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# Isoptic ruled surfaces of developable surfaces

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**Abstract.** The planar notion of isoptics cannot be carried over directly into three-dimensional spaces. Therefore, the isoptic surface of a developable ruled surface will be defined as the set of intersection lines of pairs of tangent planes that enclose a fixed angle. The existence of a one-parameter family of tangent planes of a developable ruled surface guarantees that such a set of lines is a ruled surface, while in the case of any other surface this construction would result in a complex of lines. The isoptics of quadratic cylinders and cones shall be given. Further, the isoptic ruled surfaces of developable ruled surfaces invariant under one-parameter subgroups of the Euclidean and the equiform group of motions shall be described. Moreover, the orthoptic ruled surfaces of developables with a polynomially parametrized curve of regression shall be computed. It turns out that the orthoptic ruled surfaces allow for a projective generation.

*Keywords:* isoptic ruled surface, ruled surface, developable ruled surface, helical surface, spiral surface, cylinder, cone, isoptic curve.

## 1. Introduction

The locus $c_\alpha$ of all points where pairs of tangents of a planar curve $c$ meet at a constant angle $\alpha$ is called the $\alpha$-*isoptic*. The name isoptic curve may have been used for the first time in [16]. Such curves have attracted the interest of geometers especially in the cases of low-degree curves. The isoptics of conics in Euclidean planes are well-studied and described, see for example [10, 17]. The isoptic curves of algebraic curves of degree larger than 2 are studied in detail in [9], where the degree, class, and rank of the isoptic curves depending on the class, rank, number and type of singularities of the initial curve are elaborated. Tools for the study of isoptics in the hyperbolic plane are developed in [2, 8]. Curves with a simple kinematic generation also allow for an exact computation of isoptic curves. This is especially the case for trochoidal curves, see the results in [18, 19].

The study of many *optic properties* of planar curves is based on the results on and the description of isoptic curves. For example, the equioptic curves of planar curves, *i.e.*, sets of points from which two different curves can be seen at the same angle, consider the sets of isoptic curves (for varying optic angle $\alpha$) of planar curves as level sets. Hence, the intersection of any two such curves at the same *level* yields points of the equioptic curves, see [13]. Curves

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which are their own isoptics can only be determined approximately (cf. [14]) and contain the special case of *autoevolutes* (as described in [20]). The determination of isoptic curves with a special predefined shape is a much more complicated task (see [21, 22]).

The tangents of a generic space curve $c$ are skew, with at most countably many exceptional tangents of $c$. Therefore, an isoptic curve or surface of a space curve cannot be defined by means of tangents intersecting at a constant angle. However, there are also surfaces in three-dimensional space, and an optic angle can be measured between any two tangent planes. In computer graphics, isoptic surfaces (of surfaces) have been defined by means of integral measures of tangent cones (see, *e.g.*, [1, 11]). The thus defined isoptic surfaces are the loci of all vertices of cones that are tangent to a given surface, and define a certain spatial angle measure which yields well-defined isoptic surfaces with a major flaw: They can only be found numerically and point-wise and an analytic description, *i.e.*, a handy and implicit equation or a parametrization, is missing. This prevents further study of these isoptic surfaces.

Among the huge variety of surfaces in three-dimensional space, developable surfaces play an exceptional role. Besides the fact that they are developable, *i.e.*, they can be mapped isometrically into a Euclidean plane, they have only a one-parameter family of tangent planes. To put it in other words, the set of tangent planes of a developable surface is a curve from the dual point of view. This allows us to define isoptic surfaces of developables as the set of intersection lines of pairs of tangent planes, provided that the tangent planes enclose a certain predefined angle. The isoptic surfaces of developable ruled surfaces are ruled surfaces and an analytical representation can easily be computed, at least from the theoretical point of view. This definition of isoptic surfaces includes the Thaloid surfaces that appear in [5], where a spatial analogue to the Theorem of the Angle of Circumference is presented including a spatial analogue of the Theorem of Thales. There, ruled quadrics and quartics appear as the locus of intersection lines of planes from two pencils forming a fixed angle. Thus, these loci can be viewed as the *equioptic surfaces* of two straight lines (the axes of the pencils).

The present paper is an extension of [15] and is organized as follows: Section 2 provides some general results on isoptic ruled surfaces of cylinders and cones also mentioning quadratic cylinders and cones. Section 3 is devoted to the study of isoptic ruled surfaces which are invariant with respect to special groups of transformations such as the group of Euclidean motions and the group of equiform motions. These groups are chosen since the transformations therein preserve angle measures. The huge variety of algebraic developables and their isoptic ruled surfaces will only briefly be discussed in Section 4. Upper bounds of algebraic degrees of isoptic ruled surfaces are given. Due to the complexity of the symbolic computations, we then focus on the orthoptics of curves with a monomial parametrization. This leads to a surprising result on decomposing orthoptic ruled surfaces. Finally, in Section 5, we shall provide some ideas for future research.

## 2. General results

Whenever we deal with a developable ruled surface $R$, we keep in mind that $R$ has a *curve of regression* denoted by $g$. We shall recall that $R$ and $g$ define each other mutually: The one-parameter family of tangent planes of $R$ consists of the one-parameter family of osculating planes of $g$, and *vice versa*, see [7]. The developable $R$ is enveloped by the osculating planes of $g$ and it is also swept by the tangents of $g$ since any two infinitely close osculating planes of $g$ intersect along a tangent of $g$.

The one-parameter family of osculating planes of $g$, which is at the same time the set of tangent planes of $R$ enables us to give the following definition:

**Definition 2.1** *Let $R$ be a developable ruled surface and let $\alpha \in (0, \pi)$ be the optic angle. Then, the $\alpha$-isoptic $J_\alpha$ of $R$ is the locus of intersection lines of all pairs of tangent planes of $R$ that enclose the angle $\alpha$.*

From the above definition, it is clear that the $\alpha$-isoptic of a developable (ruled surface) $R$ is a ruled surface.

If we replace the developable ruled surface $R$ in Definition 2.1 by an arbitrary surface $S$ (which is not necessarily a ruled surface), then the set of intersection lines of tangent planes fills the entire continuum of straight lines in three-dimensional space (at least partially). Adding the angle condition on any two tangent planes to be intersected, we find a three-dimensional manifold of lines. Therefore, we would have to speak about an isoptic complex of lines from all of whose lines the surface $S$ is seen at a constant angle. Thus, the isoptic loci defined in Definition 2.1 are surfaces only if $R$ is a developable ruled surface.

