have gained a little less attention than their planar counterparts.

There exist only a few notable publications on spherical trochoidal curves related to W. Jank's manuscript. In [6], we find historical remarks and a collection of known results. Maybe, it was Rudolf Bereis who first described the images of spherical trochoids under various parallel projections in [1].

This article shall first follow W. Jank's manuscript, *i.e.*, we lay down his results and his reasoning. This includes a detailed description of spherical trochoids based on a constructive approach. The kinematic generation of the top-views of spherical trochoids leads to the finding that some of these top-views are curves of constant width.

Moreover, a synthetic proof of ENNEPER's theorem on the shape of the top-views of curves of constant slope on ellipsoids of revolution (with their axis in lead direction, *i.e.*, in the direction of the projection) can be found along the way.

At the end of the manuscript, the author raised the question whether it is possible to describe planar algebraic and closed curves of constant width, *i.e.*, planar curves whose projection onto a line (within their plane) is a segment of fixed length independent of the direction of the projection, see [17]. Such curves, comparable to the example given in Fig. 14, were derived in [14]. The results therein were veryfied and improved by [12] and the related Zindler curves were described in [15]. The approaches towards curves of constant width in these references are analytic and algebraic in nature, and by no means, constructive or geometric. We shall close this gap.

The present paper is organized as follows: The remainder of this section describes the constructive treatment of spherical trochoids and discusses the kinematic generation. Special cases occur for special assumptions on the spherical radii of the rolling circles which causes special shapes of the curves and their top-views. We try to follow W. JANK's diction by trying to translate his manuscript as direct as possible. This does not necessarily include the original notation and symbols. In Sec. 2, a special spherical trochoid and its top-view are the starting point for the investigation of algebraic curves of constant width and their kinematic generation.

1.2 Generation of spherical trochoids

In the three-dimensional Euclidean space $\mathbb{R}^3$ of our perception, we distinguish a certain direction L (lead direction) and a fixed sphere Σ centered at O. Further, we assume that the equator e lies in the horizontal plane through Σ's center O (*i.e.*, in the plane orthogonal to the lead L and through O). On a fixed circle $p_0 \subset \Sigma$ (fixed polhode) with its axis parallel to L, spherical center M_0, and spherical radius $\widehat{r_0}$, we roll another circle $p \subset \Sigma$ (moving polhode) with spherical center M and spherical radius $\widehat{r}$. The

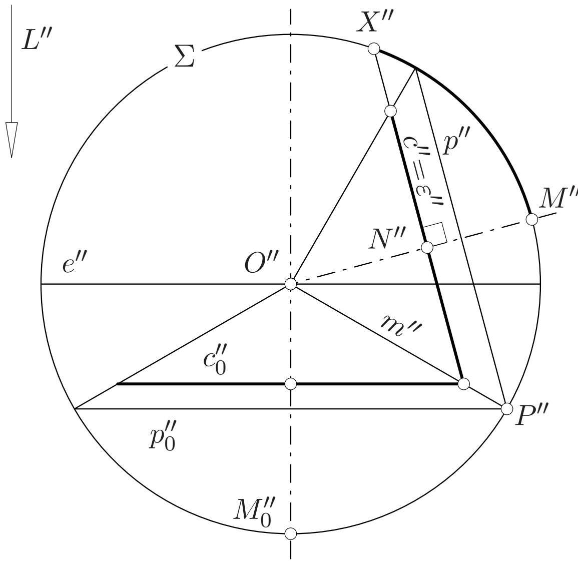

Figure 1: Front-view of the initial configuration of the rolling cones and circles.

path $l \subset \Sigma$ of an arbitrary point $X \in \Sigma$ firmly attached to p is called a *spherical trochoid* of *order 2*. Note that any point rigidly attached to p and not necessarily on Σ traces a spherical trochoid on a sphere concentric with Σ.

The spherical trochoid motion can also be considered as the glide-free rolling of the cone of revolution $\Gamma = p \vee O$ along the cone (of revolution) $\Gamma_0 = p_0 \vee P$ (sharing the vertex O) during the entire motion. The point P is the point of contact of c and c_0 and is also referred to as the spherical instantaneous pole (see Fig. 1). Γ is rolling on Γ_0 without gliding. These cones play the role of the axodes and the instantaneous axis equals the common generator $m = [O, P]$ of these two cones along which they share the tangent plane (cf. [5, 16]).

For the constructive treatment of spherical trochoids, we intersect Σ with the plane ε which is orthogonal to the axis $[O, M]$ of p and passes through X. Then, we consider the rolling of the parallel circle $c = \varepsilon \cap \Sigma$ (center $N = \varepsilon \cap [O, M]$) together with the point X on the fixed cone's parallel circle c_0 (in the plane ε_0, with the spherical radius $\widehat{r_0}$, and axis $[O, M_0]$).

We shall make explicit that each spherical (or planetary) trochoidal motion is equivalent to the (glide-free) rolling of a sphere S on two coaxial circles c_1 and c_2, see Fig. 3.

The tangent of l at X is orthogonal to the (spherical) instantaneous pole P.

