

# Higher dimensional geometries

## What are they good for?

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**Abstract.** Geometries in higher dimensional spaces have many applications. We shall give a compilation of a few well-known examples here. The fact that some higher dimensional geometries can be found within some lower dimensional geometries makes them even more interesting. At hand of some familiar examples, we shall see what these concepts in geometry can do for us. In the beginning, the meaning of dimension will be clarified and an agreement is reached about what is higher dimensional. A few words will be said about the relations and interplay between models of various geometries. To the concept of model spaces a major part of this contribution will be dedicated to. A full section is dedicated to the applications of higher dimensional geometries.

## 1 Introduction

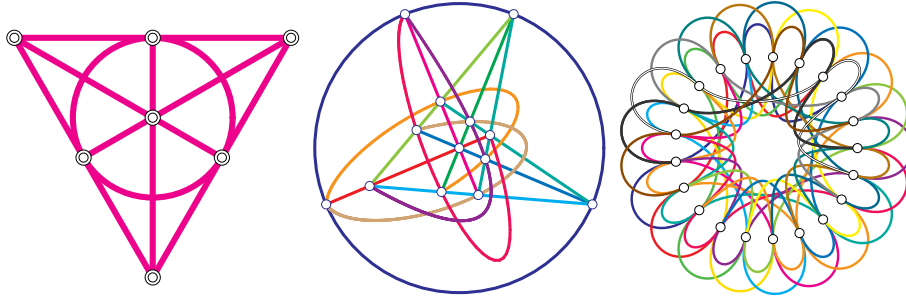
Originally, geometry was the science of measuring the land in order to calculate taxes and divide the fertile land. Later on, early cultures, *eg.*, the Egyptians, Babylonians, Greeks, ... began to detach this science from real world problems and entered the world of abstract two- and three-dimensional phenomena: things that happened in a plane or in the space of our perception, described with a new vocabulary like points, lines, angles, distances, triangles, and many, many more. Within this period, many well-known elementary geometric results were discovered and the techniques of proofs were developed. The old Greeks saw geometry rather as a philosophical discipline than as a part of mathematics. It took more than two thousand years until mathematicians and especially geometers became aware of geometries that do not fit into two or three dimensions. A major breakthrough was H. GRASSMANN's *Theory of linear Extensions* [9] in the middle of the 19<sup>th</sup> century. GRASSMANN's work was maybe not the first attempt, but it was successfully providing mathematicians and geometers with techniques that made it possible to describe higher dimensional geometries. During the end of the 19<sup>th</sup> century, a lot of work on higher dimensional geometries was done. Especially, the Italian school, mainly represented by L. CREMONA [6], G. VERONESE [26], L. BERZOLARI, and C. SEGRE (who also worked out important parts of F. KLEIN's mathematical encyclopedia [1, 24]) began to study algebraic geometries in spaces

of dimensions greater than three. At that time, higher dimensional geometries were not common sense to all mathematicians and geometers. Since history is repeating itself, a few of them even doubted in the existence of such objects. It was comparable to the physicists concept of the atom which entered the scientific stage approximately at that time: Notable scientist denied the existence of atoms using the argument that atoms cannot be seen.

## 2 A huge variety of higher dimensional geometries

### 2.1 What is a dimension?

We agree that the *dimension* is a number that counts the number of *degrees of freedom* in a geometrical object. A line is of dimension one. This may not be confused with the number of points on the line. The same is true for planes: They all are of dimension two, no matter if it is the Euclidean plane were we can choose Cartesian coordinates  $(x, y)$  in order to fix points, or if it is a projective plane were homogeneous coordinates  $x_0 : x_1 : x_2$  are suitable for describing points (still there are only two degrees of freedom, since  $x_0 : x_1 : x_2 \sim 1 : x : y$ ), or if it is a finite (projective) plane like the ones depicted in Fig. 1.



**Fig. 1.** Models of projective planes of order two, three, and four. However, these are two-dimensional geometries, even though we can give a list of its points.