We shall start with some basic examples of isoptic ruled surfaces:

**Theorem 2.1** *(1) The isoptic ruled surfaces of cylinders are cylinders.*

*(2) The isoptic ruled surfaces of cones are cones.*

*Proof:* (1) We can choose a Cartesian coordinate system $(O; x, y, z)$ such that the generators of a given cylinder $\Lambda$ are parallel to the $z$-axis and an orthogonal cross section $l$ of $\Lambda$ lies in the $[x, y]$-plane. With this assumption, the tangent planes of $\Lambda$ are also parallel to the $z$-axis and meet the $[x, y]$-plane in the tangents of $l$. The $\alpha$-isoptic surface of $\Lambda$ is the set of all intersection lines of tangent planes of $\Lambda$ that enclose the angle $\alpha$. Since all these planes are parallel to the $z$-axis, the intersection lines are also parallel to the $z$-axis. Thus, the isoptic surface $\Lambda_\alpha$ is a cylinder parallel to $\Lambda$. The $\alpha$-isoptic $l_\alpha$ of $l$ is the orthogonal cross section of $\Lambda_\alpha$ in the $[x, y]$-plane.

(2) It means no restriction to assume that $\Gamma$ is a cone centered at the origin $O$ of the coordinate system. Since the tangent planes of $\Gamma$ pass through the cone's vertex $O$, the intersection lines of any pair of tangent planes of $\Gamma$ also pass through $O$. Therefore, the tangent planes that enclose the fixed angle $\alpha$ also pass through $O$, and thus, the isoptic ruled surface $\Gamma_\alpha$ of $\Gamma$ is also a cone centered at $O$. □

### 2.1. Cylinders and cones of revolution

1. According to Theorem 2.1, the isoptic surfaces of a cylinder $\Lambda$ of revolution are cylinders $\Lambda_\alpha$. The orthogonal cross section $l$ of $\Lambda$ is a circle. Since the $\alpha$-isoptic curves of $l$ are circles $l_\alpha$ concentric with $l$, the isoptic cylinders $\Lambda_\alpha$ are cylinders of revolution coaxial with $\Lambda$.

   Let $r > 0$ denote the radius of $\Lambda$, and thus, of $l$. Then, it is elementary to show that

   $$r_\alpha = r \operatorname{cosec} \frac{\alpha}{2}$$

   is the radius of the isoptic circle $l_\alpha$ of $l$, and consequently, it is the radius of the isoptic cylinder $\Lambda_\alpha$.

2. Let $\Gamma$ be a cone of revolution with the aperture angle $0 < 2\omega < \pi$. According to Theorem 2.1, its isoptic ruled surface $\Gamma_\alpha$ is also a cone. Further, $\Gamma_\alpha$ is a cone of revolution: The cone $\Gamma$ is invariant under all rotations about its axis. Applying these rotations to a pair of tangent planes $(\tau_1, \tau_2)$ of $\Gamma$ preserves the angle $\alpha = \sphericalangle(\tau_1, \tau_2)$ enclosed by them and rotates the intersection $j = \tau_1 \cap \tau_2$ about the same axis. Thereby, $j$ sweeps a cone $\Gamma_\alpha$ of revolution with $\alpha = \sphericalangle(\tau_1, \tau_2)$.

   The aperture angle $2\omega_\alpha$ of the isoptic cone $\Gamma_\alpha$ is related to the aperture angle $2\omega$ of $\Gamma$ and the optic angle $\alpha$ by
$$\cos\omega_\alpha = \sqrt{\frac{\cos 2\omega + \cos\alpha}{1 + \cos\alpha}}. \tag{1}$$
   We shall verify this by a simple computation: Let the cone $\Gamma$ be given by its equation
$$\Gamma:\ x^2 + y^2 - \frac{z^2}{k^2} = 0 \ \text{ with } \ k = \operatorname{ctg}\omega.$$
   The tangent planes of $\Gamma$ have the equations
$$\tau(v):\ k(x\cos v + y\sin v) - z = 0 \ \text{ with } \ v \in [0, 2\pi).$$
   The rulings $e$ of the isoptic cone are the intersection lines of pairs of different tangent planes. Due to the rotational symmetry and invariance of the construction under rotations, we may assume that $e = \tau(v) \cap \tau(2\pi - v)$ for a yet undetermined $v$. Thus, $e$ has the direction $\mathbf{e} = (1, 0, k\cos v)$. The angle criterion $\sphericalangle(\tau(v), \tau(2\pi - v)) = \alpha$ yields
$$\cos\alpha = \frac{1 + k^2\cos 2v}{1 + k^2} = \frac{1 - k^2 + 2\cos^2 v}{1 + k^2}.$$
   This yields (1) since
$$\cos\omega_\alpha = \cos\sphericalangle(\mathbf{e}, (0,0,1)) = \frac{k\cos v}{\sqrt{1 + k^2\cos^2 v}}.$$

### 2.2. Quadratic cylinders and cones

1. The orthogonal cross section of a quadratic cylinder $\Lambda$ is a conic $l$ of the same affine type as the cylinder. Thus, the isoptic curves $l_\alpha$ are either the isoptics of ellipses, parabolae, or hyperbolae. Therefore, the isoptic surfaces of quadratic cylinders are cylinders with the conics' isoptics as orthogonal cross sections. If the normal forms of the equations of the quadratic cylinders are
$$mx^2 + ny^2 = 1, \quad m \cdot n \neq 0 \quad (\text{elliptic if } \ \mathrm{m}\cdot\mathrm{n} > 0, \quad \text{hyperbolic if } \ \mathrm{m}\cdot\mathrm{n} < 0),$$
$$2my = x^2, \quad m \neq 0 \quad (\text{parabolic}),$$
   then the equations of the isoptic cylinders are
$$(m+n-mn(x^2+y^2))^2\sin^2\alpha = 4mn(mx^2+ny^2-1)\cos^2\alpha \quad (\text{elliptic or hyperbolic case}),$$
$$(4x^2+(m-2y)^2)\cos^2\alpha = (m+2y)^2 \quad (\text{parabolic cases})$$
   since the equations of the isoptic cylinders agree with the equations of the isoptics of the conics. For the deduction of these equations see [17]. Hence, the *isoptic cylinders*

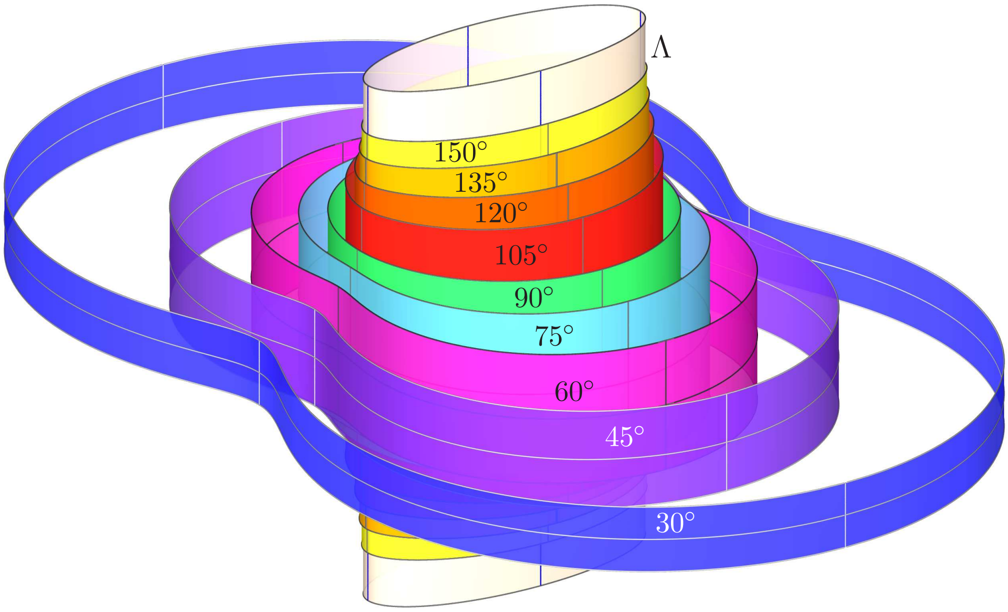


Figure 1: Isoptic cylinders $\Lambda_\alpha$ of an elliptic cylinder $\Lambda$ of various optic angles $\alpha = 30°$, $45°$, $60°$, $75°$, $90°$, $105°$, $120°$, $135°$, $150°$. The orthoptic cylinder $\Lambda_{90°}$ is a cylinder of revolution.

