MIXED INTERSECTION OF CEVIANS AND PERSPECTIVE TRIANGLES

Boris ODEHNAL
Dresden University of Technology, Germany

ABSTRACT: Any point P in the plane of a triangle $\Delta = (A,B,C)$ can be considered as the intersection of certain Cevians, whether P is a triangle center or not. Taking two centers, say Y and Z, we find six Cevians with a total of 11 points of intersection, among them Δ's vertices and Y and Z. The remaining six points form two triangles Δ_1 and Δ_2 which are both perspective to the base triangle Δ and to each other. The three centers of perspectivity are triangle centers of Δ and collinear, independent of the choice of Y and Z. Furthermore any pair out of Δ, Δ_1, and Δ_2 has the same perspectrix which yields a closed chain of Desargues' $(10_3, 10_3)$ configurations. Then special affine appearances of Desargues' configurations can be obtained by a suitable choice of Y and Z. Any choice of a fixed center Y leads to exactly one center Z as the fourth point of intersection of two conic sections circumscribed to Δ such that the perspectors P_1, P_2, and consequently P_{12} are points at infinity. For fixed center Y we obtain a quadratic transformation in Δ's plane which transforms central lines to central conic sections.

Keywords: triangle, Cevian, perspective triangles, Desargues configuration, triangle center, cross-point

1. INTRODUCTION

Many triangle centers appear as the intersection of certain Cevians. For example the incenter X_1 is the intersection of the interior angle bisectors of a triangle Δ with vertices A, B, C. The centroid X_2 can be found as the common point of the medians, whereas the circumcenter X_3 is the meet of the bisectors of Δ's edges. The orthocenter X_4 comes along as the intersection of the altitudes of Δ. Here and in the following, centers of Δ are labelled according to C. Kimberling's list, cf. [4, 6].

What happens if one makes the erroneous construction by mixing Cevians of different centers when intersecting them? Figure 1 shows such a mistake: The mixed intersection of medians and altitudes in Δ produces a triangle.

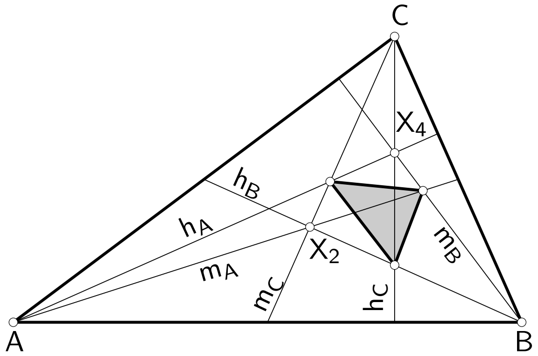

Figure 1: Is it a mistake?

Usually three arbitrarily chosen Cevians will not meet in a common point, except maybe in cases where the side lengths of Δ fulfil certain relations. However, if we start with two triangle centers, say Y and Z, and intersect the respective Cevians we obtain, besides the two centers Y and Z, the three vertices of Δ. Further we find six points which can be arranged in many ways in two perspective triangles Δ_1 and Δ_2.

In Section 2 we shall show that these triangles are perspective to each other with perspector P_{12}. They are also perspective to Δ with perspectors P_1 and P_2, cf. Figure 6. All the three perspectors are triangle centers of Δ, provided Y and Z are centers of Δ. Moreover, the perspectors P_1, P_2, and P_{12} are collinear and consequently the perspectrix is the same for any pair of triangles out of the three. So we obtain a closed chain of three Desarguesian (10_3,10_3) configurations. This will be discussed in Section 4. An Example is displayed in Figure 6. Special affine versions of $(10_3, 10_3)$ configurations are described in Section 5. In Section 3 we show that the perspector P_{12} is the crosspoint of Y and Z. This gives a new access to the crosspoints of triangle centers.

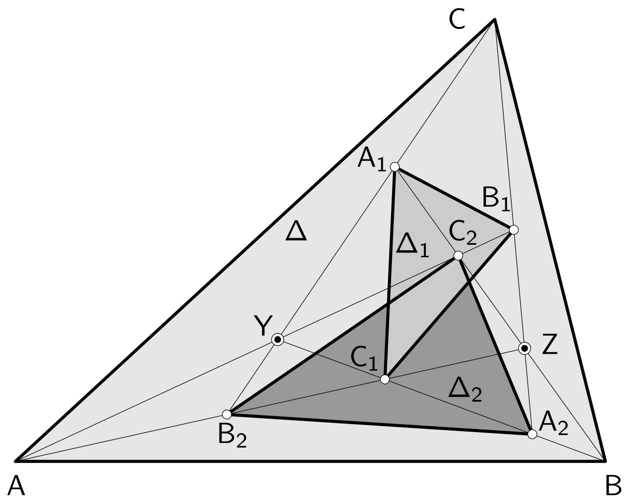

Figure 2: The triangles Δ_1 and Δ_2 whose vertices are the mixed intersections of Cevians of Y and Z.

If we fix Y then the mapping $q: \ Z \to P_{12}$ is quadratic and sends centers to centers and central lines are mapped to central conics. The mapping q is birational, *i.e.*, its inverse is also rational. It turns out that q is a composition of the isogonal conjugation with a collineation. We pay our attention to q in Section 6.