Spherical kinematics mirrors another well-known result from planar kinematics. In the Euclidean plane, the theorem by S. Aronhold and A.B.W. Kennedy (cf. [20]) states that *the instantaneous poles P_{01}, P_{02}, P_{12} of the relative motions of three moving systems Σ_0, Σ_1, Σ_2 (concentric with and congruent to Σ) are collinear.* Further the relative angular velocities ω_{01}, ω_{02} and the distances between the poles are related by

$$\overline{P_{01}\,P_{12}} : \overline{P_{02}\,P_{12}} = \omega_{02} : \omega_{01}.$$

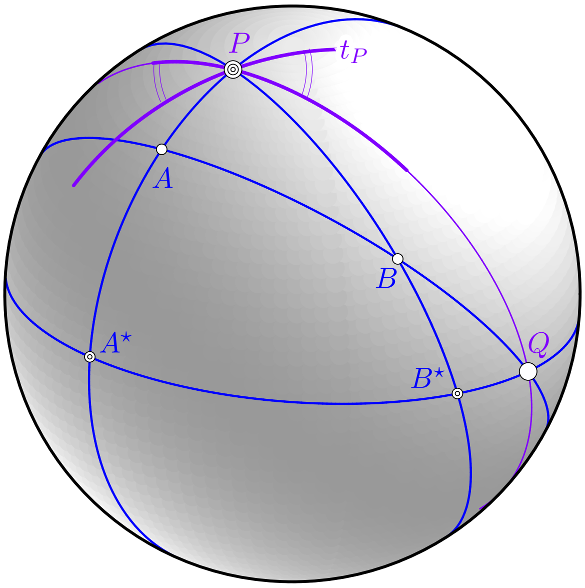

Figure 2: Construction of osculating circles of the spherical trochoid l at X according to Bobillier.

The center of the osculating circle of l at X can be constructed with the help of É. Bobillier's construction (cf. [20]) which is also valid on the sphere. This result holds also in spherical kinematics, see [5, 10, 16].

1.2.1 Top-views, trochoids of higher order

The following results on the top-views (orthogonal projections in the direction of the lead L) of spherical trochoids were deduced by W. Ströher in an analytic way (see [16]). Here, these results shall be proved by means of synthetical reasoning. In

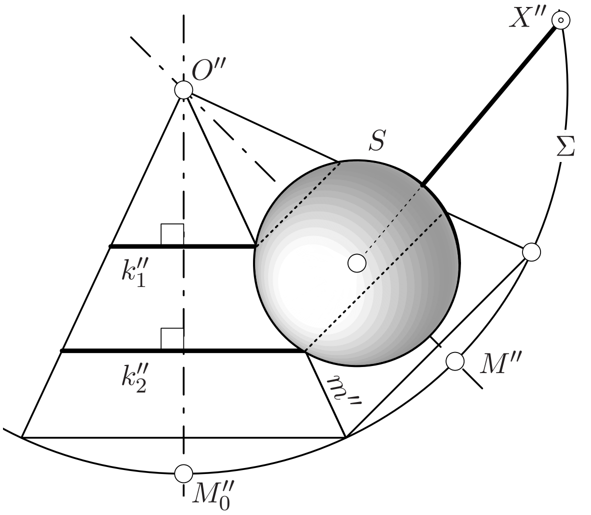

Figure 3: Alternative generation of a spherical trochoid: A sphere S is rolling on two coaxial circles k_1 and k_2.

the beginning, we recall a theorem by H. Pottmann (cf. [13] and see Fig. 4):

Theorem 1.1. *Let $\overline{k}'$ be an ellipse with center N', semi-major axis length a, and the moving point $X' \in \overline{k}'$. Assume further that the angular velocities of the rods $N'S$ and SX' in the crank slider mechanism $N'SX'$ (derived from the paper strip construction of $\overline{k}'$) are equal to $-\beta$ and β (with regard to $\overline{k}'$) and let $\overline{k}'_0$ be the ellipse's circumcircle (which is an affine image of $\overline{k}'$). Then, any two out of the following three statements are equivalent:*

- $\beta = \text{const.}$
- *$N'X'_0$ rotates with constant angular velocity, and therefore, also constant area velocity (with regard to $\overline{k}'_0$).*
- *$N'X'$ rotates with constant area velocity with respect to $\overline{k}'$.*

The top-view of the situation shown in the front-view in Fig. 1 is displayed in Fig. 5. From the latter we can deduce some results on the top-views of spherical trochoids:

Theorem 1.2. *The top-view l' of a spherical trochoid l is (in general) a trochoid of order 3 (cf. [19, 20]).*

Proof. We see that $\overline{k}'$ rotates with angular velocity α about O'. Provided that α is constant, NX rotates with constant angular and area velocity (with respect to $\overline{k}$) according to Thm. 1.1. Thus, $N'X'$ rotates with constant area velocity with respect to $\overline{k}'$. Because of the existence of the affine mapping between the ellipse and its circumcircle, $N'X'$ rotates with constant area velocity $-\beta$ with respect to $\overline{k}'$. Hence, $N'X'$ moves with constant and absolute angular velocity $\alpha - \beta(\alpha + \beta)$. □

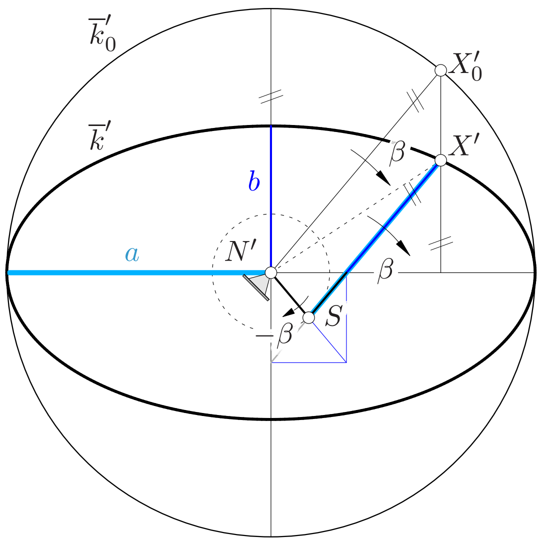

Figure 4: The crank slider mechanism and the equivalencies around an ellipse.

In [1], it is already mentioned that the top-view (orthogonal projection in the direction of the axis of the fixed cone) is a trochoid of order 3. Moreover, R. Bereis has shown that the generic orthogonal projection of a spherical trochoid of order 2 is a planar trochoid of order 5, and a generic (oblique) parallel projection results in a planar trochoid of order 8 (see also [1]). This means that the latter curves are path curves of points under planar motions which are the superpositions of 5 or 8 planar rotations (cf. [19]).