*of conics are either quartic cylinders or hyperbolic cylinders*, depending on whether the initial cylinder has an axis (line of centers of cross sections) or is parabolic. Figure 1 shows some isoptic cylinders of an elliptic cylinder $\Lambda$.

The orthoptic cylinders of a quadratic cylinder have special shapes, just like the orthoptics of conics: The orthoptic cylinders of elliptic or hyperbolic cylinders are cylinders of revolution (cf. Fig. 1, the green cylinder labeled with 90°). Since the orthoptic curve of a parabola equals the parabola's directrix, the orthoptic cylinder of a parabolic cylinder is a plane, *i.e.*, its director plane.

2. The isoptic ruled surfaces of cones are in a close relation to isoptic curves in elliptic geometry: Let $\Gamma$ be a cone centered at the origin $O$ of the coordinate system. The Euclidean unit sphere $S^2$ centered at $O$ intersects $\Gamma$ along a spherical curve $\gamma$. (Conversely, any curve $\gamma \in S^2$ defines a unique cone, provided that antipodal points are identified.)

   The tangent planes of $\Gamma$ intersect $S^2$ along great circles, *i.e.*, the straight lines in spherical geometry. Any two tangent planes $\tau_1$ and $\tau_2$ of $\Gamma$ that enclose the angle $\alpha$ intersect $S^2$ along two great circles $t_1$ and $t_2$ of $S^2$ that intersect at the angle $\alpha$. Hence, the rulings of $\Gamma_\alpha$ intersect $S^2$ in the points of the *spherical isoptic* $\gamma_\alpha$ (cf. [3]) of the *spherical image* $\gamma$ of $\Gamma$, see Figure 2.

## 3. Helical and spiral developables

The orbit of a point under one-parameter subgroup of the group of Euclidean motions is a helix and helical surfaces are swept by curves (different from point orbits) that undergo such a one-parameter subgroup. Thus, helical surfaces are invariant under a generating subgroup. The presence of a generating one-parameter motion will simplify the computation of the isoptic ruled surfaces of helical developables. This holds similarly true if we replace the Euclidean motion group by the equiform motion group. Here, the paths of points are the so-called

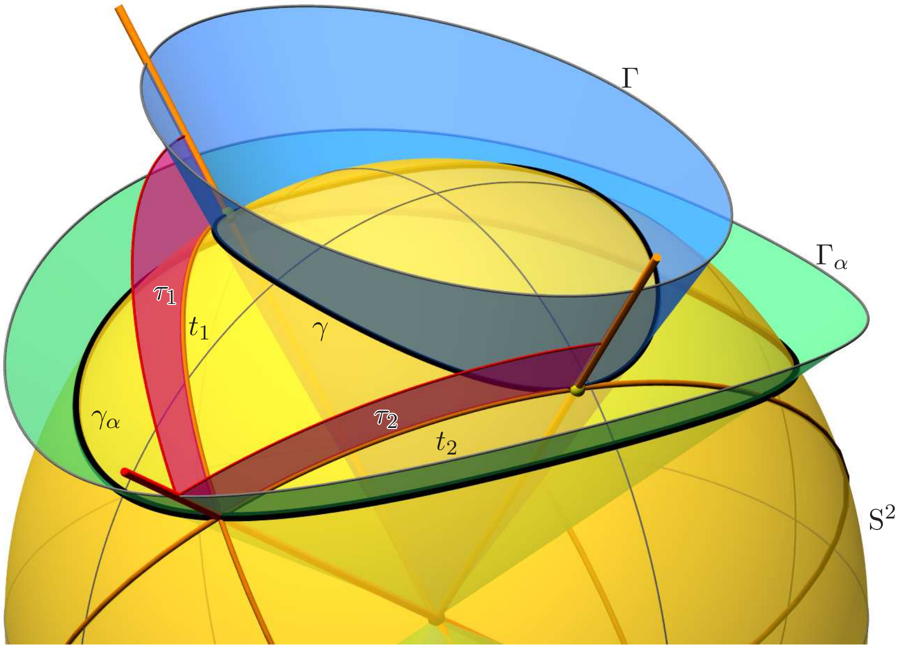


Figure 2: The spherical isoptic curve $\gamma_\alpha$ of the spherical curve $\gamma$ is the intersection of the isoptic cone $\Gamma_\alpha$ of $\Gamma$. Note that the spherical tangents (great circles) $t_1$ and $t_2$ of $\gamma$ meet at a point of the spherical isoptic $\gamma_\alpha$.

cylindro-conical spirals and the invariant surfaces are called spiral surfaces, cf. [6, 12].

Helical and spiral ruled surfaces are invariant with respect to their generating subgroups. Since tangent planes are moved along the surface and angles are not altered if they undergo Euclidean or equiform motions, we can show:

**Theorem 3.1** *The isoptic ruled surfaces of helical and spiral developables are helical and spiral ruled surfaces, respectively.*

*Proof:* Assume that $R$ is a helical developable. Its $\alpha$-isoptic ruled surface is the locus of intersection lines $j$ of such pairs $(\tau_1, \tau_2)$ of tangent planes of $R$ that fulfill $\sphericalangle(\tau_1, \tau_2) = \alpha$ with the fixed angle $\alpha \in (0, \frac{\pi}{2})$. The generating helical motion moves $\tau_1$, $\tau_2$, and $j$, such that it leaves the angle between $\tau_1$ and $\tau_2$ unchanged and $j$ sweeps a helical ruled surface. We can use similar arguments in the case of spiral developables and their isoptics. □

**Remark:**

The argumentation in the proof uses the invariance of certain curves and surfaces, the invariance of tangent planes and angles under some transformations. In principle, we used such an argument when we showed that the isoptic surfaces of cones of revolution are surfaces of the like kind. We could also have gone this way in the case of the cylinder of revolution. Note that the cylinder of revolution is invariant under two different one-parameter subgroups of the Euclidean motion group, while the cone of revolution is invariant under two different one-parameter subgroups of the group of equiform motions. ◇

It is not at all surprising that the isoptic ruled surfaces of helical or spiral developables are not algebraic: Let $\tau_1$ be a fixed tangent plane of the initial developable ruled surface $R$. It intersects another tangent plane $\tau_2 \neq \tau_1$ of $R$ at the angle $\alpha$ along a ruling $j$ of the isoptic $J_\alpha$. Then, we apply the generating motion (either Euclidean or equiform) to the tangent plane $\tau_2$

(while we keep $\tau_1$ fixed) until it reaches a position $\tau_2' \neq \tau_2$, where again $\sphericalangle(\tau_1, \tau_2') = \alpha$ holds. This is always possible because of the periodicity of the rotational component of Euclidean and equiform motions. Clearly, $j = \tau_1 \cap \tau_2 \neq \tau_1 \cap \tau_2' = j'$ and $j' \in J_\alpha$. Therefore, even in one plane $\tau_1$, we can find infinitely many rulings of the isoptic surface $J_\alpha$, which can never be the case with an algebraic surface. On the other hand, during the computation of the isoptic ruled surfaces, we will have to solve transcendental equations in order to find the parameter values corresponding to $\tau_2$, $\tau_2'$, ....

