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Loci of Poncelet triangles in the general closure case

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Abstract. We analyze loci of triangle centers over variants of two-well known triangle porisms: the bicentric and confocal families. Specifically, we evoke the general version of Poncelet's closure theorem whereby individual sides can be made tangent to separate in-pencil caustics. We show that despite the more complicated dynamic geometry, the locus of certain triangle centers and associated points remain conics and/or circles.

Mathematics Subject Classification. 51M04, 51N20, 51N35, 68T20.

Keywords. Locus, Poncelet, porism, closure, ellipse.

1. Introduction

Figure 1 (left) shows what is actually a special case of Poncelet's porism: if an N-gon can be found with all vertices on a first conic and all sides tangent to a second one (the "caustic"), then a one-parameter family of said N-gons exists, e.g., with vertex on any point on the first conic [5]. Figure 1 (right) illustrates the general case, called "Poncelet's Closure Theorem" (PCT), contemplating a porism with multiple caustics. It can be stated as follows [4]:

Theorem 1 (PCT). *Let $\mathcal{C}$ and $\mathcal{D}_i$, $i = 1, \ldots, M$ be $M + 1$ distinct conics in the same linear pencil. If an N-gon can be constructed with all vertices on $\mathcal{C}$ such that each side is tangent to some $\mathcal{D}_i$, a porism exists of such N-gons.*

Recall the pencil of two conics is a linear combination of their implicit equations [1].

Referring to Fig. 2, with PCT in mind, we analyze loci of triangle centers over variants of two well-known triangle porisms:

- the bicentric family (also known as poristic triangles or Chapple's porism): these have a fixed incircle and circumcircle; see Fig. 3.

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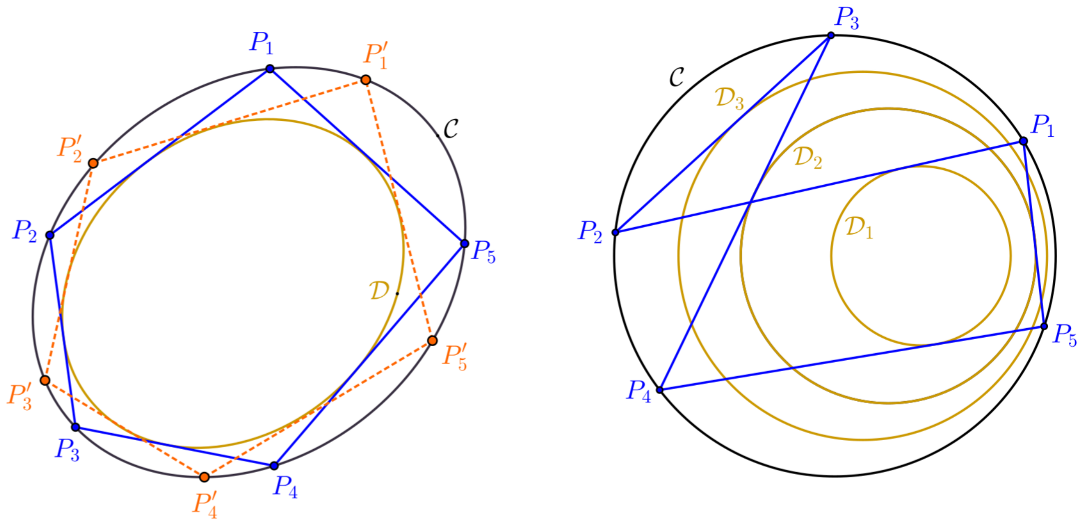

FIGURE 1 Left: A Poncelet porism of 5-gons interscribed between two ellipses $\mathcal{C}$ and $\mathcal{D}$. Video. Right: Poncelet's Closure Theorem (PCT) [4] contemplates a porism of polygons inscribed in an outer conic $\mathcal{C}$ with sides tangent to one or more in-pencil conics $\mathcal{D}_i$. Shown here is a pencil of coaxial circles. Video

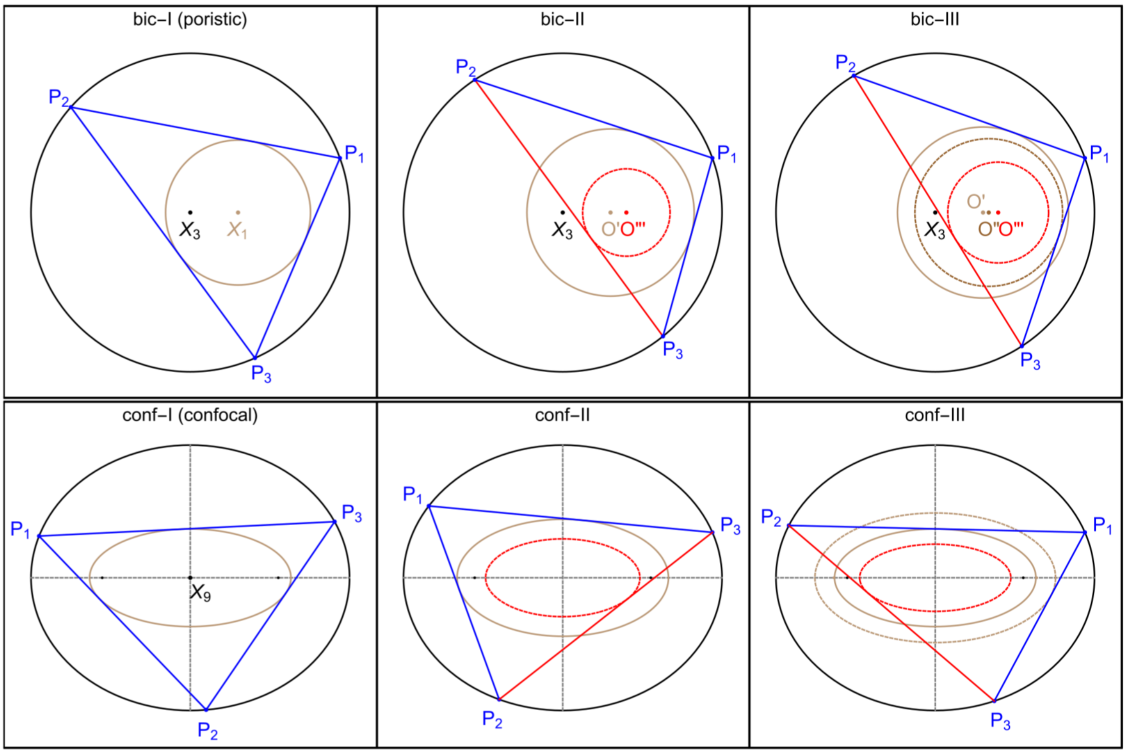

FIGURE 2 Triangle families considered herein, whose names appear at the top (bic-I, bic-II, etc.). The roman numeral after each name is the caustic count. Top row: the poristic pair (left) and its derivatives with 2, and 3 in-pencil caustics (middle and right). Bottom row: the confocal pair (left) and its derivatives with 2, and 3 in-pencil caustics (middle and right). Video

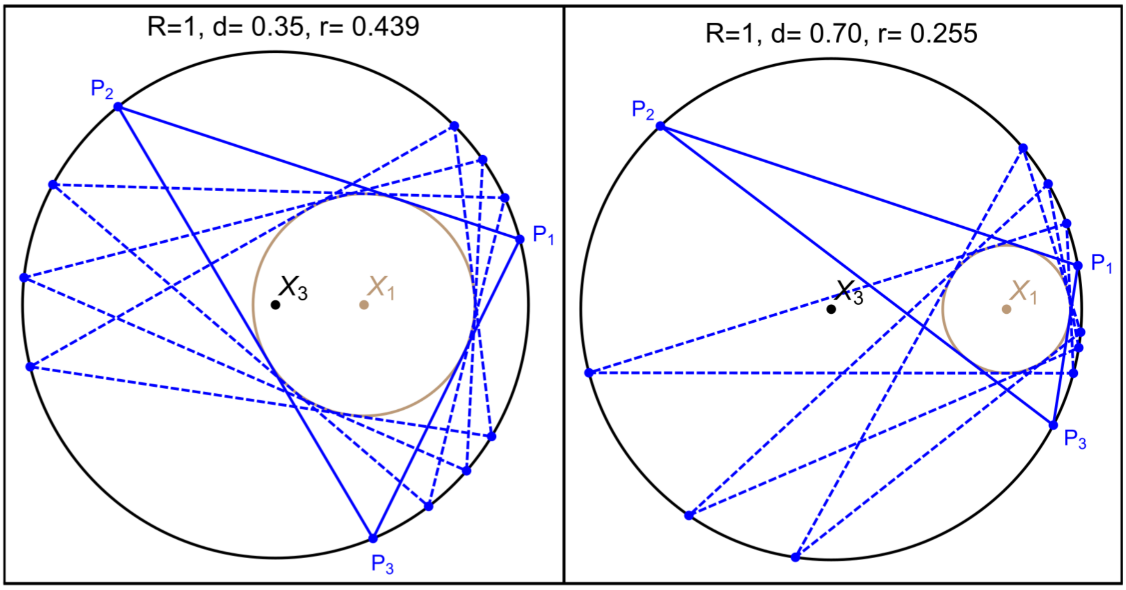

FIGURE 3 Two examples of the poristic family, i.e., a family of triangles (blue and dashed blue) inscribed in an outer circle (black) while circumscribed about an inner one (brown). Using Kimberling's notation [15], X_3 (resp. X_1) denote the circumcenter (resp. incenter). Live (color figure online)

- the confocal family (also known as the elliptic billiard): a constant-perimeter family of triangles interscribed between two confocal ellipses; see Fig. 10.

Specifically, we want to analyze loci of their triangle centers as we perturb their $N = 3$ caustic and/or add additional caustics, in the spirit of PCT. The main surprise is that the locus of certain triangle centers and excenters "survive" (remain conics) despite PCT.

Summary of the results

- For the 2-caustic poristic family; see Fig. 2 (center top):
  - Theorem 2: the locus of the incenter is still a circle.
  - Proposition 2: the locus of the barycenter is an algebraic curve of degree 6 (given explicitly in "Appendix C"). Indeed, in the standard 2-circle poristic family, this locus is a simple circle [16].
  - The locus of one of the excenters is also a circle while that of the other two are distinct non-conics.
  - Conjecture 1: experimental evidence suggests that a necessary (though not sufficient) condition for the locus of a given triangle center to be a conic is that its locus over the poristic family is a point.

- For the 2-caustic confocal family; see Fig. 2 (center bottom):
  - Proposition 4: the locus of the incenter is an ellipse only for the standard confocal pair.
  - Theorem 3: The loci of two of the excenters are the same ellipse while the third one sweeps a non-conic.
  - Corollaries 4 and 5: the elliptic locus of said excenters can assume, for certain configurations, special shapes: (i) its aspect ratio can be the reciprocal of that of the external ellipse, and (ii) it can be a circle.

In Sect. 6 we look at either a circle- or ellipse-inscribed family with three caustics (right column of Fig. 2). We find that (i) the locus of a triangle center is never a conic, and that (ii) the shape of many loci are more complicated (see Fig. 17).

Loci phenomena for all families and triangle centers considered are summarized side-by-side in Table 2 in Sect. 7.

Related work

Loci of triangle centers over poristic triangles have been studied in depth in [9, 16]. As shown there, the loci of triangle centers can be points, segments, conics, and other shapes.

Regarding the confocal family, loci of both incenter and excenters[1] are ellipses with reciprocal aspect ratios [7, 19]. Loci of other notable centers such as the circumcenter, orthocenter, etc., are also elliptic [6, 7, 10]. Certain loci are non-conic (e.g., that of the symmedian point, Fermat point, etc.) [11]. Remarkably, the locus of the mittenpunkt[2] is stationary at the common center [18].

Experiments suggests that only in the confocal pair can the locus of the incenter be a conic [14].

Article structure

The basic geometry of the poristic (resp. confocal) family is reviewed in Sect. 2 (resp. Sect. 4). In Sect. 3 (resp. Sect. 5) we analyze loci when an additional in-pencil caustic is added.

Links to simulation videos are included in the caption of most figures (a table with all videos mentioned appears in Sect. 7. As mentioned before, "Appendix C" provides an explicit parametrization for the locus of the barycenter under the bic-II family. "Appendix B" provides an explicit parametrization for bic-II and conf-II vertices. Finally, "Appendix A" compiles all main symbols used herein in a table.

[1] The intersections of external bisectors [21, Excenter].
[2] This is where lines from the excenters through the midpoints of a triangle concur [15, X(9)].

2. Review: poristic family

Referring to Fig. 3, any triangle is always "poristic" with respect to its circumcircle and incircle, in the sense that a one-parameter Poncelet family of triangles automatically exists interscribed between said circle pair.