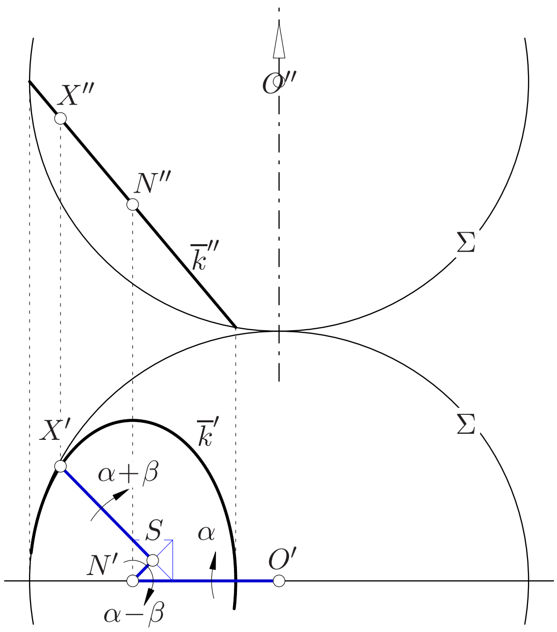

Figure 5: The top view of a spherical trochoid is a planar trochoid of order three. It can be generated by an open three-bar mechanism.

More precisely, we can infer:

Theorem 1.3. *The top-view l' of a spherical trochoid l is, in general, a trochoid of order 3, and its characteristic equals*

$$\alpha : (\alpha - \beta) : (\alpha + \beta),$$

cf. [19] and [20, p. 164]. It can be generated by the open-loop three-bar mechanism $O'N'SX'$.

In the special case $\hat{b} = \widehat{MX} = \frac{\pi}{2}$ and $N = O$, l' has the characteristic

$$(\alpha - \beta) : (\alpha + \beta). \tag{1}$$

In this case, a great circle $\overline{k}$ is rolling, taking the point $X \in \overline{k}$ with it. Hence, l a *spherical involute* of a (spherical) circle, and also, a spherical curve of constant slope. Naturally, l' is a curve with cusps gathering on a circle which is concentric with the equator's top view e'. (It is the top view of that parallel circle of Σ along which Σ's tangent planes have the same slope as l.) The vertices of l' lie on e'. By virtue of (1), l' is an epicycloid.

Referring to the very special case of spherical trochoids l as curves of constant slope on Σ, we shall point out the following: It is possible to transform the sphere Σ into ellipsoids of revolution by applying orthogonal affine mappings with the equator plane as a fixed plane (corresponding points are joined by lines orthogonal to the equator plane). Although such an orthogonal affine mapping changes the value of the slope of l, the slope remains constant. Some examples of curves of constant slope are shown in Fig. 6. Hence, we have verified that part of Enneper's theorem (see [7, p. 138] and [11, p.462]) describing the shape of curves of constant slope on ellipsoids of revolution (see Fig. 7): *The top-view (orthogonal projection in the direction of the lead L) of a curve of constant slope on an ellipsoid of revolution is an epicycloid, provided that the axis of revolution is parallel to L.*

In Fig. 8, the top-view of the case of congruent polhodes k_0 and k_1 is illustrated. In

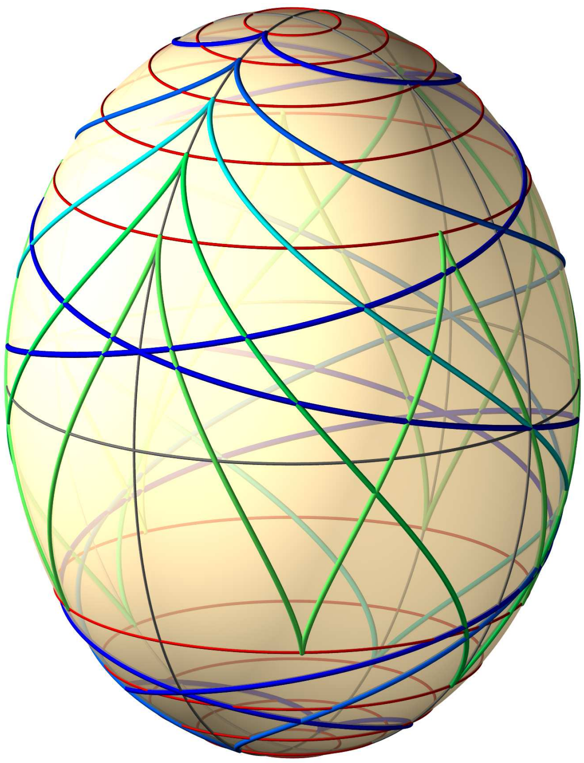

Figure 6: Some curves of constant slope on an ellipsoid of revolution with vertical axis.