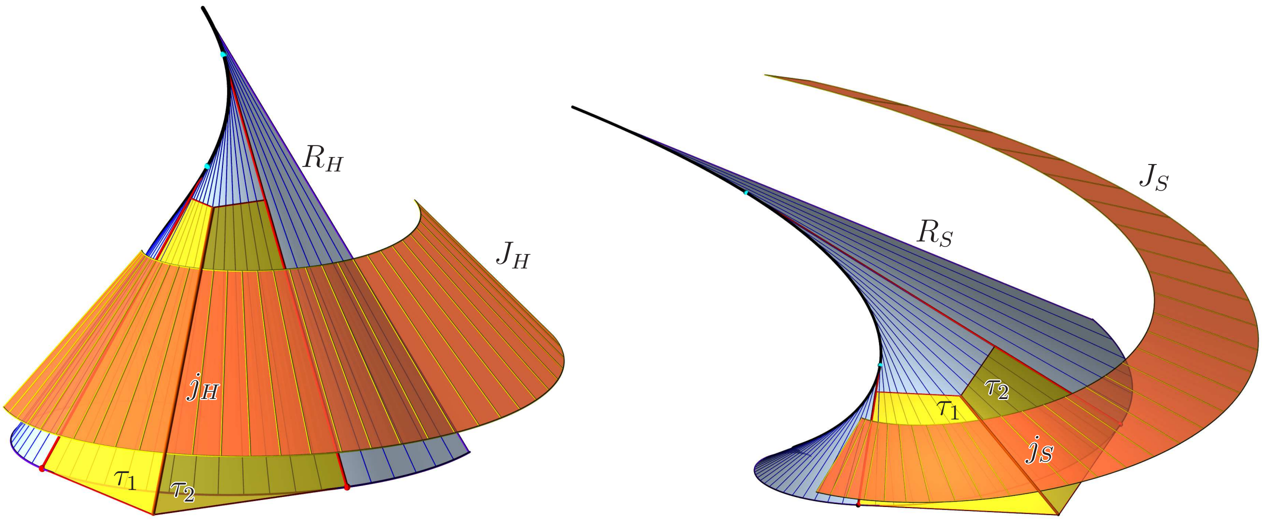


Figure 3: Left: isoptic ruled surface $J_H$ of a helical developable $R_H$. Right: isoptic ruled surface $J_S$ of a spiral developable $R_S$. Only a strip of $J_\star$ between two horizontal planes is shown.

Figure 3 shows an example of an isoptic ruled surface of a helical developable and a spiral developable.

### 3.1. Helical developables

We recall that a developable $R$ is determined by its curve $g$ of regression. So, it can be assumed that $\mathbf{g}(t) = (r\cos t, r\sin t, pt)$ with $t \in \mathbb{R}$ is a parametrization of the curve of regression and $r, p \in \mathbb{R}^+$ are the radius and the pitch, respectively. (A negative pitch $p$ changes the winding of the curve $g$, and all results turn out to be mirror images of the respective curves and surfaces with pitch $|p|$.) Obviously, $g$ is generated by the helical motion with the $z$-axis for its axis and the pitch $p$.

The one-parameter family of $g$'s osculating planes equals the one-parameter family of tangent planes of $R$. The normals of the osculating planes are the binormals of $g$, and thus, they are parallel to $\dot{\mathbf{g}} \times \ddot{\mathbf{g}}$, where dots indicates differentiation with respect to the parameter $t$ and $\times$ is the canonical exterior product of two vectors in $\mathbb{R}^3$. Hence, the unit binormal vector field along $g$ can be parametrized by

$$\mathbf{g}_3 = \frac{1}{\sqrt{p^2 + r^2}} \begin{pmatrix} p\sin t \\ -p\cos t \\ r \end{pmatrix} \quad \text{with} \quad t \in \mathbb{R}.$$

The one-parameter family of $g$'s osculating planes $\sigma$ can be given by the equations

$$\sigma(t): \; p\sin t\, x - p\cos t\, y + rz = prt$$

depending on the curve parameter $t \in \mathbb{R}$.

In order to find pairs of osculating planes of the helical developable $R$ enclosing the fixed angle $0 < \alpha < \pi$, we assume that $u \in \mathbb{R} \setminus \{0\}$. Thus, $u$ and $-u$ are two different parameters. The tangent planes $\sigma(-u) = \tau_1$ and $\sigma(u) = \tau_2$ of $R$ are different and intersect along the straight line $e$ parametrized by

$$\mathbf{e}(w) = \begin{pmatrix} \frac{ru}{\sin u} \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ 1 \\ \frac{p\cos u}{r} \end{pmatrix}, \quad \text{with w} \in \mathbb{R}. \tag{2}$$

The condition $\sphericalangle(\tau_1, \tau_2) = \alpha$ can be expressed mathematically as $\langle \mathbf{g}_3(u), \mathbf{g}_3(-u) \rangle = \cos\alpha$ (with $\langle \cdot, \cdot \rangle$ denoting the canonical scalar product of two vectors in $\mathbb{R}^3$) and relates the parameter $u$ with the optic angle $\alpha$ via

$$\cos\alpha = \frac{p^2 \cos 2u + r^2}{p^2 + r^2}. \tag{3}$$

With (2) and (3) we have the parametrization of exactly one line that is the intersection of a pair of tangent planes of $R$ which enclose the angle $\alpha$. Applying the underlying helical motion (with the $z$-axis as its axis and the pitch $p$) to the line $e$ parametrized by (2), we obtain the isoptic ruled surface $R_\alpha$.

Especially, the orthoptic ruled surfaces of helical developables are defined by

$$\cos 2u = -\frac{r^2}{p^2}$$

and turn out to be real if $|r| < |p|$.