In the following we use homogeneous trilinear coordinates $(p_0 : p_1 : p_2)$ in order to represent points in the plane of Δ. The vertices of Δ are the base points and thus their coordinate vectors are $A = (1:0:0)$, $B = (0:1:0)$, and $C = (0:0:1)$, cf. [4, 6]. Further we let $a := \overline{BC}$, $b := \overline{CA}$, and $c := \overline{AB}$ denote the side lengths of Δ. We denote the line joining two points P and Q by $[P,Q]$. A point $Z = (z_0 : z_1 : z_2)$ is a center exactly if $z_0 = f(a,b,c)$ is a homogeneous function of the side lengths a, b, and c of the base triangle Δ with $z_1 = f(b,c,a)$ and $z_2 = f(c,a,b)$. Later we use the symbol ζ in order to indicate that a homogeneous function f is transformed via the cyclic shift: If $f = f(a,b,c)$, then $f^\zeta = f(b,c,a)$. Thus a triangle center $X = (\xi_0 : \xi_1 : \xi_2)$ is characterized by $\xi_{i+1} = \xi_i^\zeta$ for $i \in \{0,1,2\}$ and i is counted modulo 3. Similarly a line in Δ's plane is called a central line if its homogeneous trilinear coordinates follow the same rules as those of centers. Note that ζ applies to any cyclically ordered triplet.

2. PERSPECTIVE TRIANGLES AND THEIR PERSPECTORS

Assume $Y = (\xi_0 : \xi_1 : \xi_2)$ and $Z = (\eta_0 : \eta_1 : \eta_2)$ are triangle centers of Δ. Then we look at the following points of intersection of Cevians through Y and Z:

$$\begin{aligned} &A_1{:=}[C,Y]\cap[B,Z], A_2{:=}[B,Y]\cap[C,Z],\\ &B_1{:=}[A,Y]\cap[C,Z], B_2{:=}[C,Y]\cap[A,Z],\\ &C_1{:=}[B,Y]\cap[A,Z], C_2{:=}[A,Y]\cap[B,Z]. \end{aligned} \tag{1}$$

We define two triangles collecting the intersection points of *wrong pairs of Cevians* by letting $\Delta_1 = (A_1,B_1,C_1)$ and $\Delta_2 = (A_2,B_2,C_2)$ as illustrated in Figure 2.

No we can show:

Theorem 2.1. *The triangles Δ_1 and Δ_2 are perspective to the base triangle Δ. The perspectors P_1 and P_2 are triangle centers of Δ.*

Proof. With the above prerequisites we find the vertices of Δ_1 and Δ_2 as

$$\begin{aligned} A_1 &= (\xi_0\eta_0 : \xi_1\eta_0 : \xi_0\eta_2),\\ B_1 &= (\xi_1\eta_0 : \xi_1\eta_1 : \xi_2\eta_1),\\ C_1 &= (\xi_0\eta_2 : \xi_2\eta_1 : \xi_2\eta_2), \end{aligned} \tag{2}$$

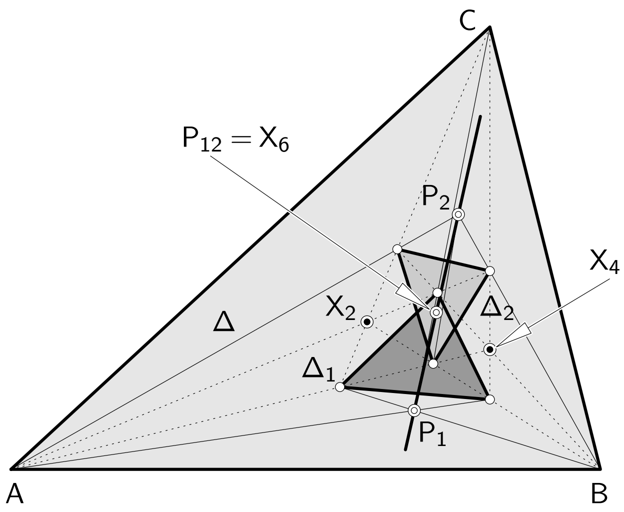

Figure 3: The three perspectors P_1, P_2, and P_{12} constructed out of the centroid $Y = X_2$ and the orthocenter $Z = X_4$.

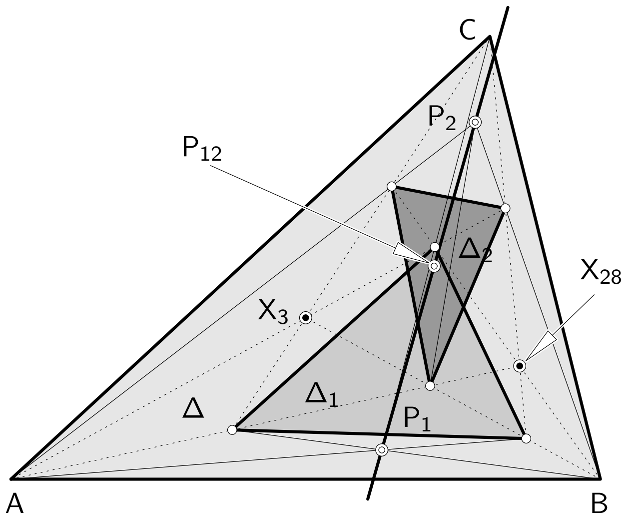

Figure 4: The three perspectors P_1, P_2, and P_{12} constructed out of the circumcenter $Y = X_3$ and the center $Z = X_{28}$.

and

$$\begin{aligned} A_2 &= (\xi_0\eta_0 : \xi_0\eta_1 : \xi_2\eta_0), \\ B_2 &= (\xi_0\eta_1 : \xi_1\eta_1 : \xi_1\eta_2), \\ C_2 &= (\xi_2\eta_0 : \xi_1\eta_2 : \xi_2\eta_2). \end{aligned} \tag{3}$$

Then we show that the lines $[A_1, A_2]$, $[B_1, B_2]$, and $[C_1, C_2]$ are concurrent by computing their trilinears and showing the linear dependency.