In the Euclidean plane and, more generally speaking, in any Euclidean space the dimension gives the number of coordinates that are necessary in order to describe points. It is a very useful and powerful result from differential geometry that any differentiable manifold can locally be mapped to a certain real vector space  $\mathbb{R}^n$ , and thus, a dimension can be assigned to the manifold. Besides the degrees of freedom of a geometrical system and the number of coordinates that are necessary to determine points, there is a more mathematical notion of dimension related to GRASSMANN'S theory of extension. The dimension of a vector space equals the number of basis vectors, *ie.*, a system of linearly independent vectors that allow a unique representation of all elements of a vectors space, the vectors. In the case of a vector space, it is assumed that coordinates are real or complex

numbers, or taken from an arbitrary field - finite or not. Things are getting more complicated, but nonetheless, more interesting, once we drop the assumption that coordinates are taken from a field. Geometries over rings are sometimes hard to handle and models need *more space, ie.*, they are higher dimensional in nature. Moreover, the set of points in geometries over rings can split into different classes and it is not so easy to compare points, see [11]. Geometries over finite fields and rings have a lot of applications, especially in physics, cf. [10, 13].

However, we shall agree that *higher dimensional geometries and spaces are those with a dimension greater than three*. These are beyond our perception, since we can move forward and backward, to the left and to the right, and up and down in the space were we live. *Time* could be considered a possible fourth dimension, but we have no sense to perceive time.

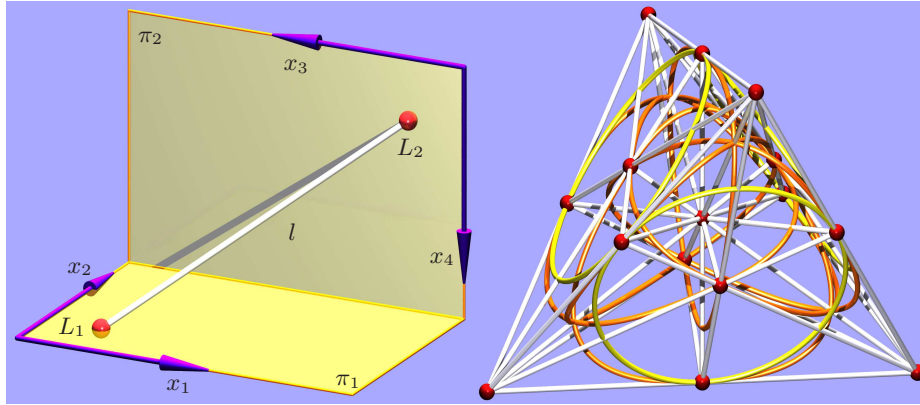
## 2.2 Some examples of higher dimensional geometries

Sometimes, the Euclidean unit sphere is called a three-dimensional object. However, this is not true in the strict sense. There are two things mixed up: The sphere itself is two-dimensional, since we need two coordinates to describe points on it: the *latitude* and the *longitude*. However, the sphere is embedded in a three-dimensional space and it is not possible to embed it into a space of lower dimension without any singularity.

Let us determine dimensions of some known geometries. A three-dimensional (affine) space is usually a space where the manifold of points is three-dimensional, *ie.*, points are determined by three coordinates  $(x, y, z)$ . From the equations of planes  $ax + by + cz - 1 = 0$  (with  $a, b, c$  taken from some field, not all simultaneously zero), we see that the same space is three-dimensional considered as the space of the planes in it. What about the lines in a three-space? As indicated in Fig. 2, any line  $l$  can be uniquely determined by its two intersection points  $L_1, L_2$  with two planes  $\pi_1, \pi_2$ . Each of these points is determined by two coordinates:  $L_1 = (x_1, x_2)$  and  $L_2 = (x_3, x_4)$ . Since these four numbers  $x_1, \dots, x_4$  can be chosen independently, there are four coordinates that describe a line. Note that the submanifold of lines that meet  $\pi_1 \cap \pi_2$  is only of dimension three. Strange to say, but the *space of lines in a three-dimensional space is four-dimensional*. The geometry of lines plays an essential role in a huge variety of applications (cf. Section 4). Naturally, there is a tremendous amount of literature dealing with line geometry, see [25, 28, 30] and the references therein.