### 3.1.1. Developable isoptic helical ruled surfaces

The isoptic ruled surface of a helical developable can also be developable. Again the underlying helical motion has the pitch $p$ and the $z$-axis for its axis. Let $e$ be a straight line at the distance $d > 0$ from the $z$-axis and assume that $k \neq 0$ is the inclination of $e$ (with respect to the $[x, y]$-plane). Then, $e$ can be parametrized by $\mathbf{e}(w) = (d, w, kw)$ with fixed $d, k \in \mathbb{R}$ (and $w \in \mathbb{R}$). We apply the underlying helical motion to $e$ and the helical ruled surface $S$ swept by $e$ is developable if, and only if, its Gaussian curvature equals zero. This yields

$$p = dk. \tag{4}$$

From (2) we can read off $d = \frac{ru}{\sin u}$ and $k = \frac{p\cos u}{r}$, where $u$ is subject to (3) in order to make $e$ in (2) a ruling of an isoptic ruled surface.

The product of these values has to be equal to the parameter $p$ of the helical motion. This yields

$$u\cos u - \sin u = 0 \Longleftrightarrow u = \mathrm{tg} u, \tag{5}$$

which is obviously independent of the parameter $p$. Since (5) has infinitely many real solutions different from 0 (cf. Figure 4, left), we can state:

**Theorem 3.2** *To each helical developable $R$ there exist infinitely many torsal (developable) helical ruled surfaces which are isoptic ruled surfaces of $R$. The optic angles can be obtained by inserting the solutions of* (5) *into* (3).

### 3.2. Spiral developables

Again, we can start with the curve of regression $g$: Let $\mathbf{g}(t) = \exp(pt)(\cos t, \sin t, 1)$ with spiral parameter $p \neq 0$ and $t \in \mathbb{R}$ be a parametrization of $g$. Note that it is always possible to choose a Cartesian coordinate such that the spiral center $O$ coincides with the origin of the coordinate system and the spiral axis is identic with the $z$-axis. Then, the $[x, y]$-plane is spiral-invariant, *i.e.*, it is left fixed under the spiral motion and the restriction of the spiral motion to this plane is a planar spiral motion (superposition of a uniform scaling and a proportional uniform rotation about the spiral center $O$, cf. [6, 12].) Further, it is sufficient to assume that $g$ is the above given cylindro-conical spiral since it is a non-trivial orbit in this group, which cannot already occur as orbit in the group of Euclidean motions.

In the same way as we have done in the case of helical developables, we compute the binormal vector field of the curve of regression and find

$$\mathbf{g}_3(t) = \frac{1}{\sqrt{(1+p^2)(1+2p^2)}} \begin{pmatrix} -p(p\cos t - \sin t) \\ -p(p\sin t + \cos t) \\ 1+p^2 \end{pmatrix}, \tag{6}$$

which yields the equations of $g$'s osculating planes

$$\sigma(t): \ p(\sin t - p\cos t)x - p(\cos t + p\sin t)y + (1+p^2)z = \exp(pt).$$

Now, we let $u \in \mathbb{R}$ be some parameter. Then, $\tau_1 = \sigma(-u)$ and $\tau_2 = \sigma(u)$ are two tangent planes of the spiral developable. These planes intersect along the lines

$$\mathbf{e}(w) = \begin{pmatrix} \frac{\sinh pu}{p\sin u} \\ 0 \\ \frac{p\sinh pu\cos u + \cosh pu \sin u}{(1+p^2)\sin u} \end{pmatrix} + w \begin{pmatrix} p \\ 1 \\ p\cos u \end{pmatrix} \quad \text{with } w \in \mathbb{R}. \tag{7}$$

We use an analogue to the angle criterion given in (3): Therefore, we compute the cosine of the angle between two different binormals (6) of $g$, say at $t = u$ and at $t = -u$. This yields

$$\cos\alpha = \frac{2p^2\cos^2 u + 1}{2p^2 + 1}. \tag{8}$$

Orthoptic ruled surfaces to spiral developables show up if $\cos 2u = -1 - \frac{1}{p^2}$. Since $p \in \mathbb{R}\setminus\{0\}$, $-1 - \frac{1}{p^2} < -1$, and thus, $u \notin \mathbb{R}$. Therefore, spiral developables do not have real orthoptic ruled surfaces.

#### 3.2.1. Developable isoptic spiral developables

In the case of spiral developables, we have to determine the torsality criterion analogous to (4). For that purpose, we have to assume that one particular ruling $e$ of the spiral ruled surface is parametrized by $\mathbf{e}(w) = (d + aw, w, kw + h)$, with $w \in \mathbb{R}$ and the same assumptions on the constants $a, d, k \in \mathbb{R}$. The additional shift $h \in \mathbb{R}$ has to be taken into account since it would be a restriction to assume that the common normal of the spiral axis (*i.e.*, the $z$-axis) and $e$ lies in the spiral-invariant plane $z = 0$. We apply the spiral motion with the origin $O$ of the coordinate system as spiral center, with $z$-axis as spiral axis, and with the spiral parameter $p \in \mathbb{R}$ to the line $e$ (7) and obtain the most general form of a spiral ruled surface (parametrization). Finally, we compute the Gaussian curvature and set it equal to zero which yields

$$ap(ah - dk) = dk - hp \qquad (9)$$

Note that for the isoptic ruled surface $u$ in (7) is subject to (8).
From (7), we have $a = p$, $d = \frac{\sinh pu}{p \sin u}$, $h = \frac{p \sinh pu \cos u + \cosh pu \sin u}{(1+p^2)\sin u}$, and $k = p\cos u$, which we substitute into (9) and find

$$p \operatorname{tg} u - \operatorname{tgh} pu = 0, \qquad (10)$$

which relates the parameter(s) $u$ chosen on the initial spiral developable with the spiral parameter $p$ (see Figure 4, right) such that the isoptic ruled surface $J_\alpha$ is developable.

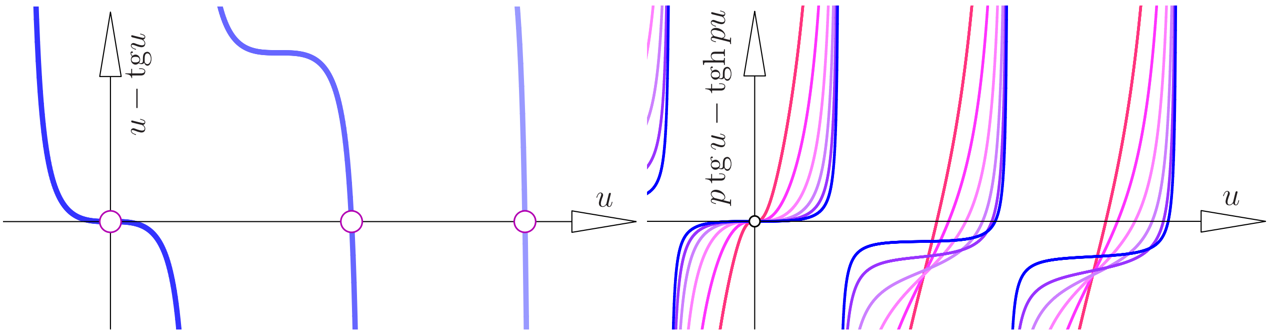


Figure 4: Left: The zeros of (5) correspond to developable isoptic helical ruled surfaces. Right: The zeros of (10) correspond to developable isoptic spiral ruled surfaces. The curves for some spiral parameters $p$ are displayed.