The perspectors P_i of Δ and Δ_i (with $i \in \{1, 2\}$) can be found as intersection of $[A, A_i]$ and $[B, B_i]$ and we arrive at

$$\begin{aligned} P_1 &= (\xi_0\xi_1\eta_0\eta_2 : \xi_1\xi_2\eta_1\eta_0 : \xi_2\xi_0\eta_2\eta_1) \\ &\text{and} \\ P_2 &= (\xi_0\xi_2\eta_0\eta_1 : \xi_1\xi_0\eta_1\eta_2 : \xi_2\xi_1\eta_2\eta_0). \end{aligned} \tag{4}$$

P_1 and P_2 are centers for ξ_i and η_i (with $i \in \{0, 1, 2\}$) are center functions, *i.e.*, they are homogeneous and cyclic symmetric in a, b, and c. For example $(\xi_0\xi_1\eta_0\eta_2)^\zeta = \xi_1\xi_2\eta_1\eta_0$ and likewise for all the other coordinate functions of P_1 and P_2, respectively. $\square$

Obviously P_1 and P_2 are triangle centers for any choice of centers Y and Z. It can be seen at once that $Y = Z$ results in $P_1 = P_2$. Furthermore if $Y = X_1$, *i.e.*, the incenter of Δ and Z is an arbitrary center not equal to X_1, then $P_1 = (\eta_0\eta_2 : \eta_1\eta_0 : \eta_2\eta_0)$ and $P_2 = (\eta_0\eta_1 : \eta_1\eta_2 : \eta_2\eta_0)$. The mappings $Z \mapsto P_1$ and $Z \mapsto P_2$ are birational for they are compositions of the isogonal conjugation $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_2x_0 : x_0x_1)$ with collineations in Δ's plane with X_1 for a fixed point and shifting Δ's vertices in clockwise or counter clockwise direction, respectively.

It is easy to show that the following holds:

Theorem 2.2. *The triangles Δ_1 and Δ_2 are perspective. The perspector P_{12} is a triangle center that is collinear with P_1 and P_2 from Theorem 2.1, except $Y = Z$, where $P_1 = P_2 = P_{12} = Y = Z$.*

Proof. The lines $[A_1, A_2]$, $[B_1, B_2]$, and $[C_1, C_2]$ are concurrent. In order to prove this, we compute the trilinear coordinates of the three lines and show that they are linearly dependent.

The point of concurrency can be computed as the intersection of any pair of the above given lines which gives

$$\begin{aligned} P_{12} = (&\xi_0\eta_0(\xi_2\eta_1 + \xi_1\eta_2) : \\ &\xi_1\eta_1(\xi_0\eta_2 + \xi_2\eta_0) : \\ &\xi_2\eta_2(\xi_1\eta_0 + \xi_0\eta_1)), \end{aligned} \tag{5}$$

which is obviously a triangle center of Δ for the coordinate function $i + 1$ is the ζ-image of the coordinate function i, for $i \in \{0,1,2\}$ and i is counted modulo 3.

The three perspectors P_1, P_2, and P_{12} are collinear, for their trilinear coordinate vectors are linearly dependent.

If $Y = Z$, then $P_1 = P_2 = Z$ as outlined above. Let $\xi_i = \eta_i$ (for $i \in \{0,1,2\}$) then $P_{12} = Z$. $\square$

At this point we shall remark that P_{12} from Equation (5) is the bicentric sum of P_1 and P_2 given in Equation (4) as defined in [3, 5].

Figures 3 and 4 show the perspectors P_1, P_2, and P_{12} for different choices of Y and Z.

Note that Y and Z are also perspectors of the following four pairs of triangles: Y connects $(\Delta, \zeta(\Delta_1))$ and $(\Delta, \zeta^{-1}(\Delta_2))$, whereas Z connects $(\Delta, \zeta(\Delta_2))$ and $(\Delta, \zeta^{-1}(\Delta_1))$, with ζ applied to the vertices of the triangles.

3. THE RELATION TO CROSSPOINTS

Equation (5) allows us to define a mapping $\oplus : (Y,Z) \mapsto Y \oplus Z = P_{12}$ for any pair of points Y and Z in the plane of Δ. In Table 1 and Table 2 the $\oplus$-images of some pairs of triangle centers X_i and X_j are given. A $\star$ indicates a yet innominate center, the numbers in the Tables 1 and 2 are the Kimberling numbers of $X_i \oplus X_j$. We have the following result:

Theorem 3.1. *The point $Y \oplus Z$ that is assigned to any pair (Y,Z) of points in Δ's plane via Equation* (5) *coincides with the crosspoint of Y and Z.*

Proof. We compare the coordinate representation of $Y \oplus Z$ given in Equation (5) with the expressions for crosspoints given in [6, p. 202] or [7]. $\square$

We shall emphasize that our access to the crosspoint differs from the definition given in [6] or [7].

