On algebraic minimal surfaces

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Abstract

We give an overiew on various constructions of algebraic minimal surfaces in Euclidean three-space. Especially low degree examples shall be studied. For that purpose, we use the different representations given by WEIERSTRASS including the so-called Björling formula. An old result by LIE dealing with the evolutes of space curves can also be used to construct minimal surfaces with rational parametrizations. We describe a one-parameter family of rational minimal surfaces which touch orthogonal hyperbolic paraboloids along their curves of constant Gaussian curvature. Furthermore, we find a new class of algebraic and even rationally parametrizable minimal surfaces and call them cycloidal minimal surfaces.

Keywords: *minimal surface, algebraic surface, rational parametrization, polynomial parametrization, meromorphic function, isotropic curve, Weierstraß-representation, Björling formula, evolute of a spacecurve, curve of constant slope.*

MSC 2010: 53A10, 53A99, 53C42, 49Q05, 14J26, 14Mxx.

1 Introduction

Minimal surfaces have been studied from many different points of view. Boundary value problems, uniqueness results, stability, and topological problems related to minimal surfaces have been and are stil topics for investigations. There are only a few results on algebraic minimal surfaces. Most of them were published in the second half of the nine-teenth century, *i.e.*, more or less in the beginning of modern differential geometry. Only a few publications by LIE [30] and WEIERSTRASS [50] give general results on the generation and the properties of algebraic minimal surfaces. This may be due to the fact that computer algebra systems were not available and classical algebraic geometry gained less attention at that time. Many of the computations are hard work even nowadays and synthetic reasoning is somewhat uncertain. Besides some general work on minimal surfaces like [5, 8, 43, 44], there were some isolated results on algebraic minimal surfaces concerned with special tasks: minimal surfaces on certain scrolls [22, 35, 47, 49, 53], minimal surfaces related to congruences of lines [25, 28, 34, 38] minimal surfaces with a given geodesic [23], minimal surfaces of a certain degree, class, or genus (whether real or not) [1, 10, 11, 19, 20, 21, 31, 41, 42, 48], minimal surfaces touching surfaces along special curves [22], minimal surfaces showing special symmetries [14, 15, 16, 17], or minimal surface which allow isometries to special classes of surfaces [4, 6, 18, 52].

The famous algebraic minimal surface by ENNEPER which is of degree 9 and class 6 attracted intensive investigation. Consequently, researchers have found different generations of this surface: as the envelope of the planes of symmetry of all points on the pair of focal parabolas

$$p_1(u) = (\tfrac{4}{3}u, 0, \tfrac{2}{3}u^2 - \tfrac{1}{3}),$$
$$p_2(v) = (0, \tfrac{4}{3}v, \tfrac{1}{3} - \tfrac{2}{3}v^2)$$

or as the unique minimal surface (22) through the rational curve

$$\gamma(t) = \left(t - \frac{1}{3}t^3, t^2, 0\right)$$

having γ's normals for its surface normals. Since γ is planar, the surface normals of the uniquely defined minimal surface form a developable surface (to be precise, a plane), and thus, γ is a planar geodesic on ENNEPER's minimal surface. The plane of γ is a plane of symmetry of ENNEPER's surface. This is a manifestation of a more general result by HENNEBERG, see [21, 24, 30, 33]:

Theorem 1.1. *A minimal surface $\mathcal{M}$ carries a planar and not straight curve c as a geodesic. If $\mathcal{M}$ is algebraic, then the involutes of c have to be algebraic or c is the evolute of a planar algebraic curve.*

We shall make use of this fact later in Sec. 7 when we construct cycloidal minimal surfaces.

A further result due to HENNEBERG (see [21, 24, 30, 33]) is the following

Theorem 1.2. *Let a minimal surface $\mathcal{M}$ be tangent to a cylinder $\mathcal{Z}$. If $\mathcal{M}$ is algebraic, then the orthogonal cross-section c of $\mathcal{Z}$ is the evolute of an algebraic curve. If c is the evolute of a transcendental curve, then $\mathcal{M}$ is also transcendental.*

However, according to a theorem by RIBAUCOUR, ENNEPER's surface, like many other minimal surfaces, appears as the *central envelope* of isotropic congruences of lines, see [25, 28, 34, 38, 45].