We can summarize:

**Theorem 3.3** *There exist infinitely many developable isoptic spiral ruled surfaces $J_\alpha$ to a given spiral developable $R$. The respective optic angles can be obtained from* (10) *and* (8).

## 4. Algebraic isoptic ruled surfaces

In this section, we describe how to compute the isoptic ruled surfaces of algebraic developables that allow a rational or even polynomial parametrization. Even with this restriction, the limits of the symbolic computational approach will become clear.
Assume that $\mathbf{g} = (g^1, g^2, g^3) : \mathbb{R} \to \mathbb{R}^3$ is a vector function with polynomial components $g^i$ (and thus, the parametrization of a space curve $g$ in Euclidean three-space). Further, we assume that all involved functions are at least two times differentiable. The algebraic degree $d$ of the algebraic curve $g$ equals $d = \max_i(\deg g^i)$.
We compute the one-parameter family of $g$'s osculating planes, *i.e.*, the dual curve $g^\star$, which can be parametrized by

$$\mathbf{g}^\star = (\det(\mathbf{g}, \dot{\mathbf{g}}, \ddot{\mathbf{g}}), -\dot{\mathbf{g}} \times \ddot{\mathbf{g}}).$$

With $\mathbf{g}^\star : \mathbb{R} \to \mathbb{R}^4$ we have a parametrization of a curve in the projectively extended dual space of the Euclidean three-space. From counting the degrees of the derivatives, we can infer that $\deg g^\star \leq 3d - 3$. This is an upper bound of the degree of $g^\star$, which will be reached only in a few cases.

By definition, the isoptic $J_\alpha$ of the developable ruled surface $R = g^\star$ is the locus of intersection lines of planes in $g^\star$ at different parameter instances. Therefore, an upper bound of the degree of the parametrization of the intersection lines equals $\deg J_\alpha \le 6d - 6$.

Let $u$ and $v$ be two different parameter values. The corresponding planes of $g^\star$ are the tangent planes $\tau_1 = \mathbf{g}^\star(u)$ and $\tau_2 = \mathbf{g}^\star(v)$. We shall simplify the notation and write $\mathbf{g}_u := \mathbf{g}(u)$ and $\mathbf{g}_v := \mathbf{g}(v)$. Thus, the rulings $j$ of the isoptic ruled surface can be given in terms of Plücker coordinates as

$$\mathbf{J} = \Big( \underbrace{(\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u) \times (\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v)}_{=\mathbf{j}}, \underbrace{(\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u) \det(\mathbf{g}_v, \dot{\mathbf{g}}_v, \ddot{\mathbf{g}}_v) - (\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v) \det(\mathbf{g}_u, \dot{\mathbf{g}}_u, \ddot{\mathbf{g}}_u)}_{=\bar{\mathbf{j}}} \Big). \tag{11}$$

Formally, the polynomials $\mathbf{g}_u$ and $\mathbf{g}_v$ are the same polynomials. Thus, each coordinate function of $\mathbf{J}$ is divisible by the factor $u - v \neq 0$ (which expresses that $\tau_1 \neq \tau_2$), and therefore, $\deg \mathbf{J} \le 5d - 6 - 1 = 5d - 7$. To be more precise, $\mathbf{J} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^6$ is the Plücker coordinate representation of the congruence of chords of the curve $g$.

From the Plücker representation $\mathbf{J} = (\mathbf{j}, \bar{\mathbf{j}})$, we can change to a parametrization of the congruence of chords based on a directrix and the vector field of the rulings as

$$\mathbf{r}(u, v; w) = \frac{\mathbf{j} \times \bar{\mathbf{j}}}{\langle \mathbf{j}, \mathbf{j} \rangle} + w\mathbf{j} \quad \text{with} \quad w \in \mathbb{R}.$$

Within this congruence, we find the isoptic ruled surfaces $J_\alpha$ of $R$ by imposing the angle condition on the pairs $(\tau_1, \tau_2)$ of tangent planes of $R$. This results in a condition on $u$ and $v$. In order to find isoptic ruled surfaces to the optic angle $\alpha = \sphericalangle(\tau_1, \tau_2)$, we have to determine either $u$ or $v$ such that

$$\cos\alpha = \frac{\langle \dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u, \dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v \rangle}{\|\dot{\mathbf{g}}_u \times \ddot{\mathbf{g}}_u\| \cdot \|\dot{\mathbf{g}}_v \times \ddot{\mathbf{g}}_v\|} \tag{12}$$

holds. Squaring the latter equation in order to make it a purely rational condition, we find that the degrees of the numerator and the denominator of the right-hand side are $8d - 12$ at most. Further reductions of the degree will appear in the case of so-called *higher degree parabolae*, *i.e.*, curves of degree $d$ which have a $(d-1)$-fold intersection with the plane at infinity.

An explicit computation of a parametrization $\mathbf{j}_\alpha$ of the isoptic ruled surface $J_\alpha$ consist of several steps of elimination, either by means of the Groebner basis algorithm or by means of resultants.

### 4.1. Some special polynomial curves

#### 4.1.1. Cubic curves

The cubic space curves and their tangent developables serve as illustrative examples. From the projective point of view, there exists only one type of (non-planar) cubic space curve, cf. [4]. However, here the affine point of view is of more importance. Thus, we distinguish between the following four types of curves:

(i) *cubic ellipses* with one real ideal point (and a complex conjugate pair),

(ii) *cubic hyperbolae* with three different real ideal points,

(iii) *cubic parabolic hyperbolae* with two different real ideal points (one of multiplicity two), and

(iv) *cubic parabolae* osculating the plane at infinity.

In any of the four cases, the developable $R$ is cubic as a one-parameter family of planes, quartic as a set of points. Thus, the intersection with any plane, and especially with the ideal plane $\omega$ is a quartic curve $r_\omega$ of degree four and class three. According to the above classification of cubic space curves, the shapes of the curves $r_\omega$ are

(i) a double line plus a pair of lines (eventually, one line from the pair coincides with the double line making it a triple line,

(ii) a quartic with three cusps,

(iii) a cusped cubic with its inflection tangent, and

(iv) a conic with a double tangent.

In the cases (i) – (iii), the angle criterion

$$\cos\alpha = \langle \mathbf{g}_3(u), \mathbf{g}_3(v)\rangle \tag{13}$$

results in an algebraic equation of degree 12 relating the parameters $u$ and $v$ on the cubic curve. The thus described algebraic curves in the parameter domain are of genus 19 in any case. Things become simpler for the cubic parabola. Then, the angle criterion defines an octic curve of genus five in the $[u, v]$-plane.

The degrees of the isoptic ruled surface to arbitrary angles $\alpha \neq 0, \frac{\pi}{2}$ become relatively high. Nevertheless, parametrizations can be derived from (11) and (13) by eliminating either $u$ or $v$.