Figure 5 shows how the crosspoint of two arbitrary points P and Q with respect to a triangle Δ is usually found: Let P and Q be any two points in Δ's plane. Let $\Delta_1 = (A', B', C')$ and $\Delta_2 = (A'', B'', C'')$ be the Cevian triangles of P and Q, respectively. Then define $\Delta_3 = (A''', B''', C''')$ by letting

$$\begin{aligned} A''' &:= [A, A''] \cap [B', C'], \\ B''' &:= [B, B''] \cap [C', A'], \\ C''' &:= [C, C''] \cap [A', B']. \end{aligned}$$

Now it turns out that Δ_1 and Δ_3 are perspective and the perspector is called the crosspoint of P and Q. Somehow this definition seems to be unsymmetric. However, if we define $\Delta_4 = (A'''', B'''', C'''')$ with $A'''' = [A, A'] \cap [B'', C'']$, and B'''' and C'''' cyclic, then Δ_4 is perspective to Δ_2. Surprisingly, the crosspoint also serves as the perspector for the latter two triangles.

Now we have a simple geometric meaning of the crosspoint of two centers. It can be found as the perspector of the two triangles Δ_1 and Δ_2 defined in Equations (2) and (3) constructed via mixed intersection of the Cevians of two certain centers. Note that Δ_1 and Δ_2 differ from the triangles which are usually involved in the construction of the crosspoint as defined in [6, p. 202] and [7]. So we have found a simple access to the crosspoints.

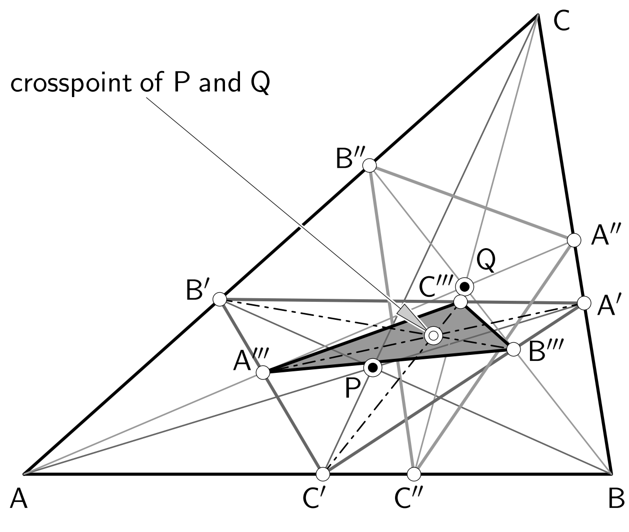

Figure 5: The usual way of finding a crosspoint of two point.

Figure 3 illustrates the case where $Y = X_2$ and $Z = X_4$. The line carrying the three perspectors P_1, P_2, and $P_{12} = X_6$ is also shown. To the best of the authors knowledge this is the only pair (X_i, X_j) where $X_i \oplus X_j = X_{i+j}$ holds. In Figure 4 we have chosen $Y = X_3$ and $Z = X_{28}$.

The mapping $\oplus$ gives a tool for the construction of triangle centers as perspectors of triangles Δ_1 and Δ_2. Therefore we find a way to construct some triangle centers for which elementary constructions are missing until now, especially those with large Kimberling number. For example $X_{4854} = X_7 \oplus X_{10}$, *i.e.*, the center X_{4854} can be found as the perspector P_{12} with $Y = X_7$ (Gergonne point of Δ) and $Z = X_{10}$ (Spieker point of Δ), cf. Table 1.

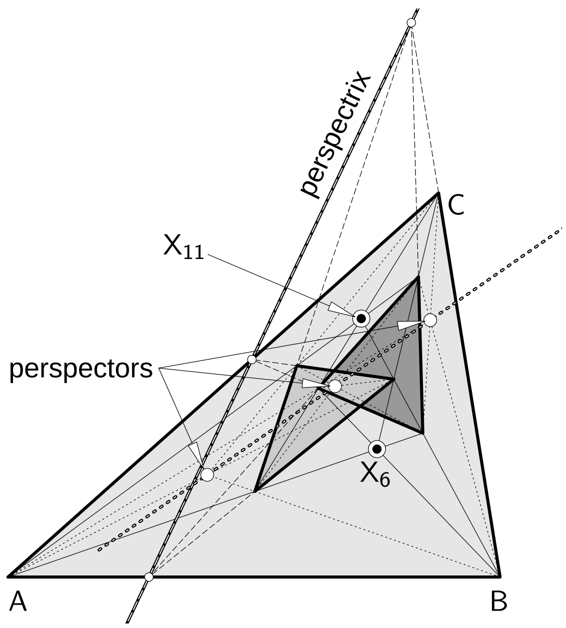

Figure 6: A closed chain of Desargues configurations with Y being the Lemoine point X_6 and Z being the Feuerbach point X_{11}.

4. CHAINS OF DESARGUES' CONFIGURATIONS

From Theorem 2.1 we know that any pair of triangles out of the triplet $(\Delta, \Delta_1, \Delta_2)$ is perspective to a certain point. According to Desargues' two triangle theorem, any pair of triangles which is perspective to a point, is also perspective to a line. Consequently, any pair of triangles out of the triplet $(\Delta, \Delta_1, \Delta_2)$ is also perspective to a line. The perspectrices shall be labelled p_1, p_2, and

Table 1: $\oplus$-compositions of some centers besides $X_1 \oplus X_2 = X_{37}$.