This fact was first described by William Chapple [3], almost 80 years prior to Poncelet generalizing it to any pair of conics in 1822 [17]. Let d, R, r denote, respectively, the distance between circle centers, the circumradius and inradius. Chapple derived[3]

$$d^2 = R(R - 2r), \quad R > 2r. \tag{1}$$

Equivalently, $1/(R-d) + 1/(R+d) = 1/r$. The abovementioned porism is sometimes called the $N = 3$ *bicentric family*, i.e., a Poncelet porism between two non-concentric circles. We will refer to it as "bic-I" for reasons that will become clear.

Adopting Kimberling's X_k notation [15] for triangle centers, the incenter X_1 and circumcenter X_3 of bic-I triangles are fixed by definition. Indeed, it has been shown that several triangle centers remain stationary over this family, while others sweep circles [16]. Another interesting fact is that this family conserves the sum of its interior angle cosines,[4] since [21, Inradius, Eqn. 9]:

$$\sum_{i=1}^{3} \cos\theta_i = 1 + \frac{r}{R}.$$

3. Bicentrics with two caustics

We now consider a slight variant of the poristic family, namely triangles $\mathcal{T} = P_1P_2P_3$, inscribed in an outer circle $\mathcal{C} = [(0,0), R]$, with two sides P_1P_2 and P_1P_3 tangent to an inner circle $\mathcal{C}' = [(d,0), r]$, where Eq. (1) does not hold.

Definition 1. The pencil of two conics $\mathcal{E}$ and $\mathcal{E}'$ is a one-parameter family of the linear combinations of their implicit equations.

We review a result, called Poncelet's Main Lemma (PML), which will simplify our constructions. This was proved by Poncelet so as to support the proof of PCT [4,17]:

Lemma 1. (PML) *Let $\mathcal{C}$ and $\mathcal{D}_i$, $i = 1, \ldots, N-1$ be N distinct conics in the same pencil. Let P_i, $i = 1, \ldots, N$ denote the vertices of a polygonal chain inscribed in $\mathcal{C}$ whose $N-1$ sides are each tangent to a $\mathcal{D}_i$. So P_1 (resp. P_N) is the first (resp. last) vertex in the chain. The open side P_NP_1 will envelop a conic $\mathcal{D}_N$, contained in the original pencil.*

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[3] Also known as "Euler's Theorem", though only published in 1765.

[4] This conservation is also manifested by the *confocal* pair, seen later in the article.

Referring to Fig. 4, Lemma 1 implies that the envelope of P_2P_3 will be a distinct circle $\mathcal{C}'''$ in the pencil of $\mathcal{C},\mathcal{C}'$, which can be regarded as a second caustic. We therefore call this family "bic-II" (bicentric family with two caustics).

Manipulation with CAS yields:

Proposition 1. *Over the bic-II family, the envelope* $\mathcal{C}''=[O'',r'']$ *of* P_2P_3 *is a circle in the pencil of* $\mathcal{C},\mathcal{C}'$ *given by*

$$O''=\left[\frac{4dR^2r^2}{(R^2-d^2)^2},0\right],$$
$$r''=\frac{R(R^4-2R^2d^2-2R^2r^2+d^4-2d^2r^2)}{(R^2-d^2)^2}=\frac{(p^2q^2-p^2-q^2)(p+q)d}{p^2q^2(p-q)},$$

where $p=(R+d)/r$ *and* $q=(R-d)/r$.

Note that $r''=0$ is achieved if $(R^2-d^2)^2-2r^2(R^2+d^2)=0$.

Recall that for even N, the diagonals of Poncelet N-periodics in the bicentric pair (with incircle and circumcircle) meet at one of the limiting points of the pair, see Fig. 4 (bottom right). The above relation is equivalent to the well-known condition for a pair of circles to admit a Poncelet family of quadrilaterals. One formulation, due to Kerawala [21, Eq. 39] is

$$\frac{1}{(R-d)^2}+\frac{1}{(R+d)^2}=\frac{1}{r^2}.$$

Theorem 2. *Over the bic-II family, the locus of the incenter* X_1 *is a circle* $\mathcal{C}_1=[O_1,r_1]$ *not in the pencil of* $\mathcal{C},\mathcal{C}'$*, given by*

$$O_1=\left[\frac{2dRr}{R^2-d^2},0\right],\quad r_1=\frac{R(R^2-2Rr-d^2)}{R^2-d^2}.$$

Referring to Fig. 5, the following proof was kindly contributed by Alexey Zaslavsky and Arseniy Akopyan, and later adapted by Mark Helman [13,22].

Proof. Let $\mathcal{T}=PAB$ be a triangle inscribed to an outer circle $\mathcal{C}=(O,R)$ with sides PA and PB tangent to an inner circle $\mathcal{C}'=(O',r)$. Let $d=|O-O'|$.

Let X_1 denote the incenter of $\mathcal{T}$. Let the interior and exterior angle bisectors at P intersect $\mathcal{C}$ at D and E, respectively. Let F be the point of contact of PA with $\mathcal{C}'$.

Since the quadrilateral $AEPD$ is cyclic, $\angle AED=\angle APD$. Also, $\angle EAD=\pi-\angle EPD=\pi-\pi/2=\pi/2=\angle PFO'$. Thus, triangles EAD and PFO' are similar, therefore $AD/DE=FO'/O'P$, so $AD/(2R)=r/O'P$. By the Trillium Theorem [21, Incenter Excenter Circle], $AD=DX_1$, so $DX_1=2Rr/O'P$.

Using the definition of the power of a point with respect to circle $\mathcal{C}=EPBDA$, $(O'P)(O'D)=R^2-d^2$, so $O'P=(R^2-d^2)/O'D$. Substituting, we have

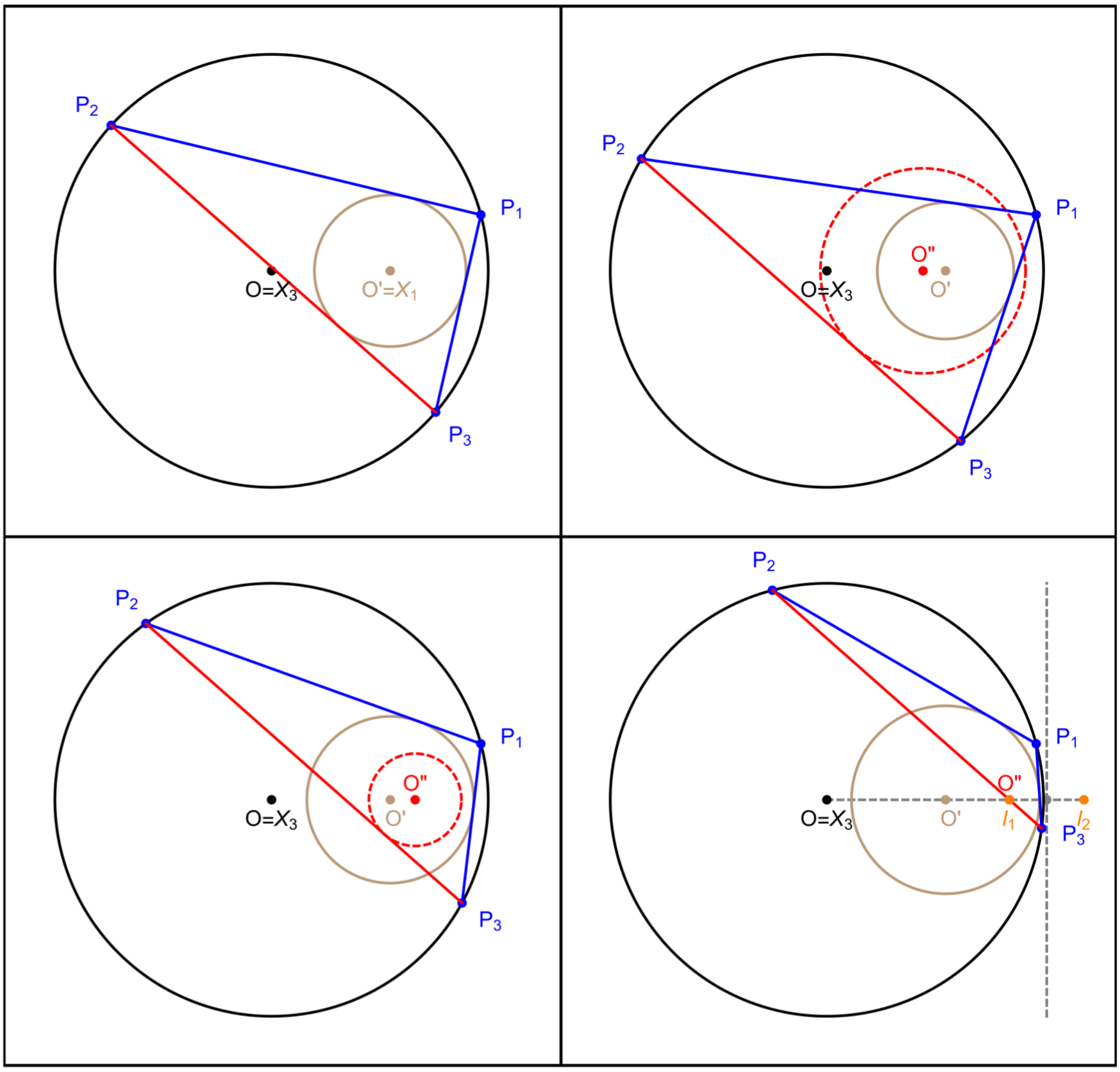

Figure 4 Top left: Chapple's porism or "bic-I" (poristic) family: triangles $\mathcal{T} = P_1P_2P_3$ with a fixed circumcircle $\mathcal{C}$ (black) and incircle $\mathcal{C}'$ (brown), centered at $O = X_3$ and $O' = X_1$ and with radii R and r, respectively. Top right: the bic-II family is obtained by decreasing r (bic-II family) while maintaining P_1P_3 and P_1P_2 tangent to $\mathcal{C}'$. Poncelet's general closure theorem predicts that the envelope of the third side (red) is a third circle $\mathcal{C}''$ (dashed red) in the pencil of $\mathcal{C}$ and $\mathcal{C}'$ centered at O''. Note X_1 (not shown) will no longer coincide with O'. Bottom left: a bic-II triangle where $\mathcal{C}'$ is exterior to the envelope of the third side (dashed red). Bottom right: a bic-II triangle such that the envelope of the third side collapses to a limiting point ℓ_1 of the $\mathcal{C}, \mathcal{C}'$ pair. Also shown are (i) the radical axis (vertical dashed gray) of $\mathcal{C}, \mathcal{C}', \mathcal{C}''$, and (ii) the second limiting point ℓ_2 of the pencil. Video (color figure online)

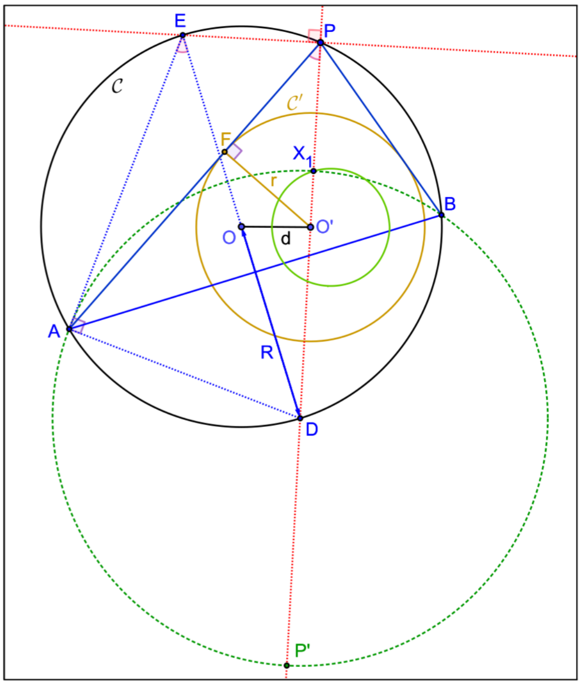

FIGURE 5 Objects and labels used in the proof of Theorem 2. A bic-II triangle PAB is shown, inscribed in an outer circle $\mathcal{C} = (O, R)$ (black) and with two sides tangent to an internal circle $\mathcal{C}' = (O', r)$ (brown). The distance between their centers is d. The "incenter-excenter" circle (dashed green) is centered at the intersection of the chord PO' with $\mathcal{C}$ and contains X_1, P' as antipodes as well as tangent chord endpoints A, B. Triangles EAD and PFO' are similar. Also shown (green) is the circular locus of X_1 over the bic-II family. Original proof by Alexey Zaslavsky and Arseniy Akopyan, adapted by Mark Helman: GeoGebra (color figure online)

$DX_1 = 2Rr(O'D)/(R^2 - d^2)$, so $DX_1/O'D = 2Rr/(R^2 - d^2)$. Finally,

$$\frac{O'X_1}{O'D} = \frac{DX_1 - O'D}{O'D} = \frac{DX_1}{O'D} - 1 = \frac{2Rr}{R^2 - d^2} - 1.$$

The above implies $O'X_1/O'D$ is constant and independent of $\mathcal{T}$. This shows that there is a homothety with center O' that sends $\mathcal{C}$ to the locus of X_1, so the claim follows. The actual values of O_1 and r_1 are obtained by applying said homothety to O and R of $\mathcal{C}$. □

Note that if Eq. (1) holds, $r_1 = 0$, i.e., the family is the standard poristic one for which the incenter locus is a point.