Many higher dimensional geometries are contained within lower dimensional (point) geometries. The geometry of circles in the Euclidean plane is three-dimensional. The center's two coordinates and the radius define a circle.

Conics in a plane can be described by an equation of the form  $a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0$  with coefficients  $a_{ij}$  from some commutative field (cf. [8]). (It is always possible to normalize an equation, *ie.*, to multiply such that one coefficient becomes unity, without altering the geometric object.) Obviously, there are five relevant numbers that determine the conic, and thus, the geometry (or manifold) of conics is five-dimensional. A useful tool for the study of conics



**Fig. 2.** Left: Lines can be determined by four coordinates determining the intersection points in two different planes. Right: Though there are only 35 lines in  $PG(2, 3)$ , the manifold of lines in a three-space is four-dimensional.

is due to G. VERONESE (see [26]): The six quadratic monomials in the conic's equation can be used as a basis in the space of conics and a mapping into a five-dimensional projective space is nearby. The manifold of singular conics is called *rank manifold* whose equation is simply given by  $\det(a_{ij}) = 0$ .

The reader may convince herself or himself by counting that the space of algebraic curves of degree  $n$  in a plane equals  $\frac{1}{2}n(n+3)$ , including the case of lines and conics. Applying knowledge from basic linear algebra, we find that the manifold of  $k$ -dimensional subspaces of a projective space of  $n$  dimensions is  $(n-k)(k+1)$ -dimensional. In any case, one has to think about a proper way of counting. The existence of a vector space model of the geometry in question simplifies the process. However, the dimension of fractals is to be computed.

### 3 Model spaces

We have seen that the geometric objects we are dealing with usually depend on a certain but fixed number of constants considered as shape parameters, varying freely within some intervals, or range even in the real or complex number field. It is nearby to use these determining constants as coordinates for these objects. The number of these constants equals the dimension of the geometry. In this section, we shall see that model spaces need not be affine, metric, or even projective.

#### 3.1 Various geometries and their models

**Circles, spheres.** Oriented circles, spheres in three-space, ..., spheres in an  $n$ -dimensional space can be mapped to points in an  $n+1$ -dimensional affine model space that is usually the Minkowski space  $\mathbb{R}^{n,1}$ , sometimes referred to as the *cyclographic model*, [4, 7]. The coordinates in the model space are simply the

sphere's center plus the radius. Signed radii can be used to express orientations. The pseudo-Euclidean metric in the model space is obtained by transferring the Euclidean *tangential distance* of spheres into the model. The metric allows us to characterize *pairs of spheres* as being in *oriented contact* and can be used to compute oriented intersection angles.

A less natural approach to a point model of the manifold of oriented Euclidean spheres uses (homogeneous) six-tuples  $(s_1, \dots, s_6)$  of real coordinates satisfying the quadratic form  $L_2^4 : s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_5^2 - s_6^2 = 0$ . The center and the radius of the sphere can be recovered from this six-tuple as long as it satisfies the quadratic form. The quadric  $L_2^4$  is called *Lie's quadric*. It is contained in a projective five-space and carries lines as maximal subspaces. This coordinatization of the manifold of spheres is more universal: Points as spheres with radius zero and planes as spheres with infinitely large radius are also described that way, see [2, 4]. Even the polar system of  $L_2^4$  has a geometric meaning: *Conjugacy* with regard to  $L_2^4$  characterizes spheres in *oriented contact*.

The cyclographic model can be linked via a stereographic projection with the *Blaschke model*, *ie.*, a cylinder model of Euclidean Laguerre geometry (oriented planes, and oriented spheres considered as the envelopes of oriented planes). BLASCHKE's cylinder is a tangential intersection of Lie's quadric, see [2, 4, 7].