$\oplus$	3	4	5	6	7	8	9	10
1	73	65	2599	42	354	3057	55	2292
2	216	6	233	39	1	9	1212	1213
3	3	185	$\star$	184	$\star$	$\star$	$\star$	$\star$
4		4	3574	51	1836	1837	1864	1834
5			5	$\star$	$\star$	$\star$	$\star$	$\star$
6				6	$\star$	$\star$	2347	$\star$
7					7	497	$\star$	4854
8						8	210	$\star$
9							9	$\star$
10								10

Table 2: More $\oplus$-compositions of centers.

$\oplus$	1	2	3	4	6	8	19	31
19	31	$\star$	$\star$	1824	1400	$\star$	$\star$	2179
20	$\star$	1249	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
21	2646	$\star$	$\star$	1858	$\star$	960	$\star$	$\star$
31	1964	$\star$	$\star$	$\star$	213	$\star$	2179	31
32	$\star$	$\star$	682	$\star$	3051	$\star$	$\star$	1918

p_{12} according to the perspectors. Since the perspectors are collinear we have:

Theorem 4.1. *The perspectrices p_1, p_2, and p_{12} fall in one line,* i.e., $p_1 = p_2 = p_{12}$. *The common perspectrix is a central line.*

Proof. The fact that all three perspectrices coincide is caused by the collinearity of the three perspectors and the fact that the perspectrix is the same for two different pairs of triangles of the triplet $(\Delta, \Delta_1, \Delta_2)$, for the set of perspective collineations with common axis and collinear centers constitute a group, cf. Figure 7. It is also possible to compute the perspectrix for any pair of triangles from our triplet.

We only have to show that the perspectrix is a central line. The computation of its homogeneous trilinear coordinates is straightforward and we find its first coordinate

$$l_0 = \frac{\xi_1 \xi_2 \eta_1 \eta_2}{\eta_1 \xi_2 - \eta_2 \xi_1}$$

and further $p_{12} = (l_0 : l_0^{\zeta} : l_0^{\zeta^2})$. Therefore it is a central line. □

The Theorems 2.2 and 4.1 tell us that there is a closed chain of Desargues configurations

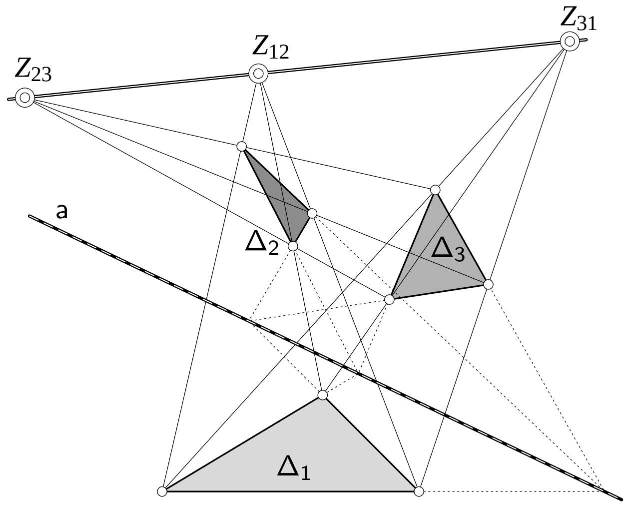

Figure 7: Chain of Desargues configurations.

with three mutually perspective triangles, three collinear perspectors, and one common perspectrix. Figure 6 shows such a chain of Desargues configurations built from $Y = X_6$ and $Z = X_{11}$.

5. SPECIAL AFFINE VERSIONS OF DESARGUES CONFIGURATIONS

Equation (5) allows us to determine special affine versions of closed Desargues' chains. A perspectors P_{12} is an ideal point, if it is contained in the ideal line with the equation

$$\mathscr{L}_\infty : \ ax_0 + bx_1 + cx_2 = 0. \tag{6}$$

If we fix one center, say Y, then the locus of points Z such that $Y \oplus Z$ is an ideal point is a conic section k_Y. This is immedeately seen, if we derive the incidence condition for $(Y \oplus Z) \in \mathscr{L}_\infty$. We insert Equation (5) into Equation (6) and find

$$k_Y : \sum_{\text{cyclic}} a\xi_0\eta_0(\xi_2\eta_1 + \xi_1\eta_2) = 0, \tag{7}$$

which is a homogeneous and quadratic equation in η_i for fixed $Y = (\xi_0 : \xi_1 : \xi_2)$. We observe that k_Y is a circumconic of Δ, *i.e.*, k_Y passes through Δ's vertices.

Figure 8 shows c_1 with some centers on it. The perspector of Δ_1 and Δ_2 is the ideal point X_{513}. In Figure 9 we have simplified the notation in order to save space, only the Kimberling numbers of centers are written. We have fixed $Y = X_2$ and thus c_2 is the Steiner circumellipse of Δ, cf. [6]. The perspector P_{12} again moves to the ideal line and $P_{12} = X_{524}$.

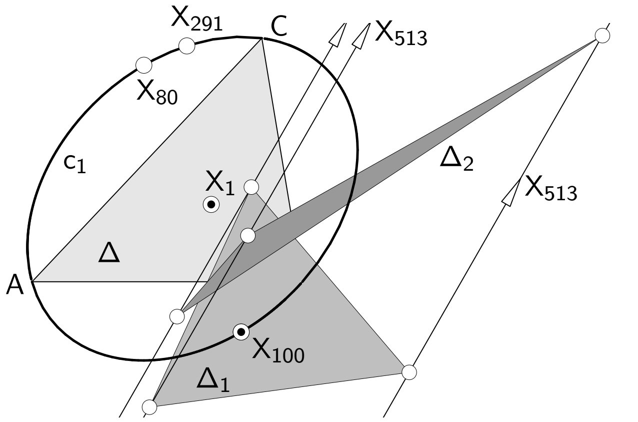

Figure 8: Special affine appearance of Desargues configuration: $X_1 \oplus X_{100} = X_{513}$ which lies on the line at infinity.