Let $\mathcal{T}'$ denote the *excentral triangle* of $\mathcal{T}$ with sides through the vertices of $\mathcal{T}$, perpendicular to the angle bisectors. The vertices of $\mathcal{T}'$ are known as the *excenters* P_i', $i = 1, 2, 3$.

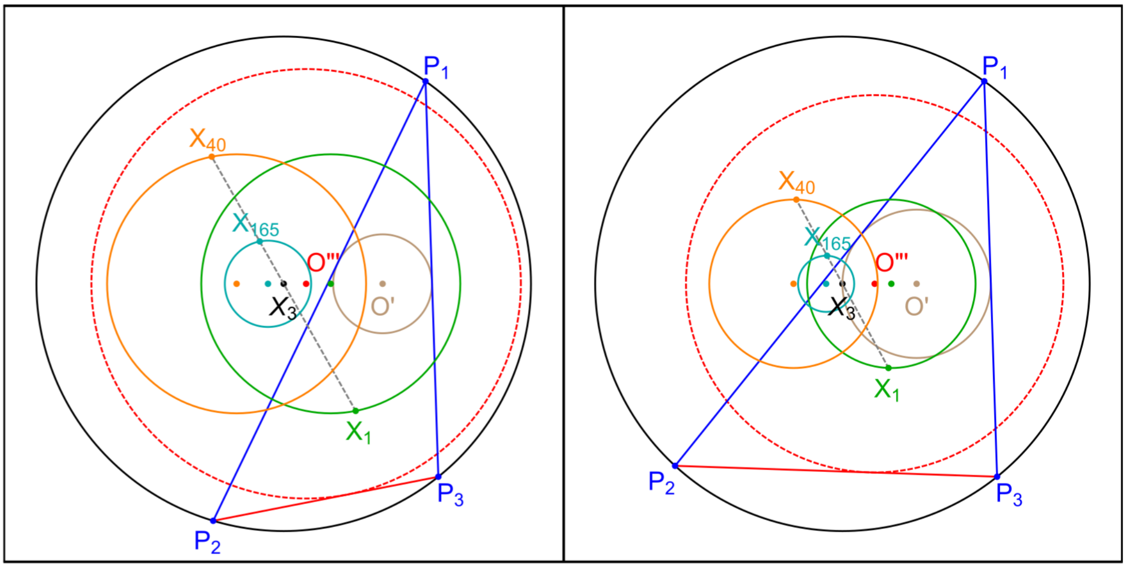

FIGURE 6 Left: A bic-II triangle $P_1P_2P_3$ inscribed in an outer (black) circle centered on X_3. Sides P_1P_2 and P_1P_3 (blue) are tangent to a first (brown) circle, centered on O'. Over the family, the third side (red) envelops a second, in-pencil circle (dashed red) centered on O_{env}. The loci of X_1 (green) and X_{40} (orange) are copies of each other reflected about O. The locus of X_{165} (light blue) is a copy of that of X_1, 1/3-scaled about O. Notice X_k, k =1, 3, 40, 165 are collinear. Right: a similar setup but with an increase in the first caustic radius and a reduction in the distance between the centers, showing all loci remain circular. Video (color figure online)

In [15], the circumcenter (resp. barycenter) of $\mathcal{T}'$ is named the Bevan point X_{40} (resp. X_{165}). It is known that X_{40} (resp. X_{165}) is a reflection of X_1 about X_3 with a scale of 1 (resp. 1/3). Referring to Fig. 6:

Corollary 1. *Over the bic-II family, the locus of X_{40} (resp. X_{165}) is a circle $\mathcal{C}_{40} = [O_{40}, r_{40}] = [-O_1, r_1]$ (resp. $\mathcal{C}_{165} = [-O_1/3, r_1/3]$).*

Corollary 2. *Over the bic-II family, the locus of any triangle center which is at a fixed proportion from X_1 and X_3 will be a circle.*

As an example, consider the circumcircle-inverse of X_1 (resp. X_{40}), called X_{36} (resp. X_{2077}) on [15]. Since the circumcircle is fixed, both their loci are automatically circles.

Referring to Fig. 7, CAS manipulation yields:

Proposition 2. *Over the bic-II family, the locus of the barycenter X_2 is an algebraic curve of degree 6.*

"Appendix C" gives an explicit expression for this locus.

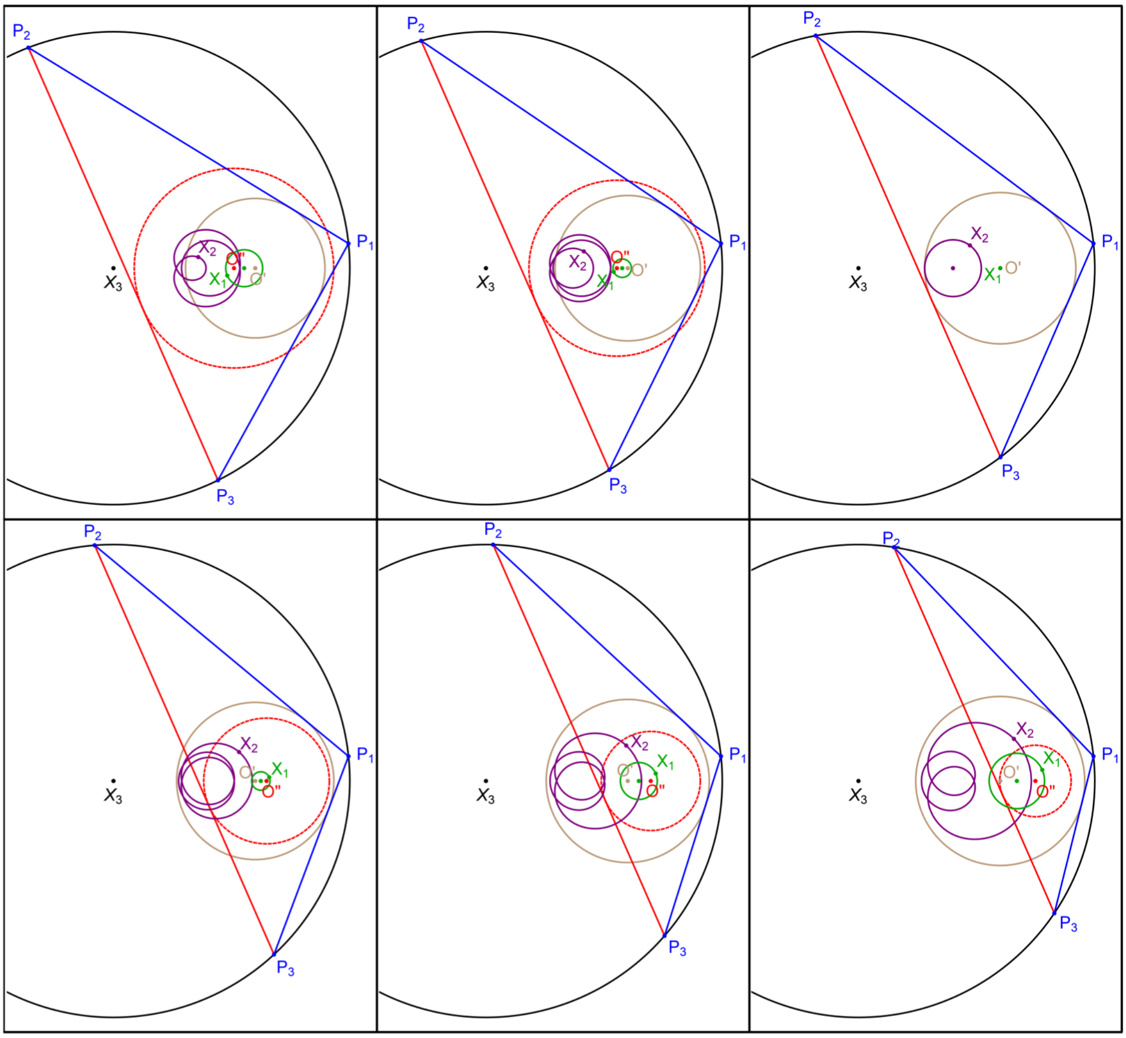

FIGURE 7 Sextic loci of X_2 under fixed R, d and with increasing r. Notice that at the poristic configuration (top right), the locus of X_2 collapses to a circle (swept three times). Video

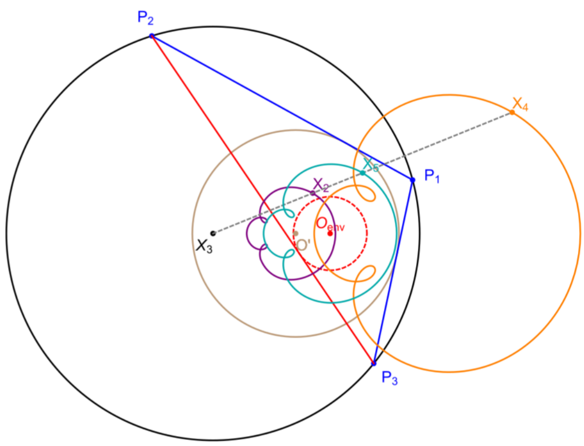

FIGURE 8 A bic-II triangle and the sextic loci of X_k, k =2, 4, 5. Along with X_3, these are collinear on the Euler line (dashed black) and at fixed proportions to X_3X_2. Therefore, the three loopy loci are homothetic. Video

TABLE 1 The first row lists triangle centers known to be stationary over bic-I (poristic family) amongst the first 1000 centers on [15]

bic-I point	X_1	X_3	X_{35}	X_{36}	X_{40}	X_{46}	X_{55}	X_{56}	X_{57}	X_{65}	X_{165}	X_{354}	X_{484}	X_{942}
bic-II	C	P	E	E	C	X	E	X	X	X	C	E	E	E
bic-III	X	P	X	X	X	X	X	X	X	X	X	X	X	X

The second (resp. third) row lists their locus type in the bic-II (resp. III) family. P,C,E,X stand for point, circle, ellipse, and non-conic, respectively

TABLE 2 Loci types for X_1, X_2, and the excenters for each family mentioned in the article

Family	Note, aka.	caustics	X_1	X_2	X_3	P_1'	P_2'	P_3'
bic-I	Poristic, bicentric	1	P	C	P	C	C	C
bic-II	Tricentric	2	C	6	P	C	6	6
bic-III	Quadricentric	3	N†	N	P	N†	N†	N†
conf-I	Confocal, elliptic billiard‡	1	E	E	E	E	E	E
conf-II	Caustics: 1 conf., 1 in-pencil	2	N†	N	N	6	E	E
conf-III	1 conf., 2 in-pencil	3	N	N	N	N	N	N

Symbols P, C, E, and 6 stand for point, circle, ellipse, and sextic curves, respectively. An "N" denotes a non-conic whose degree hasn't yet been determined. † A non-conic locus is conjectured to be convex. ‡ The mittenpunkt X_9 is stationary over conf-I [11]

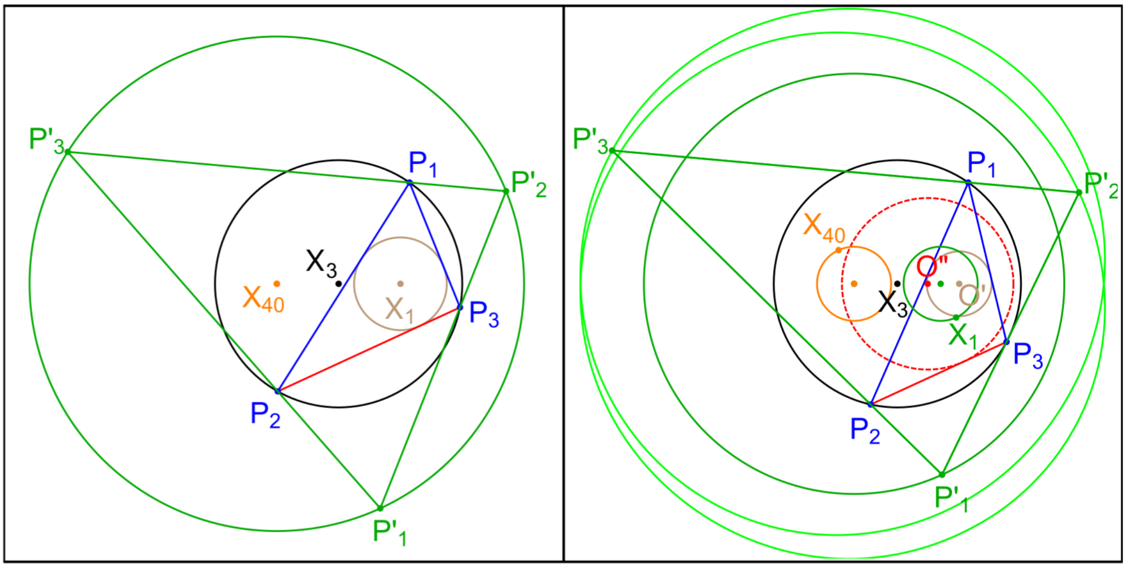

Figure 9 Left: a bic-I (poristic triangle $P_1P_2P_3$ and its excentral triangle $P_1'P_2'P_3'$ (dark green). The locus of its vertices is a circle centered on X_{40} (stationary) and with radius twice that of the circumcircle of the original family. Right: a bic-II triangle and its excentral triangle (dark green). The locus of X_1 (green) and X_{40} (orange) are identical circles (reflected about X_3). The locus of P_1' (opposite to P_1) is a circle (dark green) concentric with the locus of X_{40}. Excenters P_2' and P_3' sweep non-conics (light green) which intersect on the X_1X_3 line. Video (color figure online)

The non-conic loci of X_k, k =2,4,5,6 are shown in Fig. 8.