Among the real algebraic minimal surfaces, ENNEPER's surface has lowest possible degree 9. But there are algebraic minimal surfaces that can be found in [12, 13, 21, 30] which are of degree 3 and 4 having the equations

$$\mathcal{G}: \ (x - \mathrm{i}y)^4 + 3(x^2 + y^2 + z^2) = 0$$

and

$$\mathcal{L}: \ 2(x-\mathrm{i}y)^3 - 6\mathrm{i}(x-\mathrm{i}y)z - 3(x+\mathrm{i}y) = 0$$

with respect to a properly chosen Cartesian coordinate system. The surfaces $\mathcal{G}$ and $\mathcal{L}$ have no real equation (polynomial equation with real coefficients exclusively) and do not carry a single real point.

$\mathcal{G}$ is usually called *Geiser's surface* and $\mathcal{L}$ is named after LIE. GEISER's minimal surface is a minimal surface of revolution with an isotropic axis. Obviously, it is of degree 4 and some computation tells us that the equation of its dual surface $\mathcal{G}^\star$, *i.e.*, the surface of its tangent planes has the equation

$$\mathcal{G}^\star: \ 9w_0^2(w_1 - \mathrm{i}w_2)^4 - (w_1^2 + w_2^2 + w_3^2)^3 = 0$$

which is, therefore, of degree 6, and thus, $\mathcal{G}$ is of class 6.

Whereas LIE's surface is of degree 3 and also of class 3 since the implicit equation of the dual surface $\mathcal{L}^\star$ reads

$$\mathcal{L}^\star\colon\ 27w_0(w_2+\mathrm{i}w_1)^2+9\mathrm{i}(w_1^2+w_2^2)w_3-4\mathrm{i}w_3^3=0.$$

GEISER's surface meets the ideal plane in the same ideal line as LIE's surface does. The ideal line $x-\mathrm{i}y=0$ is a 4-fold line on $\mathcal{G}$ and a 3-fold line on $\mathcal{L}$. It is remarkable that complex (non-real) algebraic minimal surfaces have been undergoing detailed investigations, see, *e.g.*, [1, 10, 12, 13, 48].

In [30], LIE gives a result dealing with the ideal curves of algebraic minimal surfaces:

Theorem 1.3. *The intersection of an algebraic minimal surface with the plane at infinity consists of finitely many lines.*

Some of the ideal lines on a minimal surface may have higher multiplicities and pairs of complex conjugate lines can also occcur.

For the coordinatization of ideal points and lines we refer to Sec. 2.

The results on degrees, ranks, and classes of real algebraic minimals surfaces differ from the results on complex algebraic minimal surfaces. For real algebraic minimal surfaces we have (see [30])

Theorem 1.4. *The sum of the degree and class of a* real *algebraic minimal surface is at least 15.*

The two aforementioned examples of complex minimal surfaces obviously show a different behaviour.

It is well-kown (cf. [30, 33]) that 5 is the lowest possible class of a real algebraic minimal surface. HENNEBERG's surface with the parametrization

$$f(u,v)=\begin{pmatrix}\mathrm{c}_{3u}\mathrm{S}_{3v}-3\mathrm{c}_u\mathrm{S}_v\\ \mathrm{s}_{3u}\mathrm{S}_{3v}+3\mathrm{s}_u\mathrm{S}_v\\ 3\mathrm{c}_{2u}\mathrm{C}_{2v}\end{pmatrix}\tag{1}$$

is an example for that, since the implicit equation of its dual surface equals

$$\begin{gathered}u_0(u_1^2+u_2^2)^2+\\ +u_3(u_1^2-u_2^2)(3u_1^2+3u_2^2+2u_3^2)=0.\end{gathered}\tag{2}$$

The algebraic degree of HENNEBERG's surface equals 15. ENNEPER's surface is the only known example of a minimal surface where the degree and class sum up to 15: the degree equals 9 (cf. (23)), the class equals 6 (cf. (24)).