**Lines in three-space.** We have seen that lines in a three-dimensional space can be mapped to points in a four-dimensional model. This naive approach works well and is even applicable to interpolation problems (cf. [22]), but it is not as universal as the model presented in the following.

It proved useful to describe lines by Plücker coordinates  $L = (l_1, l_2, l_3, l_4, l_5, l_6)$  (see [28, 30]). Only those (homogeneous) six-tuples  $(l_1, \dots, l_6)$  that satisfy  $M_2^4 : l_1l_4 + l_2l_5 + l_3l_6 = 0$  correspond to lines in three-space, and, *vice versa*. This quadratic form is the equation of the four-dimensional model (surface) and describes a quadric  $M_2^4$  in a projective five-space. It is called *Klein's quadric* or *Plücker's quadric* and it is also the first non-trivial Grassmannian, and therefore, also denoted by  $G_{3,1}$ . It is worth to mention that the maximal subspaces contained in  $M_2^4$  are planes and that there exist two independent three-parameter families of them corresponding to ruled planes and stars of lines in three-space.

Now, the model space splits into two components: The points on  $M_2^4$  correspond to lines, while the points off  $M_2^4$  correspond to so-called *regular linear line complexes*. The latter are as important in line geometry as the lines, since they are closely related to helical motions. We shall make use of this in Section 4. The polarity with regard to  $M_2^4$  has a geometric meaning: Points *conjugate* with regard to  $M_2^4$  correspond to *intersecting lines* in three-space. Euclidean specialization of the model can achieve even more, cf. [25, 28, 30].

**The interplay between spheres and lines.** It is obvious that lines and spheres are completely different things. Allowedly, the geometries of both can be modeled within four-dimensional quadrics. However, while  $M_2^4$  carries real

planes,  $L_2^4$  carries only real lines. Nonetheless, from the view point of complex projective geometry, the two quadrics  $M_2^4$  and  $L_2^4$  can be transformed into each other by means of a collineation, *ie.*, a linear transformation in the vector space model. This mapping linking the geometry of lines and the geometry of spheres is called *Lie's line-sphere-mapping* (see [2, 7, 28]). Consequently, there is no difference between lines and spheres, at least in theory.

**Euclidean motions.** Without going too much into detail, we shall recall that Study's quadric  $S_2^6$  serves as a point model for the Euclidean motions in three-space. This quadric is ruled like  $L_2^4$  and  $M_2^4$  and carries three-dimensional subspaces, see [7, 25, 28]. Its equation equals the orthogonality condition of *dual unit quaternions*. Quadrics of the (real) projective type of  $S_2^6$  allow a definition of a so-called *triality* (that generalizes duality), cf. [3, 28].

**Subspaces of a projective space.** Klein's quadric is a very special version of a Grassmannian. In general, a Grassmannian  $G_{n,k}$  is a point model for the set of  $k$ -dimensional subspaces in an  $n$ -dimensional projective space. The dimensions of the model space are growing rapidly:  $G_{n,k}$  spans a projective space of  $\binom{n+1}{k+1} - 1$  dimensions and its inner dimension equals  $(n - k)(k + 1)$ , see [3, 7].

**Veronese varieties and rational normal curves.** As outlined earlier, conics can be studied in the *Veronese model* containing the Veronese surface  $V_2^2$  (all of whose points correspond to conics) and the rank manifold representing the singular conics. The ambient model space is five-dimensional. Clearly, the underlying concept of considering the monomials in the equation of a curve as a basis can be carried over to quadrics, cubics, any algebraic curve, and surface. Here, the *symmetric tensor product* of the underlying vector space builds the algebraic grounding. The study of Veronese manifolds  $V_1^n$ , called *rational normal curve*, being the  $n$ -fold symmetric product of a projective line is important for the study of rational curves and rational transformations, since any planar rational curve (including Bézier curves) is a projection of a rational normal curve, cf. [3].