Recall that some triangle centers with the property to always lie on a triangle's circumcircle, e.g., X_k, k=74, 98, 99, 100, 101, etc., see [21, Circumcircle] for a larger list. Let X be a triangle center not on this list. Recall also that over the poristic family, several triangle centers were identified as stationary [16]; see the first row on Tables 1 and 2. Experiments suggest that:

Conjecture 1. *Over the bic-II family, a necessary (though not sufficient) condition for the locus of X to be a conic is that its locus over the bic-I (poristic) family is a stationary point.*

Referring to Table 1, examples of triangle centers stationary over the bic-I family which do not yield conics over the bic-II family include X_k, k =46, 56, 57, 65.

Studying the excenters

Referring to Fig. 5, notice the excenter P' opposite to C is antipodal to X_1 on the "incenter-excenter" circle centered on D. Therefore, a consequence of Theorem 2 is that:

Corollary 3. *Over the bic-II family, the locus of P_1' is a circle $\mathcal{C}_1' = [-O_1, r_1']$ not in the pencil of $\mathcal{C}, \mathcal{C}'$, where*

$$r_1' = \frac{R(R^2 + 2Rr - d^2)}{R^2 - d^2}.$$

It can be shown via CAS that the two other excenters sweep a sextic. This is illustrated in Fig. 9: let P_i' denote the excenter opposite to P_i. Though the loci of P_2' and P_3' are distinct and non-conic, as shown above, that of P_1' is a circle. Note that as derived in [16], when Eq. (1) holds (poristic family), $r_1' = 2R$.

4. Review: confocal triangle family

Referring to Fig. 10, recall a special family of triangles interscribed between a pair of *confocal* ellipses $\mathcal{E}, \mathcal{E}'$. Let a, b and a', b' denote their semi-axes. The Cayley condition for any concentric, axis-parallel pair of ellipses to admit a 3-periodic family is $a'/a + b'/b = 1$ [2]. Let $\delta = \sqrt{a^4 - a^2b^2 + b^4}$ and $c^2 = a^2 - b^2$. As derived in [7], imposing confocality yields

$$a' = \frac{a\left(\delta - b^2\right)}{c^2}, \quad b' = \frac{b\left(a^2 - \delta\right)}{c^2}. \tag{2}$$

Referring to Fig. 10 (top left), this family has the special property that normals to $\mathcal{E}$ at the vertices are the bisectors, so triangles can be thought of as billiard (3-periodic) trajectories. Indeed, this system is also known as the $N = 3$ *elliptic billiard* (see [20] for an introduction of its properties). For consistency with the previous section, we shall call this family "conf-I" (confocal family with a single caustic).

Figure 10 depicts several interesting phenomena concerning conf-I triangles: (i) the mittenpunk X_9 is stationary [18]; (ii) the ratio r/R is conserved as is the sum of cosines [11], (iii) the locus of the incenter X_1 is an ellipse concentric and aligned with $\mathcal{E}, \mathcal{E}'$ [7,19]; (iv) the common locus of the excenters (vertices of the excentral triangle) is also a concentric, axis-aligned ellipse, whose aspect ratio is the reciprocal of that of (iii) [7]. Let a_1, b_1 (resp. a_e, b_e) denote the semi-axes of (iii) and (iv), respectively. These are given by [7]

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}, \quad a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}.$$

Note that these two loci are ellipses with reciprocal aspect ratios, i.e., $a_1/b_1 = b_e/a_e$ [7].

5. Confocals with two caustics

Referring to Fig. 11, we consider a variant of the conf-I family, namely triangles $\mathcal{T} = P_1P_2P_3$ inscribed in an outer ellipse $\mathcal{E}$ with semi-axes a, b. Sides P_1P_2 and P_1P_3 are tangent to an inner, confocal ellipse $\mathcal{E}'$ with semi-axes a', b'. In this case, Eq. (2) in general does not hold. Lemma 1 ensures that the envelope of P_2P_3 will automatically be a conic in the pencil of $\mathcal{E}, \mathcal{E}'$, i.e., a second elliptic caustic. We therefore call such a system "conf-II".

Remark 1. It can be shown that the only non-degenerate conics in the pencil of two confocal ones $\mathcal{E}, \mathcal{E}'$ which are confocal with the pair are $\mathcal{E}$ and $\mathcal{E}'$ themselves.

Remark 2. It can be shown that over the conf-I family, the elliptic locus of X_1 and of the excenters does not belong to the pencil of $\mathcal{E}, \mathcal{E}'$.

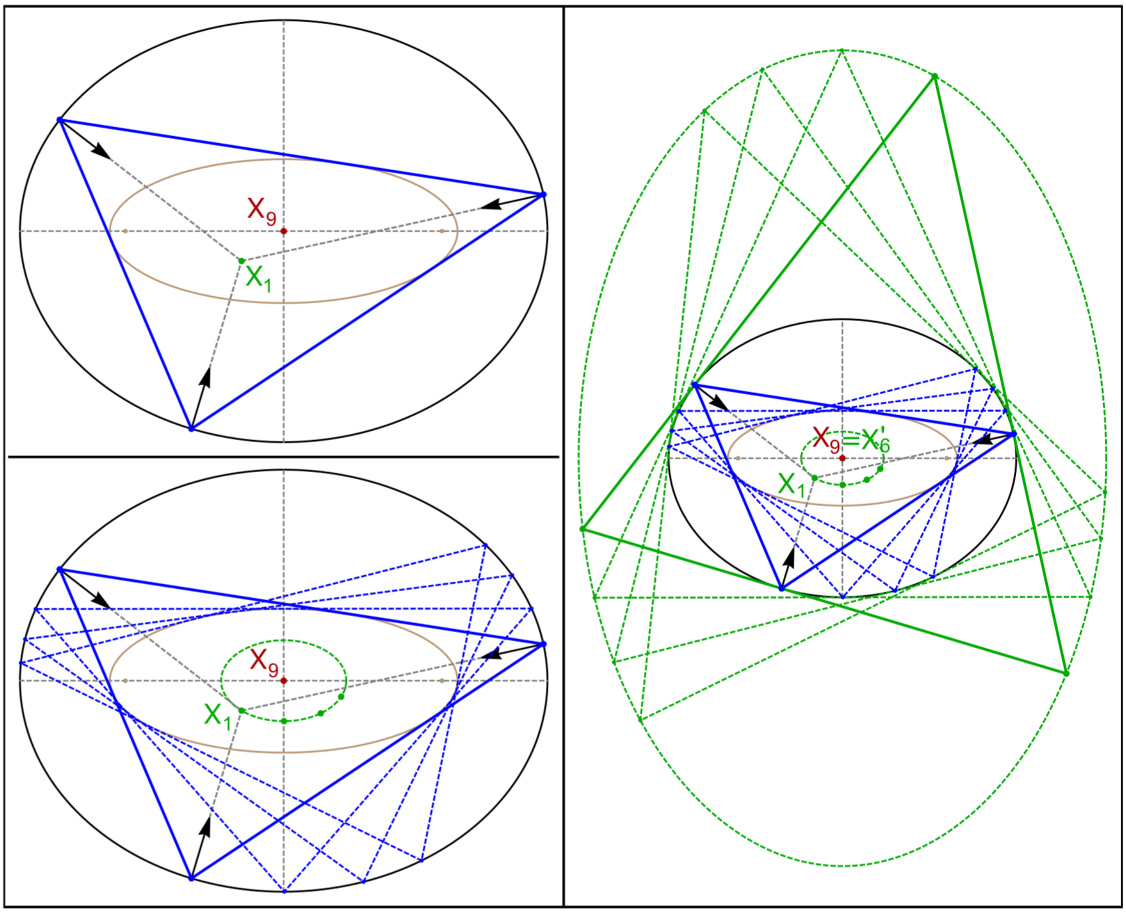

Figure 10 Top Left: A 3-periodic (solid blue) in the conf-I family. Owing to a special property of the confocal pair, internal angles are bisected by ellipse normals (black arrows); these concur at the incenter X_1. Bottom Left: superposition of 3-periodics (note: all have the same periemeter). Over the family, (i) the mittenpunkt X_9 is stationary and (ii) the incenter X_1 sweeps an ellipse (dashed green). Right: the family of excentral triangles (solid green) obtained from conf-I 3-periodics is itself a Ponceletian family. Its vertices (the excenters) sweep a concentric, axis-aligned ellipse (dashed green), whose aspect ratio is the reciprocal of the original incenter locus [7]. Live (color figure online)

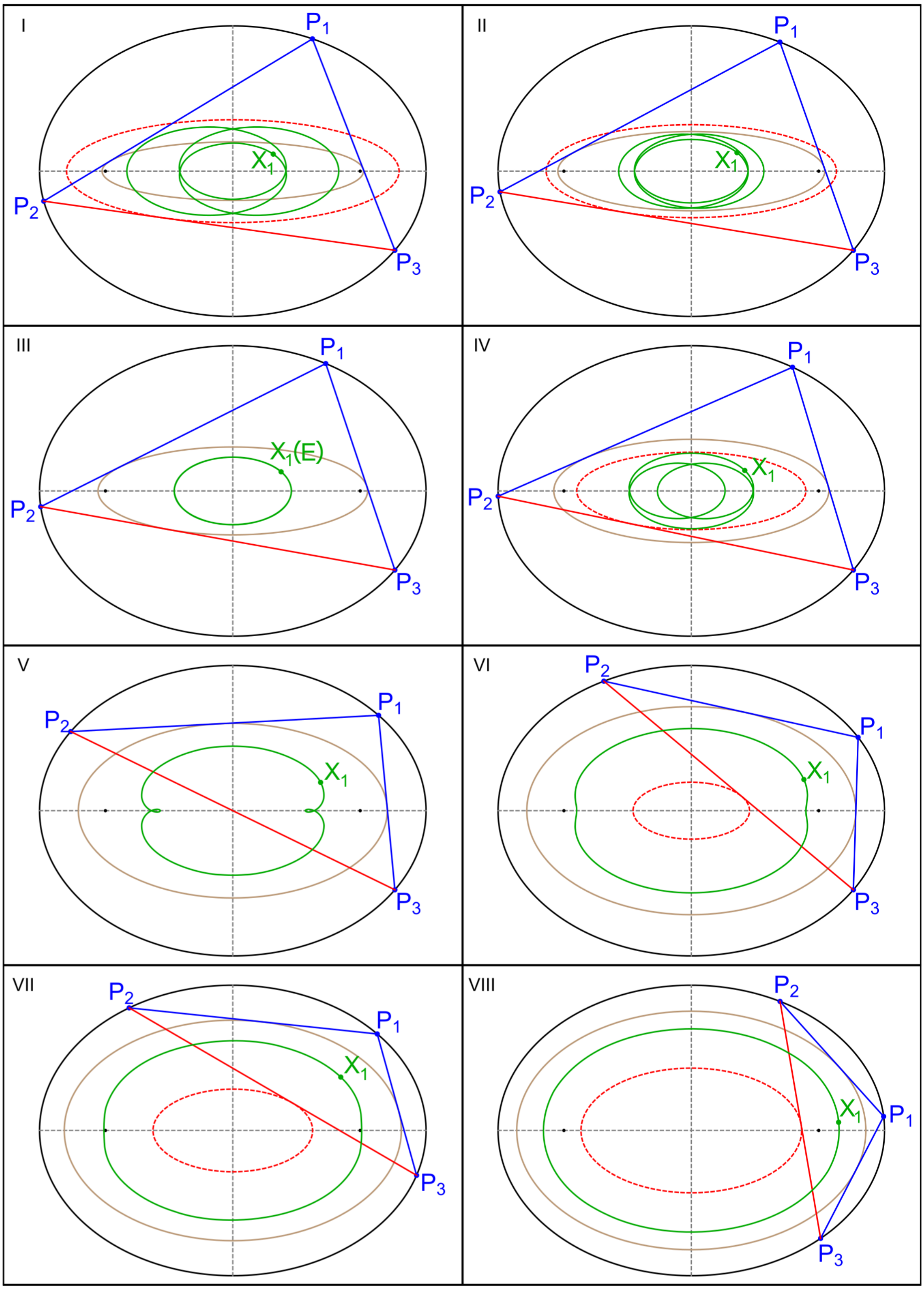

FIGURE 11 Left-to-right, top-to-bottom: the locus of X_1 (green) over conf-II triangles as the confocal caustic $\mathcal{E}'$ (brown) grows toward $\mathcal{E}$ (black). Also shown is the in-pencil envelope $\mathcal{E}''$ of P_2P_3 (dashed red). At inset III, $\mathcal{E}'' = \mathcal{E}$ and the locus of X_1 an ellipse. At inset V, $\mathcal{E}''$ is the center point. At inset VII, the locus becomes convex (see Proposition 5). Video 1, Video 2 (color figure online)

Poncelet's general theorem [4] predicts that over the conf-II family, the envelope of P_2P_3 is an ellipse $\mathcal{E}''$ which is concentric, axis-parallel, and in the pencil of $\mathcal{E},\mathcal{E}'$. Via CAS, obtain:

Proposition 3. *The semi-axes a'',b'' of $\mathcal{E}''$ are given by*

$$a''=\frac{|a\zeta|}{a^2b^2-c^2\lambda},\quad b''=\frac{|b\zeta|}{a^2b^2+c^2\lambda},$$

where $\zeta=a^2b^2-(a^2+b^2)\lambda$ and $\lambda=a^2-a'^2=b^2-b'^2$.