LIE gives also results on the class of an algebraic minimal surface:

Theorem 1.5. *The class of an orientable algebraic minimal surface is alwas even.*

HENNEBERG's surface is of class 5 and non-orientable. The rational minimal Möbius strip given in [35] is of class 15.

In Sec. 2, we introduce coordinates and define all necessary abbreviations. Then, the different parametrization techniques for minimal surfaces are collected. Proofs for these can be found in most of the standard monographs on minimal surfaces or differential geometry such as [2, 33, 46]. Sec. 3 is dedicated to ENNEPER's surface and its natural generalizations. In Sec. 4, BOUR's minimal surfaces gain attention. We show different ways to find these minimal surfaces and give estimates on the algebraic degrees of these surfaces. Then, in Sec. 5, RICHMOND's surface appears as one in a one-parameter family. Sec. 6 gives additional and apparently new results on a well-known kind of minimal surface tangent to a hyperbolic paraboloid. Sec. 7 deals with an apparently new class of minimal surfaces. The fact that cycloids (cycloidal curves with cusps) have rational normals and are algebraic as well as their evolutes and involutes are (see [32, 51, 55, 56]), allows us to construct a family of algebraic minimal surfaces that admit even rational parametrizations. We debunk their relations to curves of constant slope on quadrics of revolution.

The reasons for the interest in algebraic and, especially in rational minimal surfaces are manifold: Rational parametrizations can be converted into a geometrically favorable representation, namely into the Bézier representation. Moreover, rational parametrizations can easily be handled with computer algebra systems. This allows the computation of implicit equations of surfaces and their duals and makes them accessible for further study which is then no longer restricted to the purely differential geometric approach. The behaviour at infinity as well as other algebraic properties can be studied.

We have to confess that implicit equations of algebraic minimal surfaces will hardly show up in this paper because they can be really long. The algebraic equation of a d-dimensional algebraic variety of degree D has at most

$$q = \frac{1}{(d+1)!}\prod_{k=1}^{d+1}(D+k)$$

coefficients. In the case of the classical low degree examples by Enneper, Richmond, Henneberg, and Bour with degrees 9, 12, 15, and 16 we could expect up to 220, 455, 816, and 969 terms provided that no special coordinate system is chosen and that the equations are expanded in full length.

2 Prerequisites

Since we are dealing with minimal surfaces in the Euclidean three-space, Cartesian coordinates (x, y, z) are sufficient. Vectors and matrices are written in bold characters. The canonical innerproduct of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. The Euclidean length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is then given by $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. The induced crossproduct of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is the vector $\mathbf{u} \times \mathbf{v} \in \mathbb{R}^3$.

In the following, we shall use the abbreviations

$$\mathrm{c}_x := \cos x, \quad \mathrm{s}_x := \sin x, \ldots$$
$$\mathrm{C}_x := \cosh x, \quad \mathrm{S}_x := \sinh x, \ldots$$

for the trigonometric and hyperbolic functions whenever there is not enough space for the equations.

Sometimes, we deal with ideal points, lines, and the ideal plane. Then, we shall homogenize the underlying Cartesian coordinates by

$$x \to X_1 X_0^{-1},\ y \to X_2 X_0^{-1},\ z \to X_3 X_0^{-1}.$$

When we compute the intersection of a (minimal) surface with the ideal plane (plane at infinity), then we let $X_0 = 0$ and obtain the equation of a curve (or, more generally speaking, a *cycle* which is the union of finitely many algebraic curves) in terms of the homogeneous coordinates $(X_1 : X_2 : X_3)$ in the ideal plane. However, we shall not write this down in detail and define coordinates in the ideal plane by simply setting $X_1 = x$, $X_2 = y$, and $X_3 = z$. It is sufficient to do so, because substituting $X_0 = 0$ into the homogeneous equation returns all monomials of the highest degree of the inhomogeneous equation.