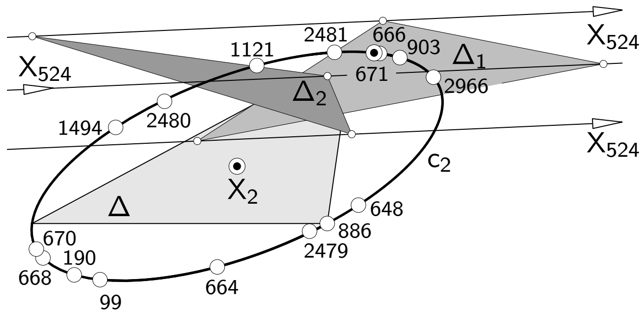

Figure 9: Special affine appearance of Desargues configuration: $X_2 \oplus X_{671} = X_{524}$ which is an ideal point.

Is it possible to find Desarguesian configurations with three perspectors on the ideal line? The answer to this question is: yes. The condition for P_1 and P_2 to lie on the ideal line $\mathscr{L}_\infty$ is given by

$$\begin{aligned} l_1 &: \sum_{\text{cyclic}} a\xi_0\xi_1\eta_0\eta_2 = 0 \\ &\quad\text{and} \\ l_2 &: \sum_{\text{cyclic}} a\xi_0\xi_2\eta_0\eta_1 = 0, \end{aligned} \tag{8}$$

where we have inserted the coordinates of P_1 and P_2 as given in Equation (4) into the Equation (6) of $\mathscr{L}_\infty$. For a fixed center $Y = (\xi_0 : \xi_1 : \xi_2)$ Equations (8) are the equations of two conic sections. Both pass through the vertices A, B, C of the base triangle and thus they have only one further common point. This yields:

Theorem 5.1. *To any fixed center Y there exists exactly one center Z such that the perspectors P_1, P_2, and P_{12} lie on the ideal line.*

Proof. We only have to show that l_1 and l_2 have precisely one center in common. Obviously, three common points fall into the vertices of Δ. It is elementary and straightforward to find the fourth intersection:

$$\begin{aligned} Z = (&\xi_0(\xi_1^2 b^2 - \xi_0\xi_2 ca)(\xi_2^2 c^2 - \xi_0\xi_1 ab) : \\ &: (\xi_1(\xi_2^2 c^2 - \xi_1\xi_0 ab)(\xi_0^2 a^2 - \xi_1\xi_2 bc) : \\ &: (\xi_2(\xi_0^2 a^2 - \xi_2\xi_1 bc)(\xi_1^2 b^2 - \xi_2\xi_0 ca)), \end{aligned}$$

which is obviously a center of Δ. $\square$

Figure 10 illustrates the contents of Theorem 5.1 with $Y = X_1$. The resulting center

$$\begin{aligned} Z = (&(b^2 - ca)(c^2 - ab) : \\ &: (c^2 - ab)(a^2 - bc) : \\ &: (a^2 - bc)(b^2 - ca)), \end{aligned} \tag{9}$$

which is not yet named, yields triangles Δ_1 and Δ_2 as defined in Equation (1), such that all the three perspectors P_1, P_2, and P_{12} are ideal points.

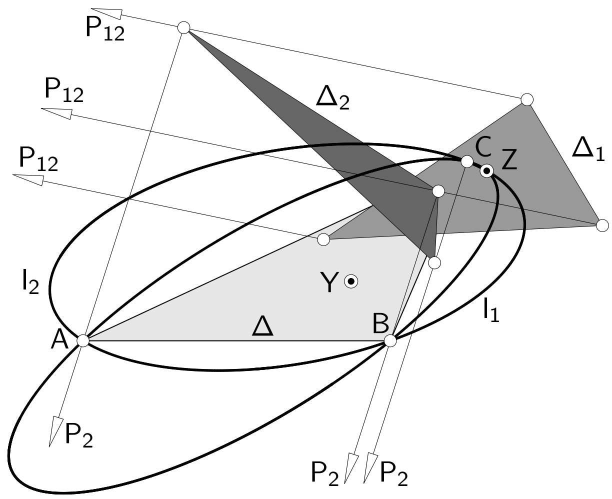

Figure 10: All three perspectors are ideal points for a suitable choice of Z (precisely that of Equation (9)) depending on Y.

6. A QUADRATIC TRANSFORMATION

The coordinates of P_{12} given in Equation (5) are quadratic in ξ_i and η_i.