Remark 3. In the conf-I family, the outer (resp. inner) ellipse is obtained when $\lambda=0$ (resp. $\lambda=a^2b^2(2\delta-a^2-b^2)/c^4$). Note that in this case, the envelope of P_2P_3 is confocal (it is the inner ellipse itself).

Proposition 4. *Over the conf-II family, the locus of X_1 is only an ellipse in the conf-I configuration, i.e., the confocal pair $\mathcal{E},\mathcal{E}'$ admits a 3-periodic family.*

Proof. Consider the triangle $P_1P_2P_3$ parametrized in Appendix 7. The locus $X_1(\lambda)$ is given by

$$X_1(\lambda)=[x,y]=\frac{s_1P_1+s_2P_2+s_3P_3}{s_1+s_2+s_3},$$
$$s_1=|P_2-P_3|,\quad s_2=|P_2-P_3|,\quad s_3=|P_1-P_2|.$$

Via CAS, obtain the following implicit equation for $X_1(\lambda)$

$$\begin{aligned}f(x,x_1,\lambda)&=\left(a^2c^4\lambda^2+2\,a^2b^2c^2\left(a^2-2\,x_1^2\right)\lambda+b^4a^6\right)x^2\\&\quad+x_1\left(2(a^2b^2-c^2\lambda)(a^4b^2-2b^2c^2x_1^2+a^2c^2\lambda)\right)x\\&\quad-a^2x_1^2\left(3a^4b^4-4b^4c^2x_1^2-4a^4b^2\lambda+6a^2b^2c^2\lambda-c^4\lambda^2\right)=0,\\g(y,y_1,\lambda)&=b^2\left(a^4b^4-2\,a^2c^2\lambda\,b^2+4\,a^2c^2\lambda\,y_1^2+\lambda^2c^4\right)y^2\\&\quad+2\,y_1\left(a^2b^2+\lambda\,c^2\right)\left(a^2b^4+2\,a^2c^2y_1^2-b^2c^2\lambda\right)y\\&\quad-y_1^2b^2\left(3\,a^4b^4+4\,a^4c^2y_1^2-4\,a^4b^2\lambda-2\,a^2c^2b^2\lambda-c^4\lambda^2\right)=0,\\E(x_1,y_1)&=b^2x_1^2+a^2y_1^2-a^2b^2=0.\end{aligned}$$

Consider the ellipse $\mathcal{I}_\lambda(x,y)=x^2/\alpha^2+y^2/\beta^2-1=0$, where

$$\alpha=\frac{a}{a^2b^2-\lambda\,c^2}\left(2\,b^3\sqrt{b^2-\lambda}+c^2\left(b^2-\lambda\right)-b^4\right),$$
$$\beta=\frac{b}{a^2b^2+\lambda\,c^2}\left(2\,a^3\sqrt{a^2-\lambda}-c^2\left(a^2-\lambda\right)-a^4\right).$$

It turns out that the parametrization of X_1 is a rational expression of the form

$$\left[\frac{x_1(a_{20}x_1^2+a_{02}y_1^2)}{c_{20}x_1^2+c_{02}y_1^2},\frac{y_1(b_{20}x_1^2+b_{02}y_1^2)}{d_{20}x_1^2+d_{02}y_1^2}\right],$$

iff $\lambda=a^2b^2(2\delta-a^2-b^2)/c^4$. This is precisely the case when the triangle $P_1P_2P_3$ is a billiard orbit.

The trace of $X_1(\lambda)$ is the ellipse $\mathcal{I}_\lambda(x,y)=0$ iff $P_1P_2P_3$ is a billiard orbit. This is obtained computing the affine curvature of $X_1(\lambda)$ and showing that it is constant only for the critical value $\lambda = a^2b^2(2\delta - a^2 - b^2)/c^4$. □

Proposition 5. *Over the conf-II family, the locus of X_1 is convex if $(a')^2 > a^2 - \lambda_o$, where λ_o is the largest non-negative root of the following polynomial which is less than a (in general there are three real roots):*

$$
\begin{aligned}
&c^8\lambda_o^5 + b^2(3a^4 - 2a^2b^2 + 3b^4)c^4\lambda_o^4 +\\
&2a^2b^4(a^4 + 5a^2b^2 - 2b^4)c^2\lambda_o^3 + 2a^4b^6(a^4+b^4)\lambda_o^2 -\\
&b^8a^6(11a^2 - 4b^2)\lambda_o + 3b^{10}a^8,
\end{aligned}
$$

where $c^2 = a^2 - b^2$.

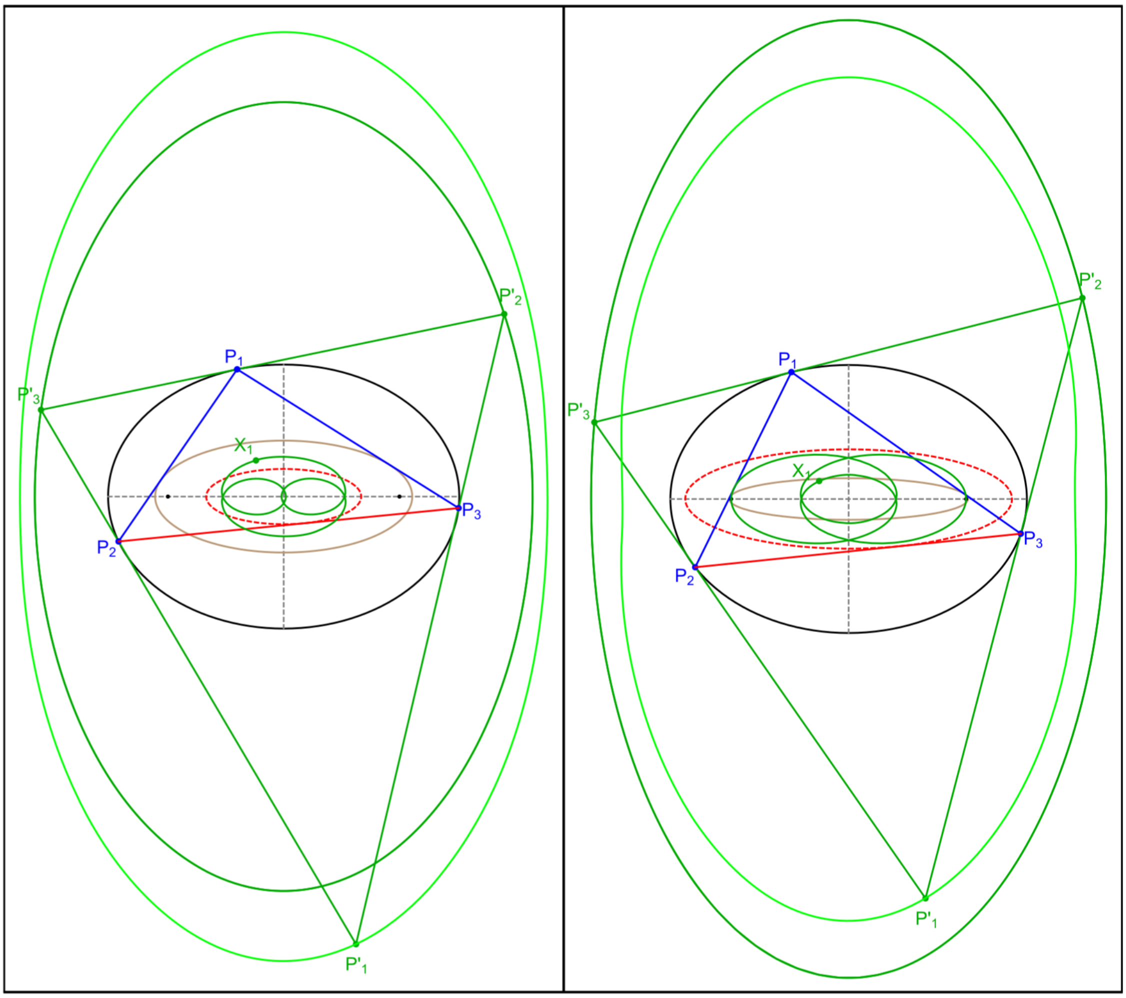

Figure 12 The non-elliptic locus of the incenter X_1 and that of the vertices P_i' of the excentral triangle (dark green) over a conf-II family with a (i) larger (left), and (ii) smaller (right) caustic than the $N=3$, conf-I one. Remarkably, excenters P_2' and P_3' survive the asymmetry sweeping a single ellipse (dark green), while P_1' sweeps a non-conic (light green) interior (resp. exterior) to the former one. Video (color figure online)

Studying the excenters

As before, let P_i' denote the excenter opposite to P_i. Let $(a')^2 = a^2 - \lambda$ and $(b')^2 = b^2 - \lambda$ the semi-axes of the confocal ellipse. Referring to Fig. 12:

Theorem 3. *Over the conf-II family, both P_2' and P_3' sweep a single ellipse $\mathcal{E}_e$, which is both concentric and axis-aligned with the Poncelet pair of ellipses. Its semi-axes a_e, b_e are given by*

$$[a_e, b_e] = \sqrt{k}\left[\frac{a}{a^2b^2 + c^2\lambda}, \frac{b}{a^2b^2 - c^2\lambda}\right],$$

where $k = \left((a+b)^2\lambda + a^2b^2\right)\left((a-b)^2\lambda + a^2b^2\right)$.

Proof. The excenter $P_2'(\lambda)$ is parametrized by

$$P_2'(\lambda) = \frac{s_1 P_1 - s_2 P_2 + s_3 P_3}{s_1 - s_2 + s_3},$$
$$s_1 = |P_2 - P_3|, \quad s_2 = |P_2 - P_3|, \quad s_3 = |P_1 - P_2|.$$

Analogously,

$$P_3'(\lambda) = \frac{s_1 P_1 + s_2 P_2 - s_3 P_3}{s_1 + s_2 - s_3}.$$

Defining the ellipse by $\mathcal{E}_\lambda(x,y) = x^2/a_e^2 + y^2/b_e^2 - 1 = 0$, with a CAS obtain that $\mathcal{E}_\lambda(P_2') = \mathcal{E}_\lambda(P_3') = 0$. □

Remark 4. It can also be shown (via CAS) that over the conf-II family, the locus of P_1' is a curve of degree 6.

5.1. Interesting excentral ellipses

The two corollaries below are illustrated in Fig. 13.

Corollary 4. *Let a, b be the semi-axes of $\mathcal{E}$. Over the conf-II family, the envelope of P_2P_3 is a point at the center of $\mathcal{E}$ for the following choice of $\mathcal{E}'$ semi-axes*

$$a' = \frac{a^2}{\sqrt{a^2+b^2}}, \quad b' = \frac{b^2}{\sqrt{a^2+b^2}}.$$

In such a case, the locus of P_2' (and P_3') is an ellipse with aspect ratio b/a.

Proof. The expressions for a', b' above are those of the confocal caustic $\mathcal{E}'$ which with $\mathcal{E}$ admits a 4-periodic family [8, App.B.3]. Indeed, opposite vertices of an even-N Poncelet family in a concentric, axis-parallel ellipse pair are antipodal with respect to the center, i.e., the major diagonals pass through the common center [12].