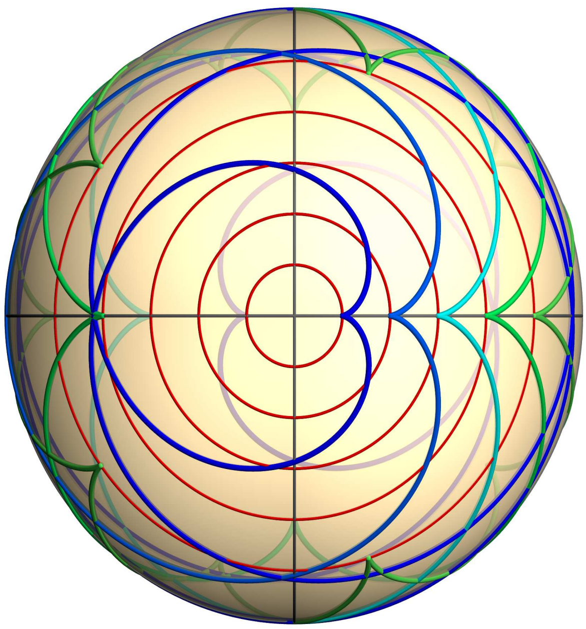

Figure 7: The top-view of the curves of constant slope on an ellipsoid shows some epicycloids.

the top-view, we can see a so-called *symmetric rolling* if we flip the moving circle k_1[1] into the horizontal plane of the fixed circle k_0. So, we see that the locus $l^{\circ\prime}$ of all points $X_i^{\circ\prime\prime}$ (*i.e.*, the orbit of $X_1^{\circ\prime}$ or $X_2^{\circ\prime}$) equals a Pascal limaçon. Further, we can deduce that the top-view l' of the spherical trochoid is also a limaçon which is a similar and smaller copy of $l^{\circ\prime}$. The mapping $\zeta:\ l^{\circ\prime} \to l'$ is a central similarity with center Z (cf. Fig. 8) and similarity factor

$$0 < \mu = \frac{1}{2}(1 + \cos\nu) < 1, \tag{2}$$

where ν is the angle enclosed by the planes of the moving circles (on Σ) and the horizontal planes.

In Fig. 9, another special case is illustrated: A great circle $\overline{k} \subset \Sigma$ is rotating about Σ's vertical axis while its radius OX rotates with the same absolute angular velocity. By rotating the initial position ε_1 (which is projecting in the front-view) into a generic position ε_2, we find that the interior angle bisector of $[O', X_1']$ and $[O', X_2^{\circ\prime}]$ equals the trace of ε_2 in the equator plane. Therefore, l' is the image of e' under a central similarity ζ with center X_1' and the similarity factor (2). Hence, l' is a circle.

In the much more special case $\nu = \frac{\pi}{2}$, we have $\mu = \frac{1}{2}$, and it is rather obvious that the latitude and the longitude of each point $X \in l$ are equal, provided that Σ is considered as the Earth and the contour for the

[1]Here, the indices 1, 2, . . . assigned to the moving circle refer to different (time) instances.

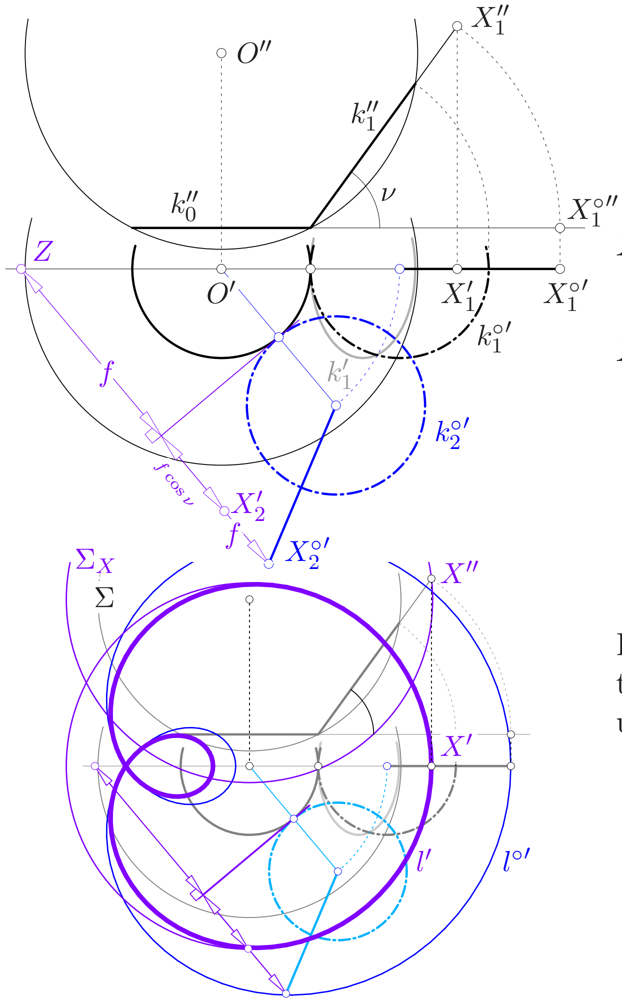

Figure 8: Top: The similarity factor between l' and $l^{\circ\prime}$ depends on the inclination of the rolling circle's plane. Bottom: The fixed and moving polhodes are congruent and the top-view shows a symmetric rolling. Therefore, l' is a Pascal limaçon, as is $l^{\circ\prime}$.

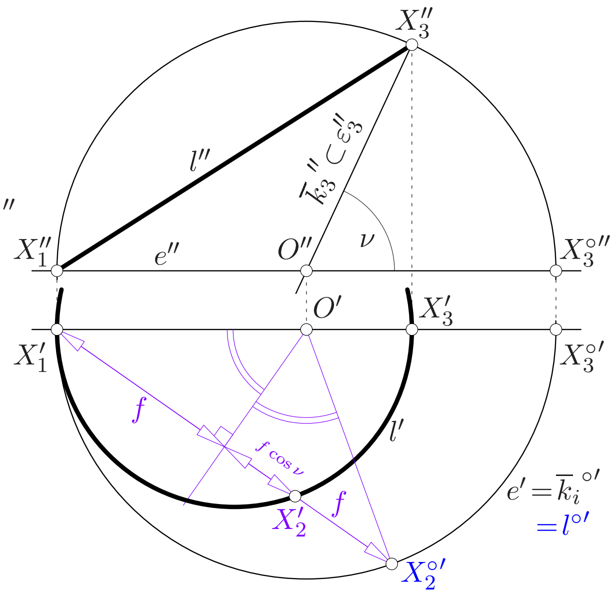

Figure 9: A very simple form of a spherical trochoid which is still a similar copy of an undistorted image: a circle.