**Segre products, flag manifolds.** Geometry models are not restricted to represent only one particular class of object. It is also possible to map combinations of objects to points. It is not at all surprising that the dimension of the model space grows with the complexity of the underlying objects. Models with the ability to simplify objects consisting of components of various classes of geometric objects can be built using Segre varieties, see [3]. This allows us to create models for example for the manifold of flags, *ie.*, a sequence of nested subspaces in some projective space, see [12, 17–19]. The flags need not be complete, some components may be missing: For example, a line element is a *partial flag*. The incidence conditions between the components give rise to equations of flag manifolds.

**Exterior algebras, Clifford algebras.** The direct sum over all model spaces that contain the Grassmannians  $G_{n,k}$  with  $k = 0, \dots, n$  (with fixed  $n$ ) is called *exterior algebra* if every summand is considered as a vector space. This  $2^n$ -dimensional vector space can be the algebraic model of a projective space (of dimension  $2^n - 1$ ) being a model for the set of all subspaces of a projective space of dimension  $n$ , cf. [3]. Sometimes, exterior algebras earn the structure of the underlying vector space. This turned out to be useful in kinematics, even in non-Euclidean geometries, see [16].

### 3.2 What makes a model space?

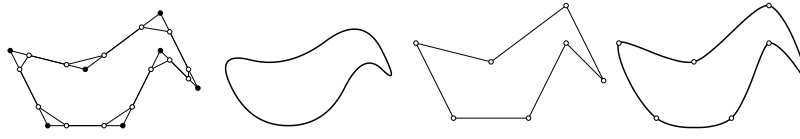
The mere fact that a geometric object can be mapped to a point in some strange high-dimensional space is not enough. The model space itself would be nothing if there is no structure in it. Sometimes, the structuring features come along with the geometries in a natural way; sometimes one has to be creative. As we have seen with lines and spheres, there is a quadratic form in the model space with a geometric meaning. At this point, we note that many of the presented model spaces can be created in a purely synthetic way. In all the aforementioned cases, we always assumed the existence of an algebraic model space, since applications need computations in almost any case. A good model space is easy to handle: It should be affine or projective, of lowest possible dimension, the coordinates should have a geometric meaning, and a metric (a quadratic form) should relate the points in the model. Transformations that act on the manifold of certain geometric objects should be easily transformed to the model space. Preferably, the induced transformations are linear in terms of the coordinates in the model space. As many properties of the underlying geometry as possible should be displayed in a very simple way in the model space.

## 4 Applications - benefiting from model spaces

### 4.1 Interpolation with ruled and channel surfaces

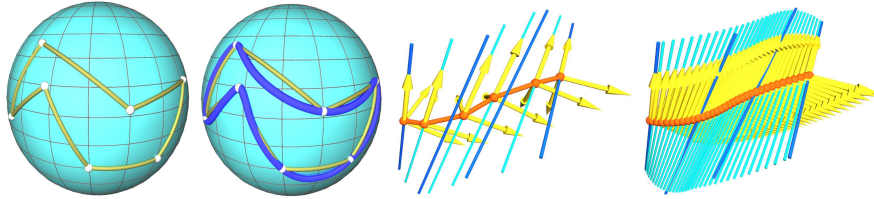
In the point models for the set of (oriented) lines/spheres in Euclidean three space, we recognize one-parameter families of (oriented) lines/spheres, *ie.*, *ruled surfaces* or *channel surfaces* as curves on  $M_2^4$  and  $L_2^4$ , respectively. So, the geometry of ruled or channel surfaces in a three-dimensional space (whether Euclidean or not) *somehow simplifies* to the geometry of curves on quadrics. The simplification is bought at the costs of more coordinates.