It is not at all surprising that the orthoptic ruled surfaces are of less degree. For the orthoptics of cubic developables, only the numerator of (13) matters. This polynomial is a full square of a sextic polynomial in the cases (i) – (iii) and a quartic polynomial in the case (iv) of cubic parabolae.

Explicit and useful parametrizations of orthoptic ruled surfaces can be given in the case of the three-parameter family of cubic parabolae given by

$$\mathbf{p}(t) = \left(At, Bt^2, \frac{1}{3AC}(B^4 + C^2)t^3\right), \tag{14}$$

$t \in \mathbb{R}$ and $A, B, C \in \mathbb{R}\setminus\{0\}$ are shape parameters. In this particular case, the angle criterion (13) factors and yields

$$((B^4 + C^2)uv + A^2B^2)(B^2(B^4 + C^2)uv + A^2C^2) = 0 \tag{15}$$

which are the equations of two hyperbolae in the $[u, v]$-plane. This means that the mapping $u \mapsto v$ that assigns to each parameter value $u$ a new parameter value $v$ such that the osculating planes at points corresponding to $u$ and $v$ are orthogonal, is a projective mapping. So we are able to state the following:

**Theorem 4.1** *The orthoptic ruled surfaces of the tangent developables $R$ of all cubic parabolae* (14) *consists of a pair of hyperbolic paraboloids.*

*Proof:* The fact that the orthoptic surface splits into two parts is caused by the splitting of the orthogonality condition (15). Each factor of (15) describes a linear rational mapping $u \mapsto v$, and thus, a projective automorphism on $p$ and on the ideal curve on $R$ (cf. [4]).

The fact that $J_{\frac{\pi}{2}}$ consists of a pair of hyperbolic paraboloids is best shown by computation. Following beaten tracks, we find the equations of the orthoptics:

$$\begin{aligned} J_{1,\frac{\pi}{2}}: &\quad 9C(B^4+C^2)(B^2x+Cz)z \quad + \quad 3A^2B^3(B^4+C^2)y \quad + \quad A^4B^6=0,\\ J_{2,\frac{\pi}{2}}: &\quad 9B^4(B^4+C^2)(B^2x+Cz)z \quad + \quad 3A^2BC^2(B^4+C^2)y \quad + \quad A^4C^4=0. \end{aligned}$$

It is left to the reader to verify that $J_{1,\frac{\pi}{2}}$ and $J_{2,\frac{\pi}{2}}$ are hyperbolic paraboloids. Hint: Look at the already factored quadratic terms. □

Note that the two hyperbolic paraboloids in Theorem 4.1 coincide if two factors of (15) are proportional. This is the case if, and only if, $C^2 = B^4$. A case with two different orthoptics is shown in Figure 5.

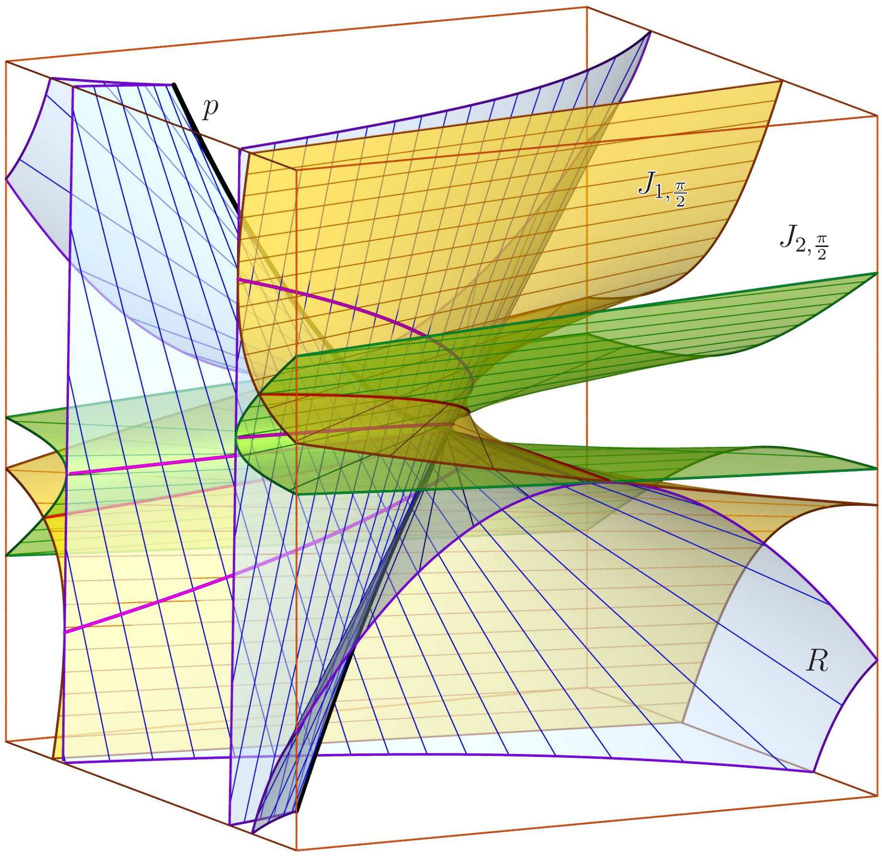


Figure 5: The orthoptic ruled surface of one of the cubic parabolas mentioned in Theorem 4.1 splits into a pair of orthoptic hyperbolic paraboloids.

### 4.1.2. A singular quartic curve

Isoptic ruled surfaces and especially orthoptic ruled surfaces can be computed to tangent developables $R$ even if their curves $g$ of regression carry singularities. We shall have a look a the following example: The parametrization of a quartic space curve of the $2^{\text{nd}}$ kind (cf. [4]) with a cusp seems to be artificial at first glance:

$$\mathbf{q}(t) = \left(\frac{1}{2}At^2, \frac{1}{3}Bt^3, \frac{B^2(1+C^2)}{8A(1-C^2)}t^4\right), \quad t \in \mathbb{R}, \tag{16}$$

$A, B \in \mathbb{R} \setminus \{0\}$ and $C \in \mathbb{R}$ are shape parameters. However, it allows us to show that the orthoptic surface has a very special shape:

**Theorem 4.2** *The orthoptic surfaces of the tangent developables of the singular quartics in the three-parameter family of quartics of the* $2^{\text{nd}}$ *kind given by* (16) *split into a pair of cubic surfaces.*

*Proof:* In this case, the angle criterion (13) splits into two quadratic factors:

$$((B^2(1+C^2)uv + 2A^2(1+C)^2)((B^2(1+C^2)uv + 2A^2(1-C)^2) = 0$$

Again, there arise two projective mappings $u \mapsto v$ which together with (11) yield parametrizations of the orthoptics. A subsequent implicitization confirms the theorem. □