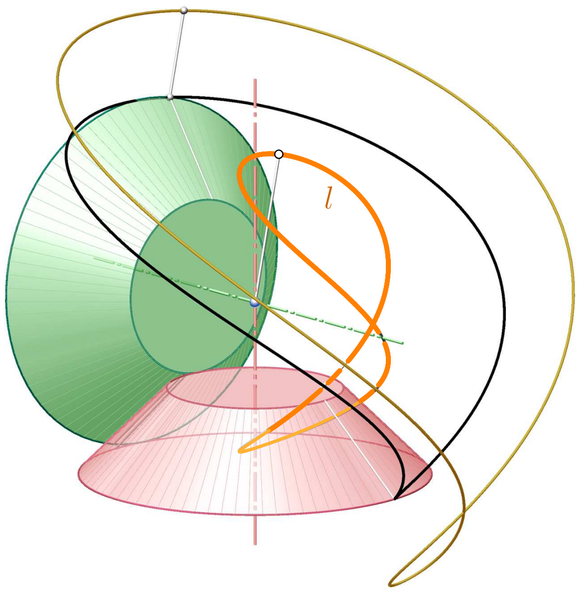

Figure 10: Viviani's curve (orange) can also be found among the spherical trochoids.

top-view is assumed to be the zero meridian. In this case, l is Viviani's curve (see Fig. 10, the orange curve l).

In Fig. 11, we recall again the constructive approach and flip the plane ε (including k, N, and X) to both sides, *i.e.*, to the interior and exterior of the sphere. For the inner version, this yields the circle k° with the center N° and radius r_1. The moving point shall be denoted by X°. The outer circle $k_\circ$ has the center $N_\circ$, the radius r_2, and the moving point shall be labelled with $X_\circ$. Then, we complete the parallelograms

$$O'N^{\circ\prime}X^{\circ\prime}Q_1 \text{ and } O'N'_\circ X'_\circ Q_2.$$

Now, we have $0 < r_0$, $0 < r_1 < r_0$, $r_1 = -r_2$, and $\alpha = r_2 = \text{const.}$, see Fig. 11. If now $O'N^{\circ\prime}N'_\circ$ rotates with the angular velocity α, then $O'Q_i$ rotates with angular velocity β_i ($i \in \{1, 2\}$), where

$$0 < \beta_1 = r_0 + r_2 \text{ and } 0 > \beta_2 = -r_0 + r_2$$

holds. According to [20, p. 151], we can see the two-fold generation of a hypocycloid z as the envelope of $n = [Q_1, Q_2, X^{\circ\prime}, X', X'_\circ]$ with the characteristic $\beta_1 : \beta_2 < 0$ (cf. [20, p. 156]). From the top-view $O'N^{\circ\prime}N'_\circ$ of the instantaneous axis, we can infer that n is orthogonal to l' at X'. Therefore, l' is the involute of z or an offset curve (parallel curve) of its similar involute. For the two instantaneous poles P_i ($i \in \{1, 2\}$) corresponding to the i-th Euler generation (cf. [20, p. 151]) of the path (or i-th generation as the envelope of a straight line) of z, we have: $\overline{OP_i} = \overline{OQ_i} \cdot \frac{r_0}{r_i}$. Further, the circle c centered at O' with radius $\overline{O'P_i}$ carries the cusps of z and the concentric circle v with radius $\overline{O'Q_i}$ carries the vertices of z

Special values of some spherical distances result in special shapes of the spherical trochoid and simplify their top-views:

Theorem 1.4. *For the following values of spherical distances $\widehat{r_0}$, $\widehat{r}$, $\widehat{a} = \widehat{M_0M}$, $\widehat{b} = \widehat{MX}$, the top-views of spherical trochoids are ordinary trochoids (of order 2):*

- *If $\widehat{r} = \widehat{b} = \frac{\pi}{2}$, l' is an epicycloid.*
- *If $\widehat{r_0} = \widehat{r}$, l' is a Pascal limaçon.*
- *In the special case $\widehat{r_0} = \widehat{r}$, $b = \frac{\pi}{2}$, l is a hippopede of Eudoxus with a circle l' for its top-view.*
- *If $\widehat{r_0} = \widehat{r}$ and $\widehat{a} = \widehat{b} = \frac{\pi}{2}$, l is Viviani's curve.*
- *If $\widehat{r_0} = \frac{\pi}{2}$, l' is the envelope of a straight line undergoing an ordinary trochoid (planetary) motion or the offset of a cycloid (cf. [20]).*

1.3 Algebraic spherical trochoids

The spherical trochoids are algebraic if the ratio $r_0 : r_1 : r_2$ is rational. With a proper scaling, we can achieve that each r_i ($j \in \{0, 1, 2\}$) is an integer.

Then, the rotation number w and the algebraic degree d of the top-view are

$$w = \frac{\beta_1 - \beta_2}{|\gcd(\beta_1, \beta_2)|} \text{ and } d = 2\left|\frac{\beta_2}{\gcd(\beta_1, \beta_2)}\right|.$$

Since the spherical curve can be considered as the intersection of the projection cyclinder and the sphere Σ, the algebraic degree of the spherical trochoid equals

$$2d = 4\left|\frac{\beta_2}{\gcd(\beta_1, \beta_2)}\right|.$$

We shall have a look at the following example, see Fig. 12. Here, a circle k is rolling on Σ's equator e and the radius of the rolling circle k is half that of e. That means $r_0 = 2$ and $r_1 = 1$, and thus, $\beta_1 = 1$, $\beta_2 = -3$, and $\alpha = -1$. Since $w = 4$ and $d = 6$, z is an astroid. Since a point on the boundary of k is moving, l' is an involute