Interpolation techniques and approximation techniques (as shown in Fig. 3) that were originally developed for affine planes and spaces (see [15]) can be adapted to arbitrary manifolds. In many cases, the adapted subdivision schemes are combinations of two operations: First, a subdivision scheme in the ambient space of the target manifold and, second, a projection onto the target manifold. In any case, one has to make sure that the projection does not fail and destroy the result. Checking the convergence of a subdivision scheme is not the problem. In the case of ruled surfaces, one can use also ordinary subdivision schemes in



**Fig. 3.** Approximating (left) and interpolatory subdivision scheme (right).

order to first refine the striction curve and then refine the spherical image of the rulings. However, this is only one possibility, cf. [20]. Fig. 4 shows the action of an interpolatory scheme on the sphere and applied to a finite sequence of lines (discrete ruled surface). Subdivision of the motion of the Sannia frame is obtain as a byproduct, see [20]. Since there is only a small difference between



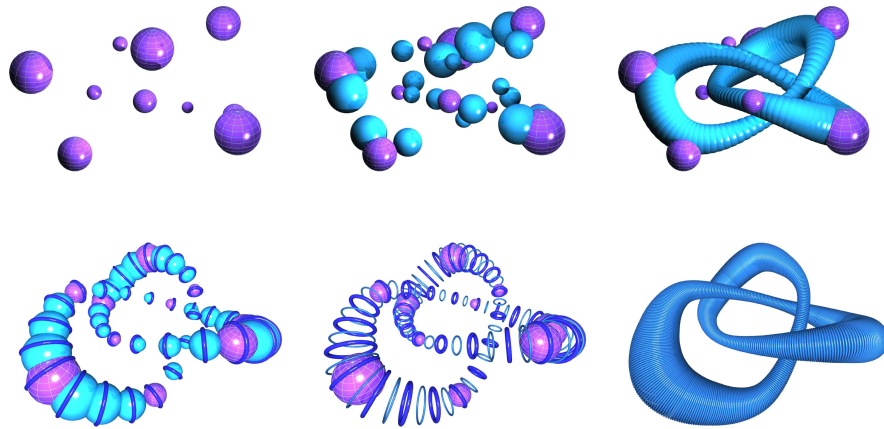
**Fig. 4.** Above: SLERP - spherical linear interpolation on the sphere. Below: Subdividing ruled surface data is also suitable for discrete motions (here the Sannia motion) and uses SLERP for the direction of the rulings.

the geometry of oriented lines and oriented spheres in Euclidean three-space, the algorithms developed for ruled surfaces apply nearly in the same way to channel surfaces, see Fig. 5. Even the characteristic circles on a channel surface are accessible to modified subdivision schemes that take place on a six-dimensional cone-shaped variety which can be obtained as a projection of the Grassmannian of two Lie quadrics, see [5]. Algebraic techniques like the ones used for  $G^r$  or even  $C^r$  interpolation of data from curves (as described in [15]) need only some minor modifications in order to apply to interpolation problems with ruled/channel surfaces (cf. [21]). This allows us to perform  $G^r$  interpolation of data that stems from ruled/channel surface data, see Fig. 6.

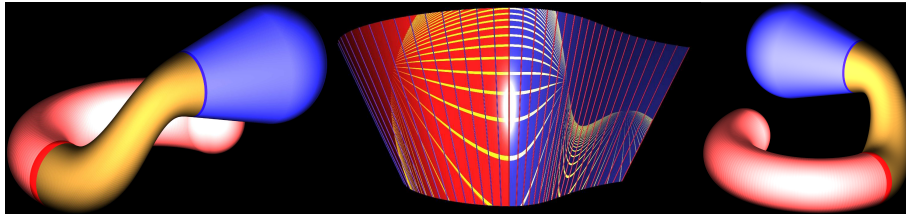
## 4.2 Recognition and reconstruction of surfaces

Surface recognition benefits from line geometry as well as from line element geometry (cf. [14, 23, 18]). It is well-known that the normals of helical surfaces (including surfaces of evolution and cylinders) are contained in a *linear line complex*. The Plücker coordinates of the lines of such a three-dimensional submanifold of  $M_2^4$  fulfill a *linear* homogeneous equation. Once a surface is captured by a laser scanner, the point cloud allows an estimation of the surface normals.