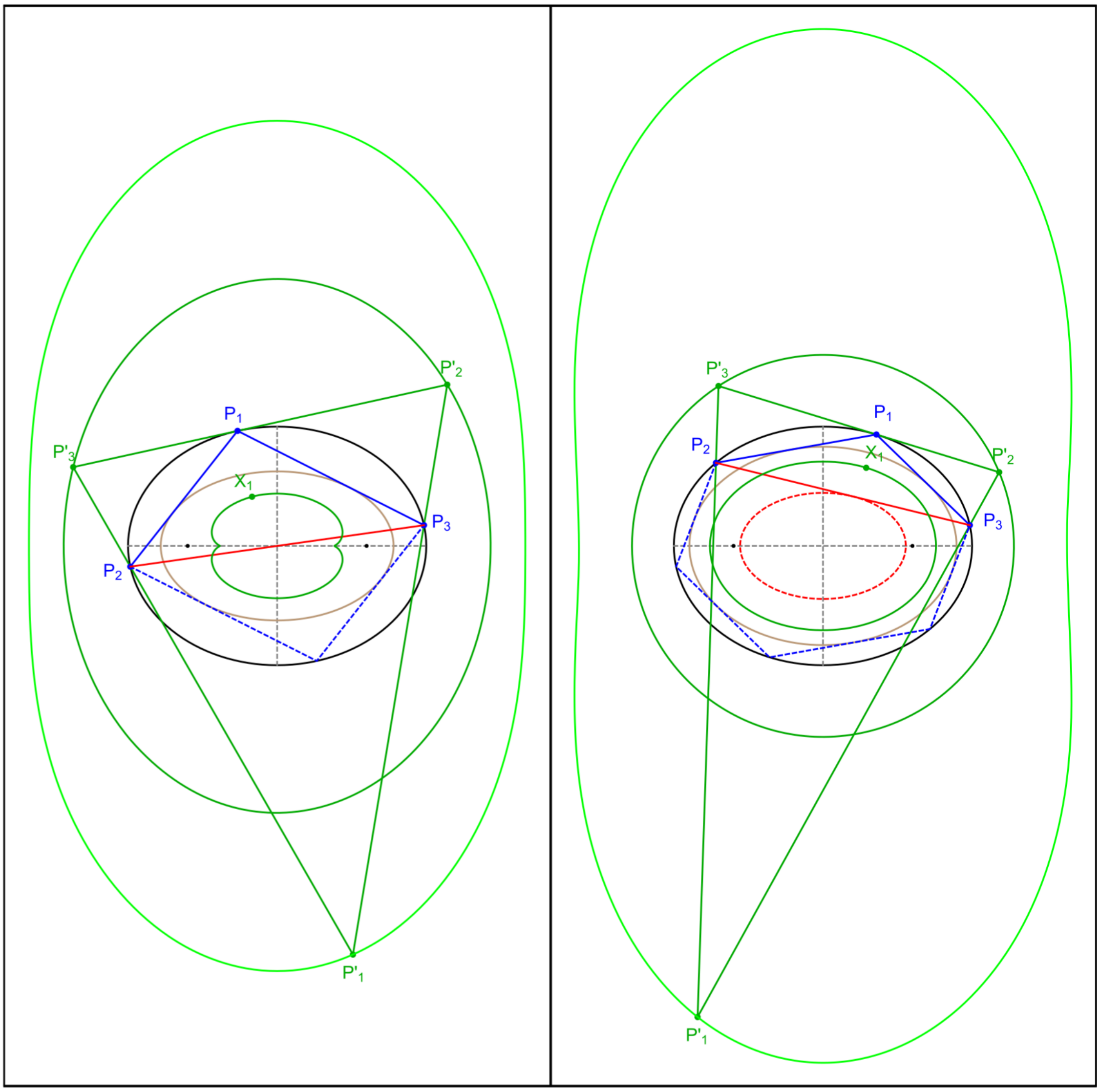

Figure 13 Left: the conf-II caustic $\mathcal{E}'$ (brown) is chosen such that the envelope of P_2P_3 (red) is the center of the system. Note that $\mathcal{E}'$ is the caustic required for the $N = 4$ trajectory (dashed blue). In such a case, the axis ratio of the elliptic locus of excenters P_2', P_3' (dark green) is the reciprocal of that of $\mathcal{E}$ (black). Right: The caustic (brown) is now chosen such that if one were to complete the Poncelet iteration about $\mathcal{E}'$ one would obtain an $N = 6$ trajectory (dashed blue). In such a case, the locus of P_2', P_3' is a circle! Note that in both cases the locus of P_1' (light green) and that of the incenter X_1 (dark green interior to $\mathcal{E}'$) are non-conics. Video (color figure online)

Under a', b' in the proposition, λ in Theorem 3 is given by

$$\lambda = \frac{a^2b^2}{a^2+b^2}.$$

Carrying out the simplifications, obtain $a_e/b_e = b/a$. □

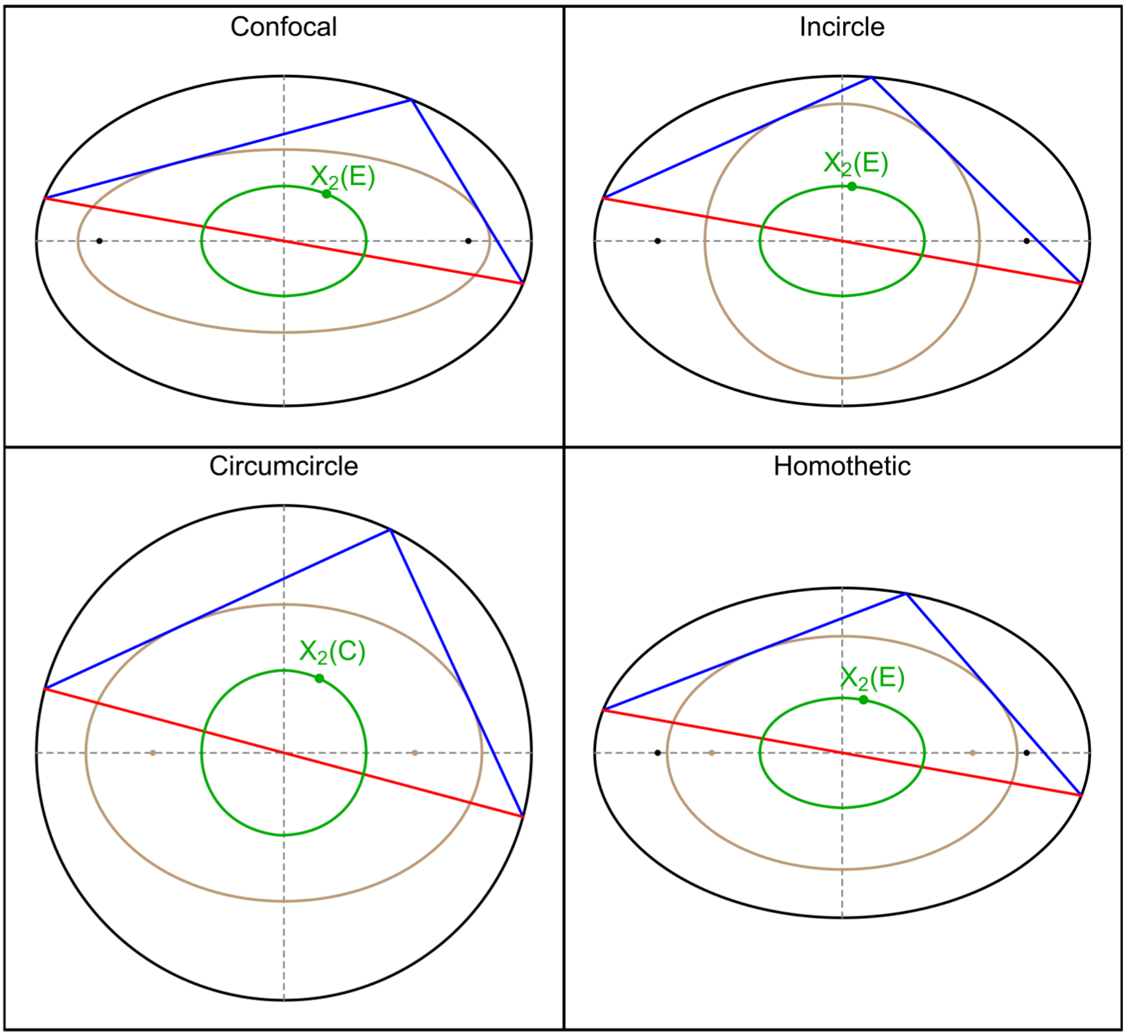

FIGURE 14 In the conf-II family (top left), where the caustic is selected such that the envelope of the free side (red) is the origin, the locus of X_2 is an ellipse. Indeed, the locus of X_2 is an ellipse for any triangle families inscribed in an outer ellipse and with two sides tangent to an inner concentric, axis-parallel one, such that the third side's envelope is the origin. Video (color figure online)

Corollary 5. *The locus of P_2' and P_3' is a circle concentric with the $\mathcal{E}, \mathcal{E}'$ when*

$$a' = \frac{a\sqrt{a(a+2\,b)}}{a+b}, \quad b' = \frac{b\sqrt{b\,(2\,a+b)}}{a+b}, \quad \lambda = \left(\frac{ab}{a+b}\right)^2.$$

Note that in such a case, a', b' are precisely the semi-axes of the elliptic billiard $N=6$ family [8, Appendix B.5].

5.2. The barycenter locus

That the locus of the barycenter X_2 (and many other triangle centers) is an ellipse over the conf-I family has been established [11]. For the general conf-II case, it is not a conic. As shown in Fig. 14 (top left), it is a conic if the caustic is such that the envelope of P_2P_3 is the center point. Indeed:

Proposition 6. *Over a family of triangles $P_1P_2P_3$ inscribed in an outer ellipse $\mathcal{E}$, where sides P_1P_2 and P_1P_3 are tangent to an inner, concentric, axis-parallel ellipse $\mathcal{E}'$, and the third side P_2P_3 envelops the center point, the locus of X_2 will be homothetic to $\mathcal{E}$, at $1/3$ scale.*

The proof was kindly contributed by Mark Helman [13].

Proof. Let P_4 be the reflection of P_1 about the center O. Since an even-N Poncelet family in a concentric, axis-parallel ellipse pair has (i) parallel opposite sides, and (ii) diagonals which pass through O [12], $P_1P_2P_3P_4$ will be a parallelogram and P_1P_4 will be such a diagonal. So O is the midpoint of P_2P_3, meaning X_2 of $P_1P_2P_3$ will be two-thirds of the way between P_1 and O. This sends the outer ellipse to a copy of itself scaled by $1/3$. □

The elliptic locus of X_2 is shown over other concentric, axis-parallel "half $N = 4$" families in Fig. 14.