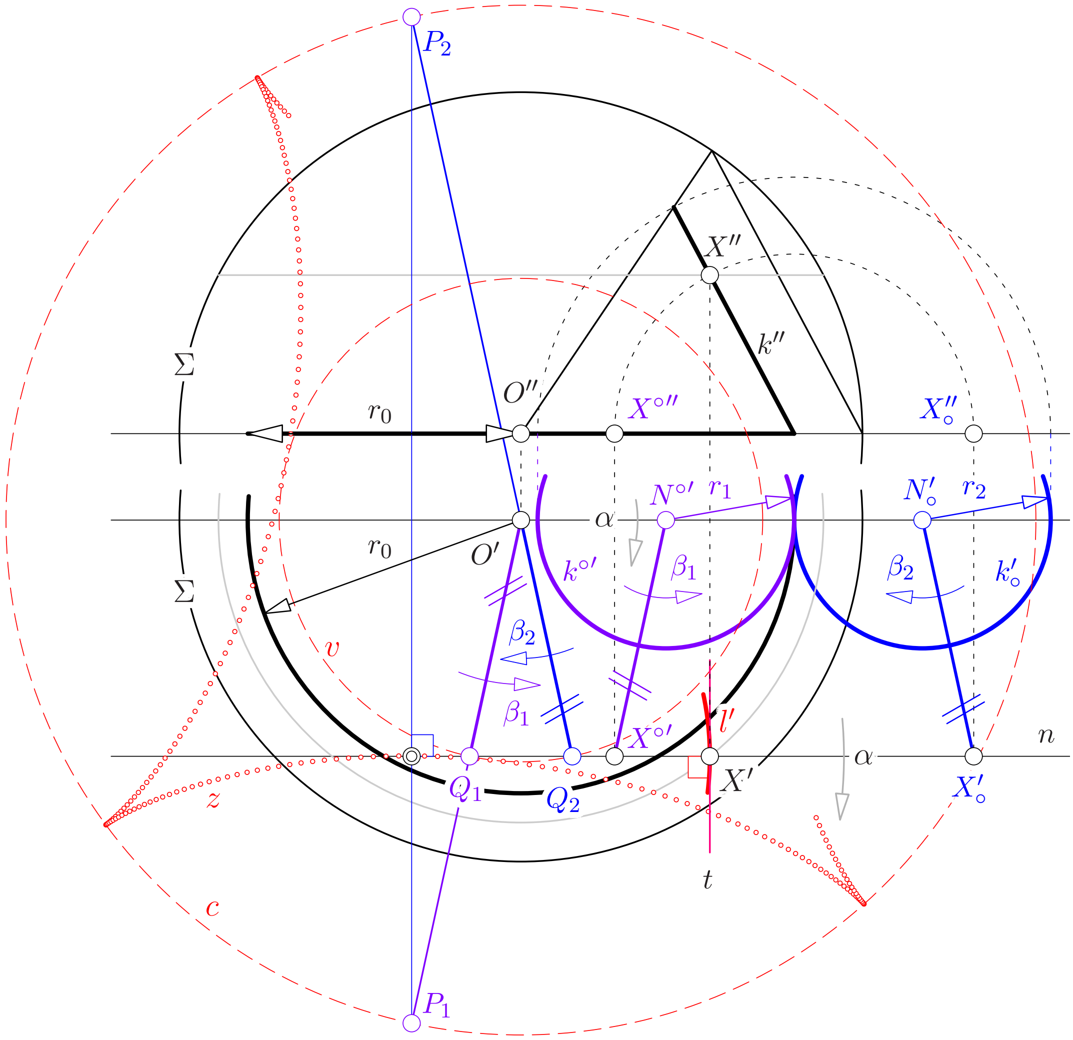

Figure 11: The top-view l' of a spherical trochoid is the involute of a hypocycloid z. The two different flips of k''s plane are displayed in different colors (blue = to the outside, violet = to the inside).

of z with two cusps X_1' and X_3' of the third kind[2].

The initial position of the rolling circle shall be labelled with k_1.

2 Some algebraic curves of constant width

A further example shall be illustrated in Fig. 14. Here, we have chosen $r_0 = 3$ and $r_1 = 1$. Therefore, $\beta_1 = 2$, $\beta_2 = -4$, and $\alpha = -1$. This yields $w = 3$ and $d = 4$ which makes z a Steiner hypocycloid. In this case, l' is a closed algebraic curve of constant width. This raises the question, if spherical trochoids can be generated such that their top-views are curves of constant width.

As mentioned earlier, the top-view l' of the spherical trochoid is the involute of a cycloid. It is well-known (see [4, 8, 9, 18]) that the involute of a cycloid is a trochoid, and moreover, it is also the envelope of a straight line under a trochoidal motion. Therefore, it is nearby to look for curves of constant width among trochoidal, and eventually, among higher order trochoidal curves.

Up to scale and w.r.t. a properly chosen Cartesian coordinate system, the curve z in Fig. 14 can be parametrized as

$$z(t) = 2\mathrm{e}^{2\mathrm{i}t} + \mathrm{e}^{-4\mathrm{i}t}, \quad t \in [0, \pi[$$

and l' allows the representation

$$l'(t) = \frac{2}{3}\mathrm{e}^{2\mathrm{i}t} - \frac{1}{3}\mathrm{e}^{-4\mathrm{i}t} - d\mathrm{e}^{-\mathrm{i}t}. \qquad (3)$$