Assume $Y = (\xi_0 : \xi_1 : \xi_2)$ is an arbitrary point in Δ's plane, not necessarily a center. Then q is a quadratic mapping in Δ's plane. The base points of q are the vertices of Δ. The associated net of conic spanned by three pairs of lines whose equations are given by the coordinate functions of P_{12} from Equation (5) and we have

$$\begin{aligned} q_0 :\ \xi_0\eta_0(\xi_2\eta_1 + \xi_1\eta_2) &= 0, \\ q_1 :\ \xi_1\eta_1(\xi_0\eta_2 + \xi_2\eta_0) &= 0, \\ q_2 :\ \xi_2\eta_2(\xi_1\eta_0 + \xi_0\eta_1) &= 0. \end{aligned} \tag{10}$$

Obviously these are three pairs of lines, each containing a side line of Δ and a further line passing through the harmonic conjugates Y'_A, Y'_B, Y'_C of the vertices of Y's Cevian triangle (Y_A, Y_B, Y_C) with respect to Δ's vertices, *i.e.*, the pairs of lines are

$$\begin{aligned} q_0 &:= [B,C] \cup [A, Y'_A], \\ q_1 &:= [C,A] \cup [B, Y'_B], \\ q_2 &:= [A,B] \cup [C, Y'_C]. \end{aligned}$$

Note that the harmonic conjugates Y'_A, Y'_B, Y'_C gather on the trilinear polar line of Y with respect to the base triangle Δ. Figure 11 shows the base points and exceptional lines of q for an arbitrary point Y.

It is a well-known result from algebraic geometry that q is birational, *i.e.*, its inverse is also rational, for it has three base points, which appear as the intersections of the three degenerate

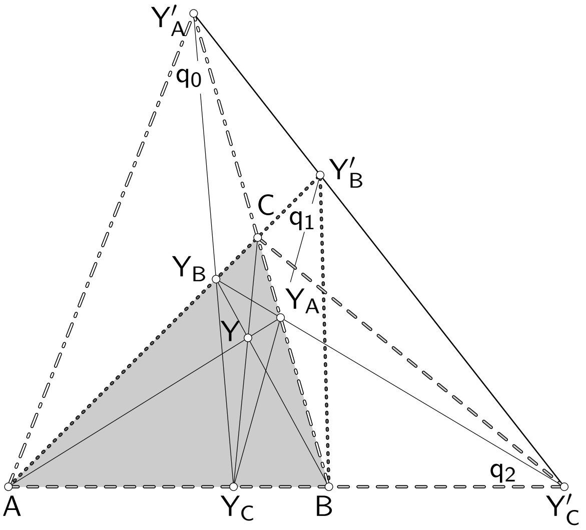

Figure 11: Base points and exceptional lines of q for an arbitrary point Y.

conic sections q_i defined by the three coordinate functions (10) of q, cf. [2].

Since Equation (5) can be rewritten in the form

$$P_{12} = \begin{bmatrix} 0 & \eta_0\eta_1 & \eta_0\eta_2 \\ \eta_0\eta_1 & 0 & \eta_1\eta_2 \\ \eta_0\eta_2 & \eta_1\eta_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1\xi_2 \\ \xi_2\xi_0 \\ \xi_0\xi_1 \end{bmatrix},$$

we can conclude that q is a composition of the isogonal conjugation and a collinear transformation that depends on Y.

The mapping q transforms central lines to central conics. We show that the image of a central line is given by the following conic section:

Theorem 6.1. *Assume that $g = (g_0 : g_1 : g_2)$ is a central line,* i.e., *its coordinate functions g_i are cyclic symmetric in a, b, c and $\sum_{\text{cyclic}} X_0 g_0 = 0$ is its equation. Further let $Y = (\eta_0 : \eta_1 : \eta_2)$ be a fixed triangle center for Δ. Then $q(g)$ has the equation*

$$\begin{aligned} \sum_{\text{cyclic}} \eta_1^2\eta_2^2(g_0\eta_0 - g_1\eta_1 - g_2\eta_2)X_0^2 \\ -2g_0\eta_0^3\eta_1\eta_2X_1X_2 = 0. \end{aligned} \tag{11}$$

Proof. Assume $g = (g_0 : g_1 : g_2)$ is a central line. Then $g(\lambda : \mu) = \lambda(g_2 : 0 : -g_0) + \mu(g_1 : -g_0 : 0)$ with $(\lambda : \mu) \neq (0 : 0)$ is a parametrization of g. We substitute this parametrization into (5) and obtain a parametrization which is obviously homogeneous and quadratic in $(\lambda : \mu)$. Thus it describes a conic section. We keep in mind that η_i are fixed values for a fixed center Y. By eliminating λ and μ we arrive at Equation (11). $\square$

Figure 12 shows the q-image of the Euler-line e with $Y = X_1$. The quadratic mapping q is also applied to some of the triangle centers on e.

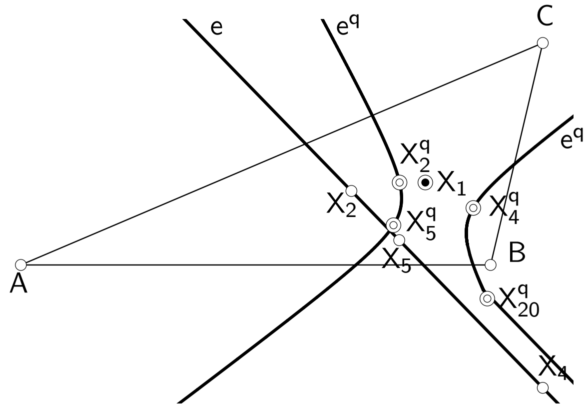

Figure 12: The quadratic mapping q and the image of the Euler-line e with $Y = X_1$ together with some centers on e and their q-images.

Any conic section through the base points of q is mapped to a straight line. This is shown in Figure 13, where $Y = X_2$. Δ's circumcircle u is mapped to aline. In this case the line at infinity is mapped to the Steiner inellipse.