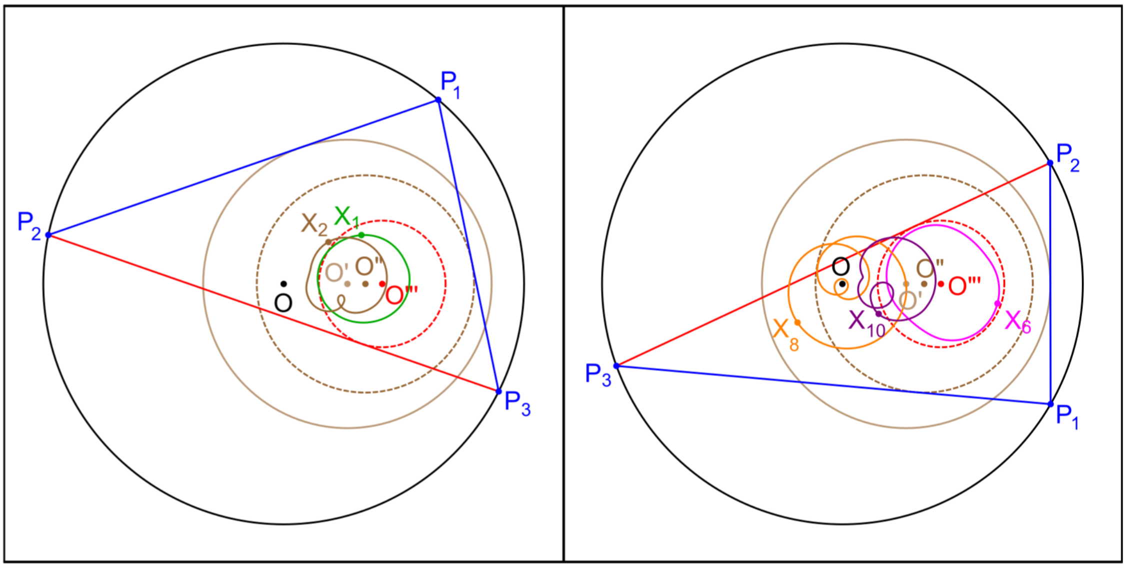

FIGURE 15 Non-conic loci of triangle centers over bic-III triangles, vertices lie on an outer circle (black) while each side is tangent to a distinct in-pencil circle (brown, dashed brown, and dashed red). Left: loci of the incenter X_1 (green) and of the barycenter X_2 (brown). Experimentally, the locus of X_1 is always convex. Right: loci of the symmedian point X_6 (pink), Nagel point X_8 (orange), and Spieker center X_{10} (purple). Though in the current choice the locus of X_6 looks convex, in general it is not. Video (color figure online)

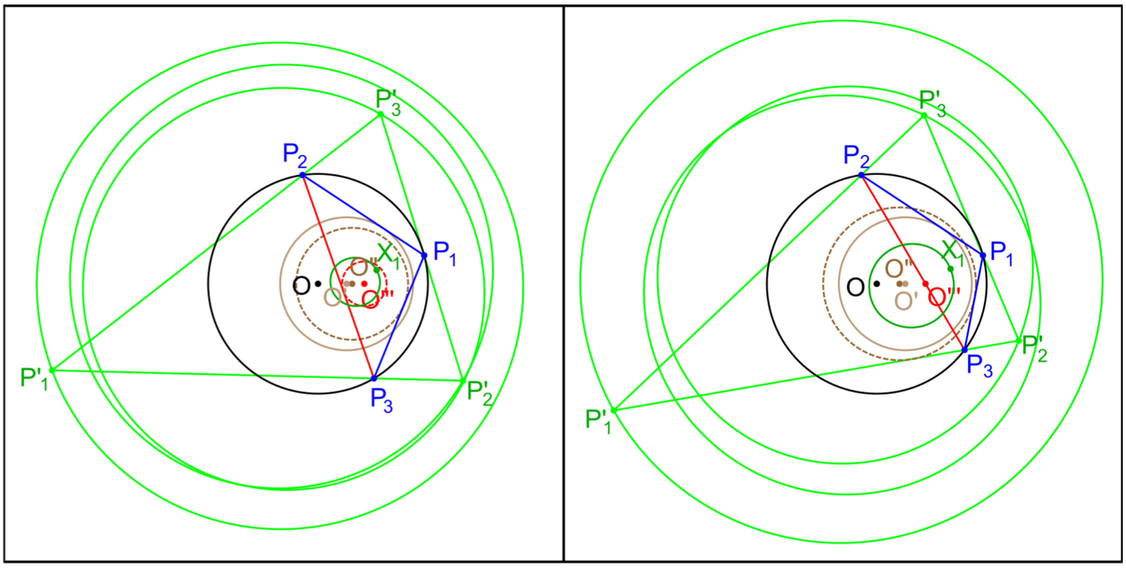

FIGURE 16 Left: bic-III triangles $P_1P_2P_3$, where the envelope of P_2P_3 (dashed red) is a circle with non-zero radius. The loci (light green) of the three excenters P_i' are all non-conics, as is that of the incenter X_1 (dark green). Right: two caustics (brown and dashed brown) are selected so that the envelope of P_2P_3 is the (internal) limiting point of the pencil. Even in this case, the loci of all excenters are non-conics, though that of P_1' is numerically very close to a circle. Video (color figure online)

6. Families with three caustics

6.1. Adding another caustic to bic-II

We now consider a natural extension to the bic-II family, namely, a family of triangles $P_1P_2P_3$ inscribed in an outer circle $\mathcal{C}$ where P_1P_2 (resp. P_1P_3) is tangent to a first circle $\mathcal{C}'$ (resp. second circle $\mathcal{C}''$). In particular, $\mathcal{C}''$ is picked to be in the pencil of $\mathcal{C}, \mathcal{C}'$. Note that by Lemma 1, the third side P_2P_3 will be tangent to a third, in-pencil circle $\mathcal{C}'''$. We term this the "bic-III" family.

For the purposes of the next set of results, assume all four in-pencil circles in a bic-III family are distinct. Referring to Fig. 15, experimental evidence suggests:

Conjecture 2. *Over the bic-III family, the locus of X_1 is a convex curve.*

Conjecture 3. *Over the bic-III family, the locus of a triangle center is never a conic.*

Referring to Fig. 16, we also find that:

Conjecture 4. *Over the bic-III family, the excenters sweep three distinct non-conic curves.*

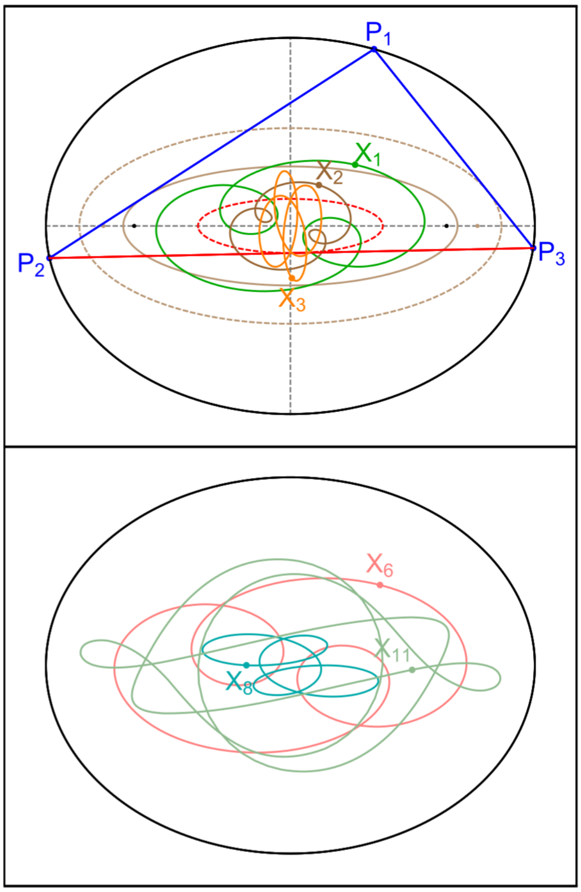

FIGURE 17 Left: A conf-III family inscribed in an outer ellipse (black) and with each side tangent to a separate in-pencil caustic (brown, dashed brown, and red). Only the first of these (brown) is confocal. The non-conic, convoluted loci of X_k, $k = 1, 2, 3$ are shown (green, brown, green). Right: the same setup, now showing the whirly loci of X_k, $k = 6, 8, 11$. For simplicity, the three inner caustics have been omitted (color figure online)

6.2. Adding another in-pencil caustic to conf-II

We call "conf-III" a family of triangles inscribed in an ellipse $\mathcal{E}$, with a first side tangent to a confocal caustic $\mathcal{E}'$, and a second one tangent to a distinct ellipse $\mathcal{E}''$ in the pencil spanned by $\mathcal{E}, \mathcal{E}'$ (it is therefore non-confocal). Again, Lemma 1 ensures that the third side will envelop a third ellipse in the same pencil. As in the bic-III family, no triangle centers produce conic loci. As shown in Fig. 17, these can be quite convoluted.

6.3. Choosing one of four triangles

Referring to Fig. 18, there are four ways in which a bic-III triangle can be chosen, namely, two tangents of P_1P_2 over $\mathcal{C}'$, and two tangents of P_1P_3 over $\mathcal{C}''$. As it turns out, these only produce two distinct envelopes for P_2P_3. For each choice, a triangle center such as the incenter or barycenter will sweep a distinct locus.

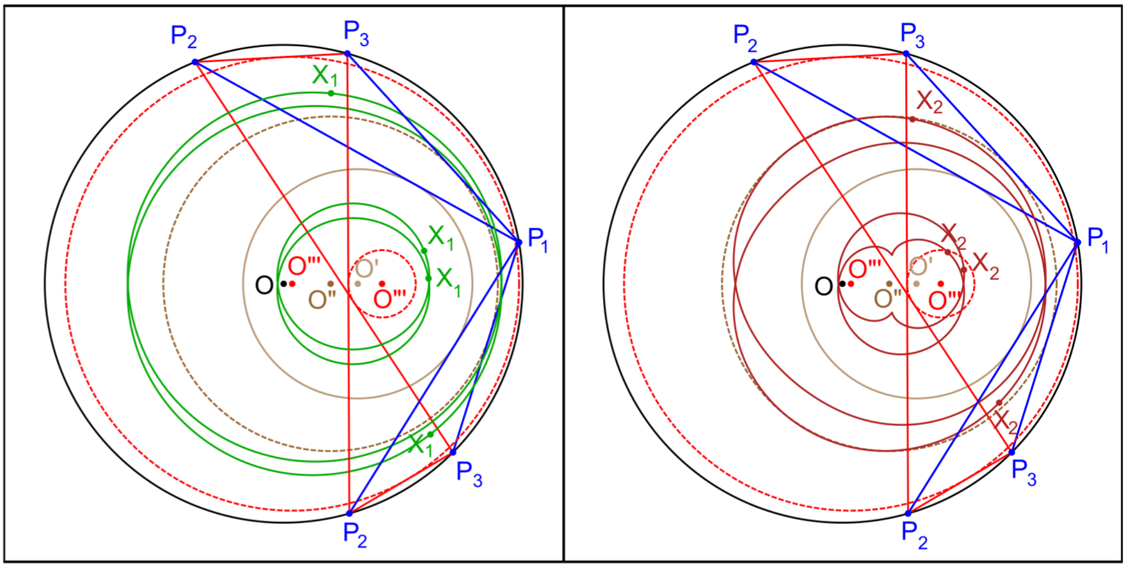

FIGURE 18 The four possible choices of P_1P_2 and P_1P_3 over the bic-III family, and the locus of the incenter (green, left), and the barycenter (brown, right), traced out under each choice. Note: only two distinct envelopes (dashed red) of P_2P_3 are produced. Video 1, Video 2 (color figure online)

TABLE 3 Table of videos. The last column is clickable and provides the YouTube code

id	Fam.	Title	youtu.be/< . >
01	n/a	Poncelet's porism with 2 ellipses, $N = 7$	kzxf7ZgJ5Hw
02	n/a	Poncelet's closure theorem (PCT)	L5A_S4VQLiw
03	all	$N = 3$ families with multiple caustics	8HXgkuY-nFQ
04	bic-II	Bicentric family with 1,2 caustics	OM7uilfdGgk
05	bic-II	Circular loci of $X_k, k =$1,40,165	qJGhf798E-s
06	bic-II	Loci of X_1 and X_2 over varying caustic radius	3dnsWPlAmxE
07	bic-II	Loci of $X_k, k = 2, 4, 5$ over varying caustic radius	6Fqp6Z1Q-0A
08	bic-II	Loci of the excenters (a circle and 2 non-conics)	qdqIuT-Qk6k
09	conf-II	Confocal family with 1,2 caustics	C14TL430UBc
10	conf-II	Locus of X_1 under sweep of confocal caustics	kCY6KHFDV2M
11	conf-II	Locus of X_1 and excenters	kCY6KHFDV2M
12	conf-II	Loci of the excenters with the $N = 4, 6$ caustics	wB9bVkY9rqU
13	bic-III	Non-conic loci of $X_k, k =$1, 2, 6, 8, 10	cvBOA7LmlZc
14	bic-III	Non-conic loci of incenter and excenters	A_-U2VvM5kY
15	bic-III	Loci of X_1, X_2 under 4 possible triangle choices	E1Rcu38MePQ
16	bic-III	Loci of X_6, X_8 under 4 possible triangle choices	u1_uANWDNr8
17	n/a	Locus of X_2 over "half $N = 4$" families	6yNod1LFVrY
18	n/a	Loci of X_{11} and X_{59} over varying circular caustic	2VqgB6KvP2g

7. Summary and videos

Animations illustrating some of the above phenomena are listed in Table 3.

Acknowledgements

We would like to thank Alexey Zaslavsky for the proof of Theorem 2 and Mark Helman for proposing the variant we use here. Also invaluable has been Arseniy Akopyan's generous availability to review and answer dozens of questions about many of the results herein. The first author is fellow of CNPq and coordinator of Project PRONEX/ CNPq/ FAPEG 2017 10 26 7000 508.

Appendix A: Symbol table

Table 4 lists most symbols used herein.