The curve l' is an involute of z and the choice of real constant d determines the starting point of the involute. We shall use the support function $h:\ \mathrm{S}^2 \to \mathbb{R}$ which assigns to each point on the unit circle the oriented distance of the curve's tangent from the origin of the coordinate system. From the parametrization of z, we obtain the unit normal vector field $\mathbf{n} = (\sin t, \cos t)$. Now, the support function h equals the canonical scalar product of the position vector $\mathbf{l'} = (\operatorname{Re} l', \operatorname{Im} l')$ of the points of l' (from (3)) with the corresponding unit normal. This yields $h = \langle \mathbf{n}, \mathbf{l'} \rangle = d - \frac{1}{3}\cos 3t$ which agrees, up to a scaling, with the support function used in [14] to compute a closed algebraic curve of constant width. It is necessary and sufficient that h fulfills

$$\begin{aligned} &h(t)+h(t+\pi)=\text{const.}, && \text{const. width} \\ &\dot h(t)+\dot h(t+\pi)=0, && \\ &h(t)-h(t+2\pi)=0, && \text{closedness} \end{aligned} \qquad (4)$$

besides some conditions on continuity and differentiability (which are always fulfilled in the case of trochoidal curves). The dot indicates differentiation w.r.t. the parameter t.

It is a matter of fact that functions that fulfill (4) can be expanded in Fourier series

$$\begin{aligned} h(t) &= a_0 + \sum_{k=1}^{n}(a_k\cos kt + b_k \sin kt) = \\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}(a_k - \mathrm{i}b_k)\mathrm{e}^{\mathrm{i}kt} + (a_k+\mathrm{i}b_k)\mathrm{e}^{-\mathrm{i}kt}, \end{aligned} \qquad (5)$$

where $n \in \mathbb{N}^\times$ and $a_k, b_k \in \mathbb{R}$ (not all zero at the same time). Fourier series are to be

[2] Cusps of the first and second are characterized by the initial terms of their local expansions (t^2, t^3) and (t^2, t^4), respectively. The expansion at a cusp of the third kind starts with (t^3, t^4). In German such a point is called *Spitzpunkt*.

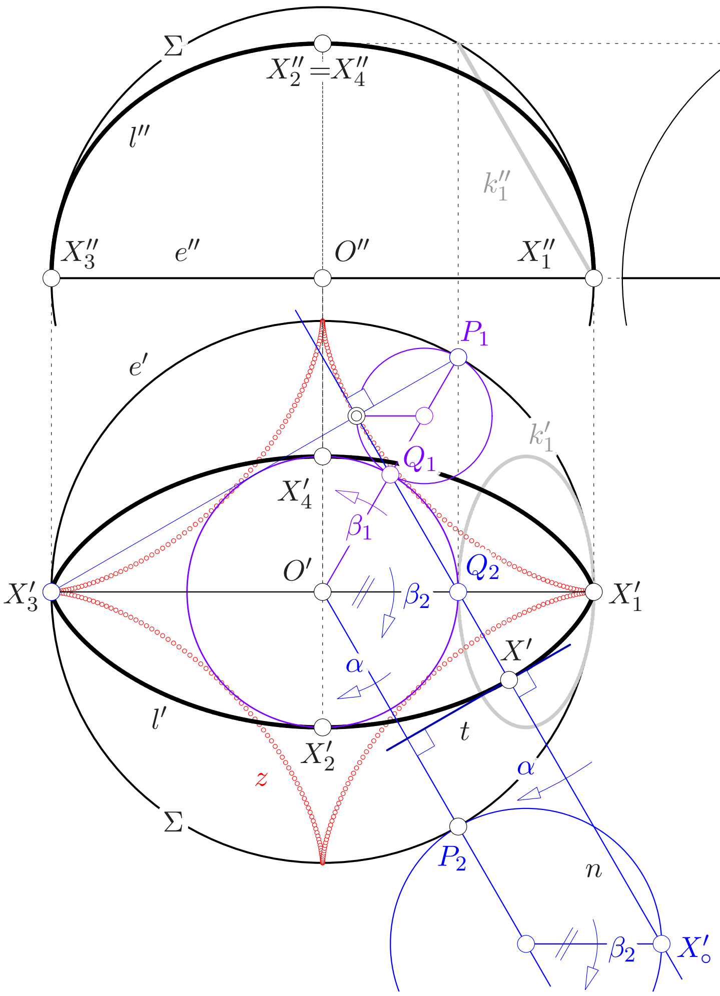

Figure 12: The spherical trochoid with $r_0 = 2$, $r_1 = 1$, and thus, with $\beta_1 = 1$, $\beta_2 = -3$, and $\alpha = -1$ is mapped to a sextic curve l' in the top-view with two cusps of the third kind at X_1' and X_3', to the upper half l'' of a doubly covered cubic (with an ordinary node) in the front-view, and to a part l''' of Neil's parabola in the left-side view.

preferred for they naturally fulfill the third condition in (4). An alternatively, Chebyshev polynomials were used in [15].

Closed algebraic curves of constant width whose support functions can be given as a finite Fourier series are always rational and their representations can always be converted into an equivalent series of complex exponential functions

$$l'(t) = h(t)\mathrm{e}^{\mathrm{i}t} + \dot{h}(t)\mathrm{e}^{-\mathrm{i}t} \tag{6}$$

with h from (5). Hence, these curves are higher trochoids of order n and first and intensively studied in [19]. They allow for a generation as the superposition of n independent rollings in $n!$ ways which includes the two-fold generation of ordinary trochoids (were $n = 2$). Further, they can be generated by closed n-bar linkages.

The example of a closed algebraic curve of constant width given in [14] can be described by the support function

$$h = 9 + \cos 3t$$

and is an algebraic curve of degree 8. It admits a rational parametrization, and thus, it has to have the maximum number of singularities two of which are the absolute points of Euclidean geometry (pair of complex conjugate ideal points, ordinary double points with self-osculation) and three of which are real isolated ordinary double points on the curves' lines of symmetry. In [12], the authors modified the support function to

$$\widetilde{h} = 8 + \cos 3t$$

in order to remove the isolated double points. This particular choice of the support function pushes the isolated double points to points on the curve, and thus, they become cusps of the third kind (see [2, 3, 4, 18]).