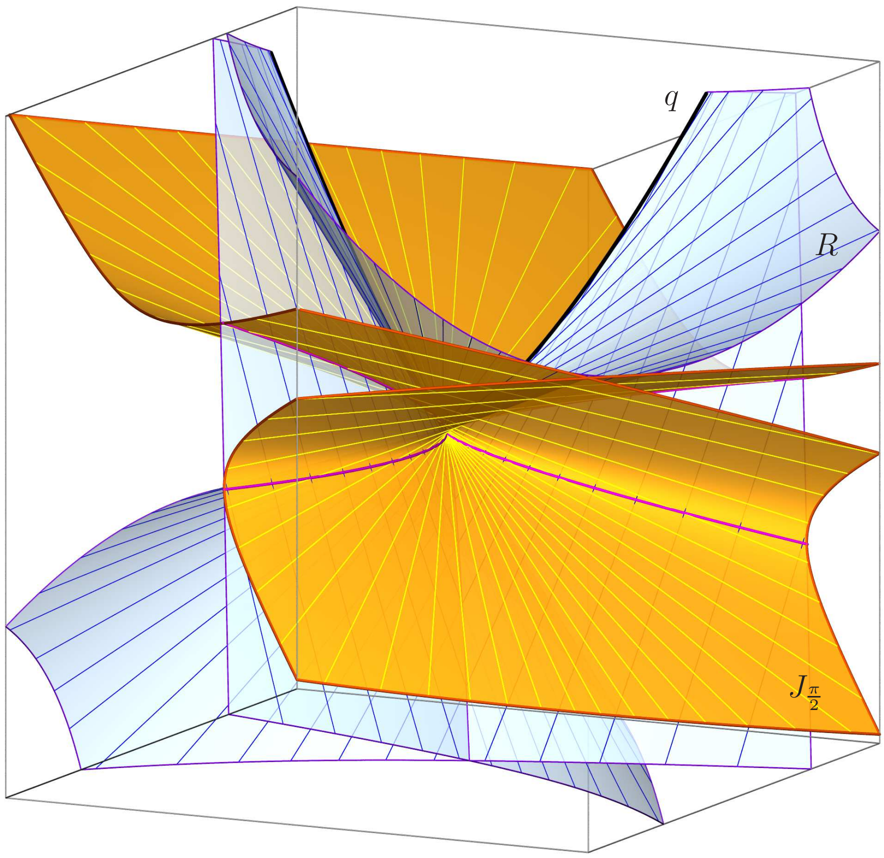


Figure 6: The orthoptic ruled surface of a certain member of the family of quartics from Theorem 4.2 is a cubic ruled surface with multiplicity two.

Figure 6 shows the tangent developable of that curve $q$ in (16) where the two cubic orthoptics coincide. This occurs if, and only if, $C = 0$.

### 4.2. General remarks on simple polynomial curves

In this section, we only investigate orthoptic surfaces.

Among the space curves with polynomial parametrization, the curves with monomial coordinate functions are distinct. We have seen that the orthogonality criteria for osculating planes of the curves (14) and (16) factor and split into bivariate polynomials that describe hyperbolae in the parameter domain. Thus, a linear rational mapping (or a projective mapping) $u \mapsto v(u)$ is established that assigns to each point on the initial curve $g$ exactly one point with the orthogonal osculating plane.

The cubic parabolas and the cusped quartics we met are not the only curves with that special behaviour. Assume that

$$\mathbf{g}(t) = \left(at^i, bt^j, ct^k\right), \quad \text{with } t \in \mathbb{R} \tag{17}$$

is the parametrization of the curve $g$ of regression of a developable ruled surface $R$. Here, $1 \leq i < j < k \in \mathbb{N}$ and $a, b, c \in \mathbb{R}$. Now, only the numerator of the angle criterion (12) or (13) matters. The binormals of the curves (17) are parallel to

$$\mathbf{g}_3(t) = \left(bcjk(k-j)t^{k-i}, caki(i-k)t^{k-j}, abij(j-i)\right). \tag{18}$$

Here, we have canceled the dispensable factor $t^{i+j-3}$ since $i+j-3$ is at least 0, and the cancellation of positive powers of $t$ does not alter the direction of $\mathbf{g}_3$. It may ignore multiple rulings and tangent planes of $R$ at $t=0$.

With (18), the numerator of (12) becomes

$$(bcjk)^2(k-j)^2(uv)^{k-i} + (caki)^2(i-k)^2(uv)^{k-j} + (abij)^2(j-i)^2 = 0, \tag{19}$$

where $k-i > k-j > 1$, due to the initially made assumptions.

It is nearby to interpret (19) as curves in the $[u,v]$-plane. Since $k-i = \max(k-i, k-j)$, the curves (19) are of degree $2(k-i)$. These curves are always singular. Moreover, they split into at most $k-i$ equilateral hyperbolae (degree = 2) in the parameter plane, *i.e.*, in the $[u,v]$-plane. The hyperbolae have the equations $uv = z_\mu$, where $z_\mu$ is a root of the polynomial equation (19) with previously substituted $z = uv$.

All the components of (19) may be complex. Only if $k-i$ is odd, we have at least one real root $z_\mu$, and thus, the existence of a real orthoptic of the developable defined by (17) is guaranteed. Multiple components of (19) (corresponding to multiple roots $z_\mu$) determine orthoptic surfaces with that particular multiplicity.

We shall summarize our results:

**Theorem 4.3** *The orthoptic ruled surfaces of the developables with monomial curves* (17) *of regression consist of at most (not necessarily real) $k-i$ components. Each of these components is generated by a projective mapping $u \mapsto v(u)$ acting on the curve of regression and assigning to each point (and osculating plane) the corresponding point with the orthogonal osculating plane. The rulings of the orthoptic surface are the intersection lines of corresponding planes.*

## 5. Conclusion, future work

The class of isoptic ruled surfaces of developables invariant under certain groups of motions is not very rich. However, this is not the case with algebraic developables. There is a huge variety of algebraic space curves and corresponding tangent developables that allow for a computation of orthoptics and general isoptics. Clearly, we are restricted to low degree examples, but this could be a minor flaw. For example: The computation of isoptics to the developables of cubic parabolas is sufficient if we want to study the behaviour of isoptic ruled surfaces in the vicinity of a generic regular generator of a developable since the local cubic approximant of a space curve in the vicinity of a regular, non-inflection, and non-handle point is a cubic parabola.

The monomial curves parametrized by (17) are very special local expansions of space curves. Provided that $\kappa \neq 0$ is the curvature, $\dot{\kappa} \neq 0$ is the derivative of the curvature, and $\tau \neq 0$ is the torsion of a space curve in the vicinity of a regular, non-inflection, and non-handle point, then the local expansion of the curve $c$ equals

$$\mathbf{c}(t) = \left( s - \frac{\kappa^2}{2} s^3, \frac{\kappa}{2} s^2 + \frac{\dot{\kappa}}{6} s^3, \frac{\kappa\tau}{6} s^3 \right)$$

if we ignore terms in $s$ of degree larger than 3. In this case, the curve in the $[u, v]$-plane described by the angle criterion (12) does not degenerate and split into several low degree factors. It is of genus 1 if $\kappa, \dot{\kappa}, \tau \neq 0$. Nevertheless, it would be interesting to see the isoptic ruled surfaces of the tangent developable of $c$ since these surfaces describe the generic isoptics of a developable ruled surface in the vicinity of an ordinary, regular ruling of a developable.

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