TABLE 4 Symbols used in the article

Symbol	Meaning
$\mathcal{C}, \mathcal{C}'$	Outer and inner circle in a poristic pair
R, r, d	Radii of $\mathcal{C}$ and $\mathcal{C}'$ and distance between their centers
$\mathcal{C}''$	Circular envelope of side P_2P_3 in the pencil of $\mathcal{C}, \mathcal{C}'$
$\mathcal{E}, \mathcal{E}'$	Outer and inner ellipses in confocal pair
$\mathcal{E}''$	Non-confocal envelope of side P_2P_3 in the pencil of $\mathcal{E}, \mathcal{E}'$
O, O'	Centers of either $\mathcal{C}, \mathcal{C}'$ or $\mathcal{E}, \mathcal{E}'$
a, b	Major and minor semi-axes of $\mathcal{E}$
c	Half the focal length of $\mathcal{E}$
a', b'	Major and minor semi-axes or $\mathcal{E}'$
X_1	Incenter
X_2	Barycenter
X_3	Circumcenter
X_4	Orthocenter
X_5	Euler's circle center
X_6	Symmedian point
X_9	Mittenpunkt
X_{40}	Bevan point

Appendix B: Family parametrizations

Bic-I

This family is also known as poristic or bicentric triangles. To obtain its vertices, use the parametrization for bic-II (see below), setting $d = \sqrt{R(R-2r)}$.

Bic-II

Let $P_i = [x_i, y_i]$ denote the vertices of a bic-II triangle, $i = 1, 2, 3$. Then:

$$
\begin{aligned}
x_2 &= \frac{(2ry_1(R^2-d^2)\Delta + (2dR^2-(R^2+d^2)x_1)(\Delta^2-r^2))}{(R^2+d^2-2dx_1)^2},\\
y_2 &= \frac{(4R^2d-2(R^2+d^2)x_1)r\Delta - y_1(R^2-d^2)(\Delta^2-r^2)}{(R^2+d^2-2dx_1)^2},\\
x_3 &= x_2 - \frac{4(R^2-d^2)y_1r\Delta}{(R^2+d^2-2dx_1)^2},\\
y_3 &= y_2 - \frac{4(2R^2d-(R^2+d^2)x_1)r\Delta}{(R^2+d^2-2dx_1)^2},
\end{aligned}
$$

where $\Delta = \sqrt{R^2+d^2-2dx_1-r^2}$.

Bic-III

Consider the pencil of circles $\mathcal{C}_u = (1-u)\mathcal{C}_1 + uC_2$ where

$$\mathcal{C}_1 : \ (x-d)^2 + y^2 = r^2, \ \ \mathcal{C}_2 : \ x^2 + y^2 = R^2.$$

The center of $\mathcal{C}_u$ is $(d(u), 0) = (d(1-u), 0)$. The radius $r(u)$ of $\mathcal{C}_u$ is given by

$$r(u) = \sqrt{d^2u^2 + (R^2-d^2-r^2)u + r^2}.$$

Let $P_i = [x_i, y_i]$ denote the vertices of a bic-III triangle, $i = 1, 2, 3$, where P_1P_2 is tangent to $\mathcal{C}_1$ and $P_2P_3(u)$ tangent to $\mathcal{C}_u$. Then

$$
\begin{aligned}
x_2(u) &= \frac{(2r(u)y_1(R^2-d(u)^2)\Delta(u) + (2d(u)R^2-(R^2+d(u)^2)x_1)(\Delta(u)^2-r(u)^2))}{(R^2+d(u)^2-2d(u)x_1)^2},\\
y_2(u) &= \frac{(4R^2d(u)-2(R(u)^2+d(u)^2)x_1)r(u)\Delta(u) - y_1(R(u)^2-d(u)^2)(\Delta(u)^2-r(u)^2)}{(R^2+d(u)^2-2d(u)x_1)^2},\\
x_3(u) &= x_2(u) - \frac{4(R^2-d(u)^2)y_1r(u)\Delta(u)}{(R^2+d(u)^2-2d(u)x_1)^2},\\
y_3(u) &= y_2(u) - \frac{4(2R^2d(u)-(R^2+d(u)^2)x_1)r(u)\Delta(u)}{(R^2+d(u)^2-2d(u)x_1)^2}.
\end{aligned}
$$

where $\Delta(u) = \sqrt{R^2+d(u)^2-2d(u)x_1-r(u)^2}$.

Conf-I

This family is also known as the "elliptic billiard". Its vertices can be obtained from the conf-II family (see below), setting a', b' as in Eq. (2).

Conf-II

$$
\begin{aligned}
P_1 &= [x_1, y_1],\\
P_2 &= \left[\frac{2\,a\alpha_3\,y_1\Delta - \alpha_1\,\alpha_2 x_1}{W}, \frac{-2\,b^2\,\alpha_2\,x_1\Delta - a\alpha_1\,\alpha_3\,y_1}{aW}\right],\\
P_3 &= \left[\frac{-2\,a\alpha_3\,y_1\Delta - \alpha_1\,\alpha_2 x_1}{W}, \frac{2\,b^2\,\alpha_2 x_1\Delta - a\alpha_1\,\alpha_3\,y_1}{aW}\right],\\
W &= \frac{\alpha_2^2 x_1^2}{a^2} + \frac{\alpha_3^2 y_1^2}{b^2},\;\; \Delta^2 = \left(a^2 b'^2 - a'^2 b'^2\right) x_1^2 + \left(a^2 a'^2 - \frac{a^2 a'^2 b'^2}{b^2}\right) y_1^2,\\
\alpha_1 &= a^2(b^2 - b'^2) - a'^2 b^2,\;\; \alpha_2 = (a^2 - a'^2) b^2 + a^2 b_c^2,\\
\alpha_3 &= a^2(b^2 - b'^2) + a'^2 b^2.
\end{aligned}
$$

Appendix C: Locus of barycenter over bic-II

Consider the bic-II family, parametrized by $P_1 = [x_1, y_1] = R[\cos t, \sin t]$. The locus of the barycenter X_2 of $P_1P_2P_3$ is given by

$$
X_2 = \left[\frac{-4\,d^2 x_1\,y_1^2 - (d^4 + (6\,R^2 - 4\,r^2)\,d^2 + R^4 - 4\,R^2 r^2)\,x_1 + 4\,R^2 d(d^2 + R^2 - 2r^2)}{3\,\left(R^2 + d^2 - 2\,dx_1\right)^2},\right.
\left. -\frac{\left(-4\,d^2 x_1^2 + 8\,d^3 x_1 - 3\,d^4 + (-2\,R^2 + 4\,r^2)\,d^2 + R^4 - 4\,R^2 r^2\right) y_1}{3\,\left(R^2 + d^2 - 2\,dx_1\right)^2}\right].
$$

Using the method of resultants, eliminate x_1, y_1 from the system $X_2 - [x, y] = 0$, $x_1^2 + y_1^2 - R^2 = 0$. Then X_2 will be given by an implicit polynomial equation $f(x, y) = 0$ of degree 6. Explicitly:

$$
\begin{aligned}
&729(x^2+y^2)^3 - 972dx(x^2+y^2)((4R^2+5d^2-4r^2)y^2 + (4R^2-3d^2-4r^2)x^2)\\
&-81(x^2+y^2)((R^6 - 86R^4d^2 - 8R^4r^2 + 121R^2d^4 + 128R^2d^2r^2 + 16R^2r^4 - 36d^6\\
&-72d^4r^2 - 96d^2r^4)x^2 + (R^6 - 42R^4d^2 - 8R^4r^2 - 63R^2d^4 + 128R^2d^2r^2 + 16R^2r^4\\
&-4d^6 + 24d^4r^2 - 32d^2r^4)y^2) + 108dx((3R^8 - 45R^6d^2 - 28R^6r^2 + 81R^4d^4\\
&+44R^4d^2r^2 + 80R^4r^4 - 39R^2d^6 + 20R^2d^4r^2 - 112R^2d^2r^4 - 64R^2r^6\\
&-36d^6r^2 + 64d^2r^6)x^2 + (3R^8 - 45R^6d^2 - 28R^6r^2 + 81R^4d^4 + 76R^4d^2r^2 + 80R^4r^4\\
&-39R^2d^6 - 44R^2d^4r^2 + 16R^2d^2r^4 - 64R^2r^6 - 4d^6r^2 + 64d^2r^6)y^2)\\
&-36d^2((9R^{10} - 36R^8d^2 - 90R^8r^2 + 54R^6d^4 - 18R^6d^2r^2 + 312R^6r^4 - 36R^4d^6\\
&+306R^4d^4r^2 - 372R^4d^2r^4 - 480R^4r^6 + 9R^2d^8 - 198R^2d^6r^2 + 96R^2d^4r^4\\
&+256R^2d^2r^6 + 384R^2r^8 - 36d^6r^4 + 96d^4r^6 - 64d^2r^8)x^2\\
&+(9R^{10} - 36R^8d^2 - 102R^8r^2 + 54R^6d^4\\
&+126R^6d^2r^2 + 392R^6r^4 - 36R^4d^6 + 54R^4d^4r^2\\
&-116R^4d^2r^4 - 544R^4r^6 + 9R^2d^8 - 78R^2d^6r^2 + 160R^2d^4r^4 + 128R^2r^8 - 4d^6r^4\\
&+32d^4r^6 - 64d^2r^8)y^2) + 48R^2d^3r^2(3R^2 - 3d^2 + 4r^2)(3R^6 - 9R^4d^2 - 28R^4r^2\\
&+9R^2d^4 - 4R^2d^2r^2 + 80R^2r^4 - 3d^6 + 32d^4r^2 - 32d^2r^4 - 64r^6)x\\
&-16R^2d^4r^4((R+d)^2 - 4r^2)((R-d)^2 - 4r^2)(3R^2 - 3d^2 + 4r^2)^2 = 0.
\end{aligned}
$$

References

[1] Bix, R.: Conics and Cubics. Springer, Basel (2006)

[2] Cayley, A.: On the porism of the in-and-circumscribed triangle, p. 1. *Quart. J. Pure Appl. Math.* (1857)

[3] Chapple, W.: An essay on the properties of triangles inscribed in, and circumscribed about two given circles. In: Miscellanea Curiosa Mathematica, The Gentleman's Magazine, vol. 4 (1749). https://archive.org/details/miscellaneacuri01unkngoog/page/n142/mode/2up

[4] del Centina, A.: *Poncelet's porism: a long story of renewed discoveries i.* Arch. Hist. Exact Sci. **70**(2), 1–122 (2016). https://doi.org/10.1007/s00407-015-0163-y.

[5] Dragović, V., Radnović, M.: Poncelet porisms and beyond: integrable billiards, hyperelliptic Jacobians and pencils of quadrics. In: Frontiers in Mathematics. Springer, Basel (2011). https://books.google.com.br/books?id=QcOmDAEACAAJ

[6] Fierobe, C.: On the circumcenters of triangular orbits in elliptic billiard. *J. Dyn. Control Syst.* (2021). arXiv:1807.11903

[7] Garcia, R.: *Elliptic billiards and ellipses associated to the 3-periodic orbits.* Am. Math. Mon. **126**(06), 491–504 (2019)

[8] Garcia, R., Reznik, D.: Invariants of self-intersected and inversive N-periodics in the elliptic billiard (2020). arXiv:2011.06640

[9] Garcia, R., Reznik, D.: *Related by similarity I: poristic triangles and 3-periodics in the elliptic billiard.* Int. J. Geom. **10**(3), 52–70 (2021)

[10] Garcia, R., Reznik, D., Koiller, J.: Loci of 3-periodics in an elliptic billiard: why so many ellipses? (2020). arXiv:2001.08041

[11] Garcia, R., Reznik, D., Koiller, J.: New properties of triangular orbits in elliptic billiards. *Math. Mon.* (2020)

[12] Halbeisen, L., Hungerbühler, N.: *A simple proof of Poncelet's theorem (on the occasion of its bicentennial).* Am. Math. Mon. **122**(6), 537 (2015)

[13] Helman, M.: Private communication (2021)

[14] Helman, M., Laurain, D., Garcia, R., Reznik, D.: Center power and loci of poncelet triangles (2021). arXiv:2102.09438

[15] Kimberling, C.: Encyclopedia of Triangle Centers (2019). https://faculty.evansville.edu/ck6/encyclopedia/ETC.html

[16] Odehnal, B.: *Poristic loci of triangle centers.* J. Geom. Graph. **15**(1), 45–67 (2011)

[17] Poncelet, J.-V.: Traité sur les propriétés projectives des figures. Bachelier, Paris (1822). https://gallica.bnf.fr/ark:/12148/bpt6k9608143v.texteImage

[18] Reznik, D., Garcia, R., Koiller, J.: *Can the elliptic billiard still surprise us?* Math. Intell. **42**, 6–17 (2020)

[19] Romaskevich, O.: *On the incenters of triangular orbits on elliptic billiards.* Enseign. Math. **60**, 247–255 (2014)

[20] Tabachnikov, S. Geometry and Billiards, Student Mathematical Library (Mathematics Advanced Study Semesters, University Park, PA), vol. 30, American Mathematical Society, Providence (2005). https://www.personal.psu.edu/sot2/books/billiardsgeometry.pdf

[21] Weisstein, E.: Mathworld. MathWorld—A Wolfram Web Resource (2019). https://mathworld.wolfram.com

[22] Zaslavsky, A., Akopyan, A.: Private communication (2021)

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