The choice of a support function of the form (3) (such that it fulfills (4)) leads in any case to a curve of constant width which allows for a kinematic generation by means of sufficiently many rotations. These curves can always be interpreted as the top-view of spherical curves. Depending on whether $\sqrt{1 - l'(t)\overline{l'}(t)}$ can be written as a finite sum of exponential functions (or trigonometric functions) or not, the curve l allows for a kinematic generation by means of superposed rollings on a sphere. The *order* of the spherical trochoid l will, in general, be higher than 2.

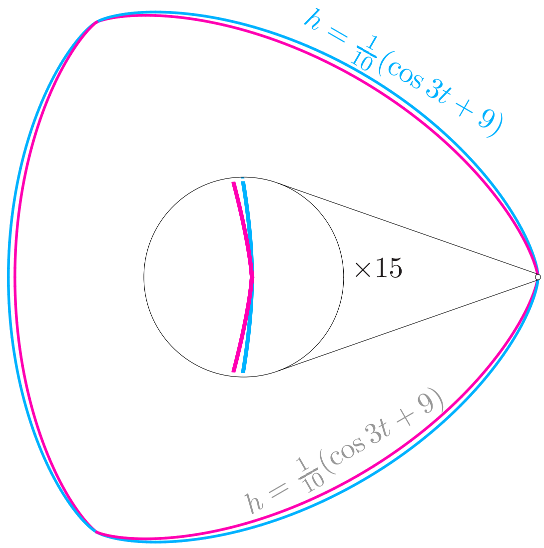

Figure 13: Two curves of constant width (similar to those mentioned in the text and scaled to equally sized circumcircles. The vicinity of the right vertex is enlarged by the factor 15 in order to display the differences between the two curves.

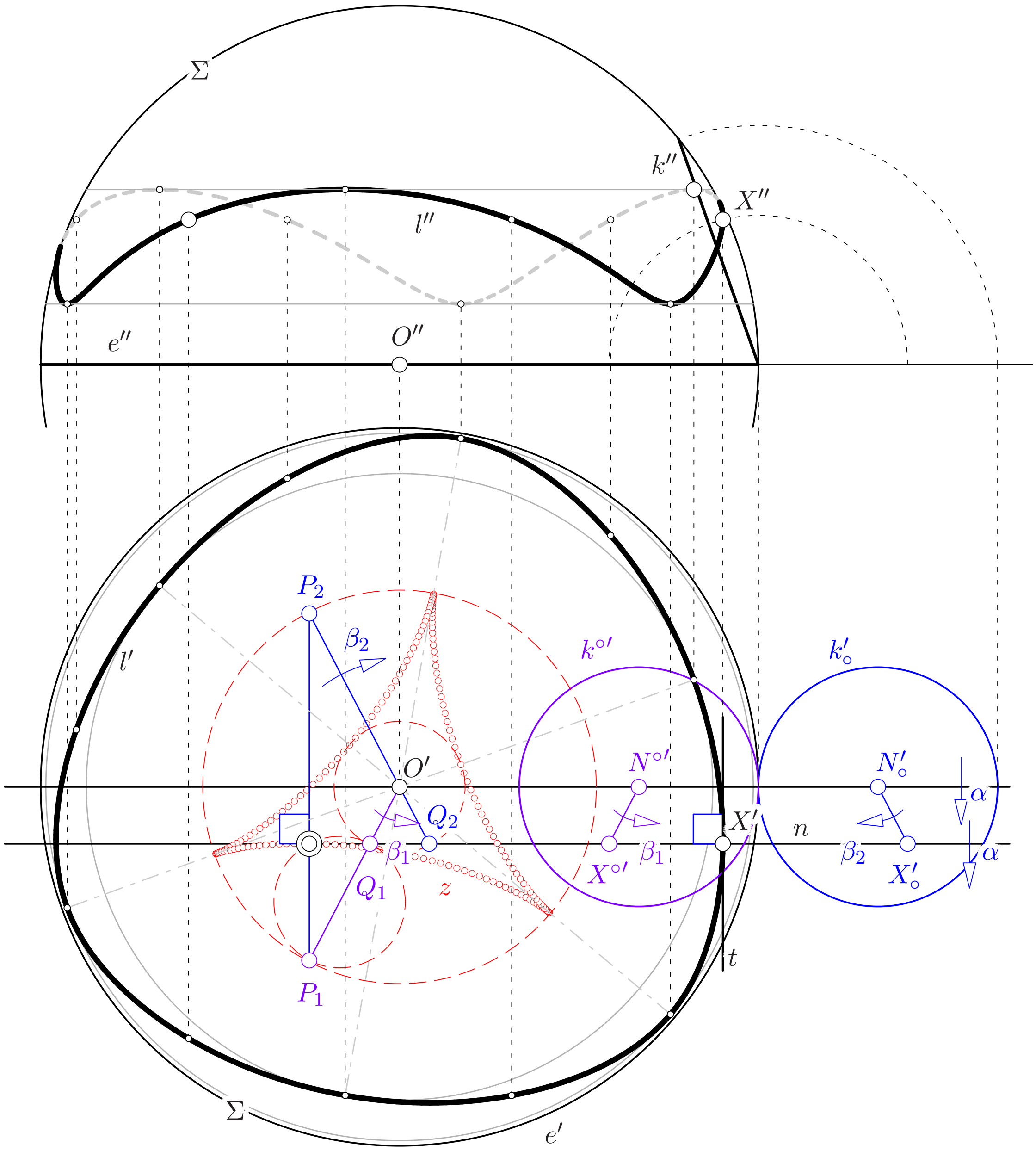

Figure 14: The top-view $l'$of a spherical trochoid may even be a closed and algebraic curve of constant width.

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