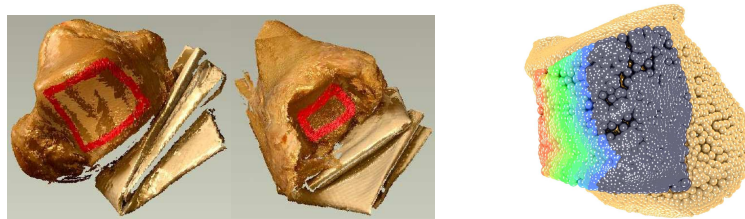




**Fig. 5.** Above: An approximating subdivision scheme applied to a discrete channel surface. Below: The set of characteristic circles of a channel surface can also be refined.

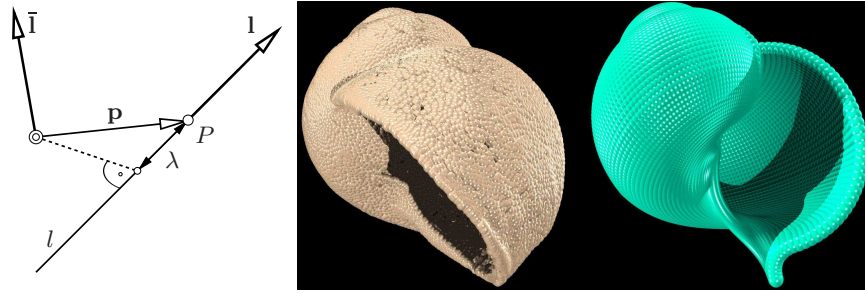


**Fig. 6.** Hermite interpolation of ruled and channel surfaces uses a projective five-space.



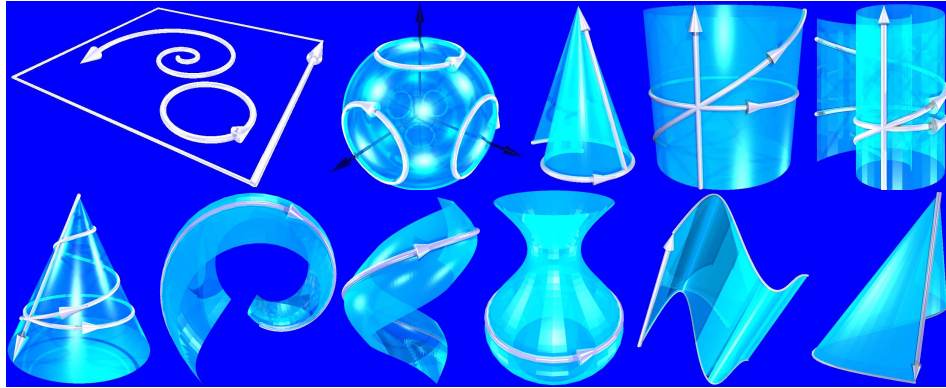
**Fig. 7.** Left and middle: Scans of the articulate surfaces of the ankle joint. Right: The gliding of the contact surfaces generates a helical motion.

Fitting linear subspaces to point data in the model space is a simple task and the computation of the axis and the pitch of the helical motion generating the scanned surface part is straight forward. Fig. 7 shows a comparison of the two flanks of the human ankle joint. The comparison of the surfaces was only possible once the motion defined by the flanks was known. For objects composed of many different surfaces, a segmentation is necessary.



**Fig. 8.** Left: The geometric meaning of the coordinates of a line element. Middle: Data from a snail shell can be recognized as a part of a spiral surface. Right: reconstruction.