The determinant of the hessian of (11) equals $(2\eta_0\eta_1\eta_2)^5 g_0g_1g_2$. So the conic section is singular (splits off into two lines) if Y lies on one side line of Δ. This is the case for example for a right angled triangle and $Y = X_3$ or even $Y = X_4$. In general neither $g_i = 0$, for g is a central line. Exceptions occur for special triangles.

In Figure 14 the action of q on Δ's incricle i is illustrated. We have chosen four different centers Y as pivot for q, namely X_1 (incenter), X_2 (centroid), X_3 (circumcenter), and X_4 (orthocenter). The four quartic curves showing up as as the four different q-images of i are labelled according to Y. Note that each of these quartics has

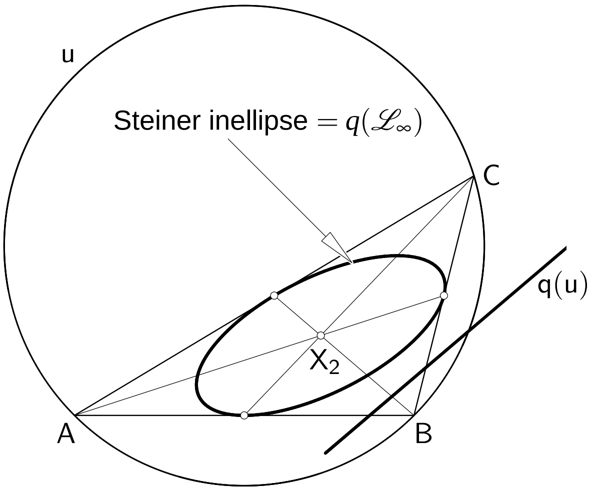

Figure 13: The quadratic mapping q with $Y = X_2$ shows the Steiner inellipse as the image of the ideal line.

three ordinary cusps in points on the exceptional lines. The cusps are the q-images of the contact points of i with Δ's side lines.

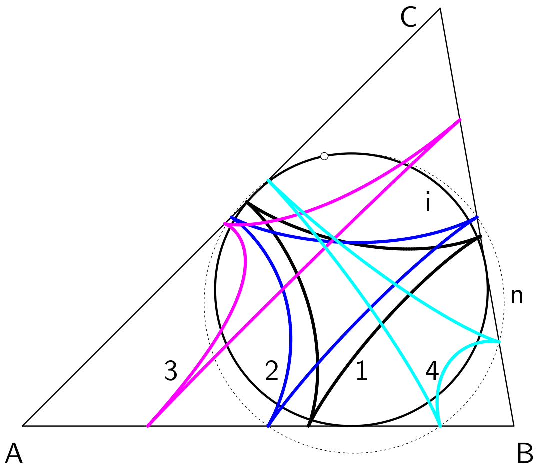

Figure 14: Four different q-images of the incircle i.

Figure 15 shows the image of the ninepoint circle n of Δ under the same four quadratic transformations. Here we observe ordinary nodes on the image curves corresponding to the two different transversal intersection of n with the exceptional lines of q.

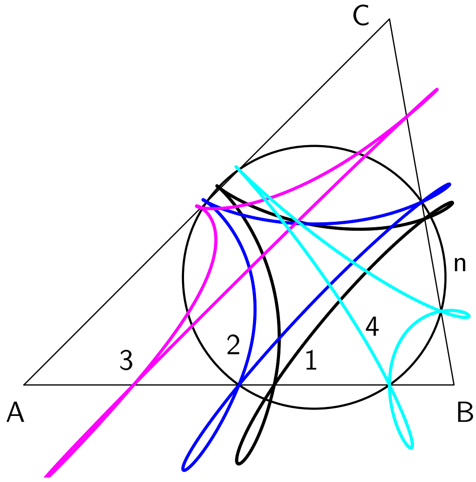

Figure 15: Four different q-images of the ninepoint circle n.

7. CONCLUSION

The study of mixed intersections of Cevians of triangle centers of a triangle Δ led directly to closed chains of Desarguesian $(10_3, 10_3)$ configurations. These chains consist of three such configurations and it is still an open question, if there are such chains involving more then three Desargues figures. However, these chains where byproducts.

The circumcircle u of Δ is only one prominent example of conic sections circumscribed to Δ. It is more or less an easy task to find the q-images with arbitrary center as a pivot that map u or other circumscribed conic sections to well-known central lines. We have skipped this lengthy and less fruitful enumeration and tried to give an example.

The quadratic mapping q appears with pivots X_1 and X_2. Any center can serve as a pivot of the transformation. Any conic section, whether it is circumscribed or inscribed to Δ, any central conic can be transformed via a certain q. All the image curves are rational. Therefore none of the well-known triangle cubics (see for example [1]) can be obtained as some q-image of a certain conic section, for all these are elliptic. If a line splits off from the q-image of a conic section, then there remains at most a rational cubic curve.

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ABOUT THE AUTHOR

Boris Odehnal, PD PhD, is currently associate professor at the TU Dresden. His research interests are line and sphere geometry, higher dimensional geometries, non-Euclidean geometries, projective geometries, algebraic curves and varieties, differential geometry of special curves and surfaces, and triangle geometry. He can be reached by e-mail: Boris.Odehnal@Mailbox.tu-dresden.de or boris@geometrie.tuwien.ac.at or under the postal address: TU Dresden, Institute of Geometry, 01062 Dresden, Germany.