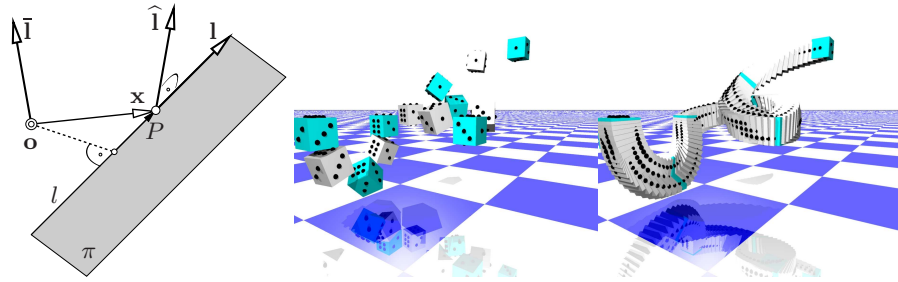
An obvious extension of line geometry is called *line element geometry*. It is the geometry of pairs  $(l, P)$  where  $l$  is a straight line in Euclidean three-space with a point  $P$  on  $l$ . Since the lines in Euclidean three-space can be identified with a subset of  $M_2^4$  and one further parameter is needed in order to fix  $P$  on  $l$ , we end up with a quadratic cone  $L_2^5$  erected over  $M_2^4$  serving as a point model for the set of (oriented) line elements in Euclidean three-space (cf. [18]). The coordinates  $(\mathbf{l}, \bar{\mathbf{l}}, \lambda) \in \mathbb{R}^7$  of a line element satisfy  $\langle \mathbf{l}, \bar{\mathbf{l}} \rangle = 0$  and  $\lambda \in \mathbb{R}$ . A projective version can be found in [19]. Fitting linear subspaces in the geometry of line elements works well. The reconstruction uses the determined generating Euclidean or equiform motion (*ie.*, a combination of a Euclidean motion and a homothety) to find a profile curve, see Fig. 8. In contrast to line geometry, a wider class of surfaces (including spiral surfaces, *ie.*, shells of molluscs) can be detected (cf. Fig. 9), since the group of Euclidean motions is a subgroup of the group of equiform motions.



**Fig. 9.** In line element geometry, 11 classes of surfaces can be detected (from top-left to bottom-right): planes, spheres, spiral cones, cylinders of rev., spiral cylinders, cones of rev., spiral surfaces, helical surfaces, surfaces of rev., generic cylinders and cones.

### 4.3 Interpolation of poses by smooth motions

A complete flag (*ie.*, a plane  $\pi$  containing a line  $l$  with an incident point  $P \in l$ ) in Euclidean three-space determines a Euclidean motion (not necessarily unique). Thus, any model of the geometry of flags in Euclidean three-space can be used for the design of one-parameter families of Euclidean motions. Either by means of adapted subdivision schemes (like in [20]) or by algebraic techniques (like in [21]). Fig. 10 (left) shows how flags can be coordinatized using a vector  $(\mathbf{l}, \hat{\mathbf{l}}, \lambda) \in \mathbb{R}^{10}$ . The equation of the flag manifold in the model space  $\mathbb{R}^{10}$  is obtained by the natural constraints to which the flag coordinates are subject to:  $\langle \mathbf{l}, \hat{\mathbf{l}} \rangle = \langle \mathbf{l}, \hat{\mathbf{l}} \rangle = 0$  (with  $\|\mathbf{l}\| = \|\hat{\mathbf{l}}\| = 1$ ). Combined subdivision techniques like those from [20] apply here as well, see Fig. 10. A different approach is presented in [27].



**Fig. 10.** Left: coordinatization of flags. Right: a Euclidean motion interpolating given poses. The interpolation uses the rationally parametrized manifold of flags.

## 5 Conclusion

We have seen that higher dimensional geometries occur frequently and enable us to do interpolation with ruled/channel surfaces, surface reconstruction, motion planning and interpolation, and subdivision in spaces of geometric objects. These are only a few and we haven't treated *shape spaces* that are models for *moving and deformable objects*. Kinematics uses the various model spaces for motion design and the analysis of mechanisms.

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