In the following, we collect some results and representations of minimal surfaces that will be useful for the generation of algebraic minimal surfaces. These representations are well-known and proofs can be found in the literature, see, *e.g.*, [2, 27, 30, 33, 36, 46].

2.1 Björling's problem

Let $\gamma: I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve and let $\nu: I \to \mathrm{S}^2$ be a smooth unit vector field along γ with $\langle \gamma', \nu \rangle \equiv 0$, *i.e.*, ν is perpendicular to γ in the entire interval I. Both are considered to have complex continuations. A real parametrization $\mathbf{f}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ of the uniquely defined real minimal surface $\mathcal{M}$ through γ with its normals along γ parallel to ν is then given by

$$\mathbf{f}(u,v) = \mathfrak{Re}\left(\gamma(t) - \mathrm{i}\int_{t_0}^{t} \nu(\theta) \times \mathrm{d}\gamma(\theta)\right). \qquad (3)$$

We call the pair (γ, ν) a *scroll* and it is the envelope of the one-parameter family of planes $\langle \nu(t), \mathbf{x} - \gamma(t) \rangle = 0$. The curve γ shall henceforth be called the *spine curve* of the scroll.

Since γ and ν are considered to have complex continuations, the parameter t in (3) is assumed to be a complex parameter. Subsequent to the integration, t is replaced by $t = u + \mathrm{i}v$ and finally the real part of the vector function in $\mathbb{C}^3$ is extracted. Formula (3) is called *Björling formula*, see [2, 27, 33, 36], and was first published by H.A. Schwarz in [44]. Actually, the Björling formula is just the solution of a problem posed by E.G. Björling in 1844.

The Björling formula can be a starting point for the construction of algebraic minimal surfaces, but it has a big disadvantage like all other integral formulae: Antiderivatives of rational or algebraic functions may sometimes be not rational or even algebraic.

A remarkable application of the Björling formula (3) may be its application to non planar curves. The following result is due to Lie, see [30]:

Theorem 2.1. *The minimal surface that touches the evolute $c^\star$ of an algebraic space curve c exactly at the centers of curvature of c is algebraic.*

However, the algebraic degree of the surface generated according Thm. 2.1 may not only be high, it may even be hard to determine.

As an application of Thm. 2.1, we can give the following low degree example: We choose the PH-curve (for details and definition see [9])

$$\mathbf{c}(t) = (6t, 6t^2, 4t^3), \quad t \in \mathbb{R}. \qquad (4)$$

Its evolute is then parametrized by

$$\mathbf{c}^\star(t) = \begin{pmatrix} -12t^3 \\ 3 - 12t^4 + 6t^2 \\ 16t^3 + 6t \end{pmatrix}, \quad t \in \mathbb{R} \qquad (5)$$

and the normals $\nu(t)$ are $\lambda \mathbf{c}_1 = (1, 2t, 2t^2)$ with $\lambda = 1 + 2t^2$. The requirements for the application of the Björling formula are met since $\langle \dot{\mathbf{c}^\star}, \mathbf{c}_1\rangle = 0$. A real parametrization of the real minimal surface on the scroll $(\gamma, \nu) = (\mathbf{c}^\star, \mathbf{c}_1)$ is found with (3) and reads

$$\begin{aligned}\mathbf{f}(u,v) = 12\begin{pmatrix} 4uv(u^2-v^2) \\ 6u^2v^2-u^4-v^4 \\ 0 \end{pmatrix} + \\ +12\begin{pmatrix} 3uv^2-u^3 \\ v^3-3u^2v \\ \frac{4}{3}u^3-4uv^2 \end{pmatrix} + 6\begin{pmatrix} 2uv \\ u^2-v^2-v+\frac{1}{2} \\ u(2v+1) \end{pmatrix}.\end{aligned} \tag{6}$$

Figure 2.1 shows the minimal surface parametrized by (6) together with the curves $\mathbf{c}$ and $\mathbf{c}^\star$. Implicitization shows that

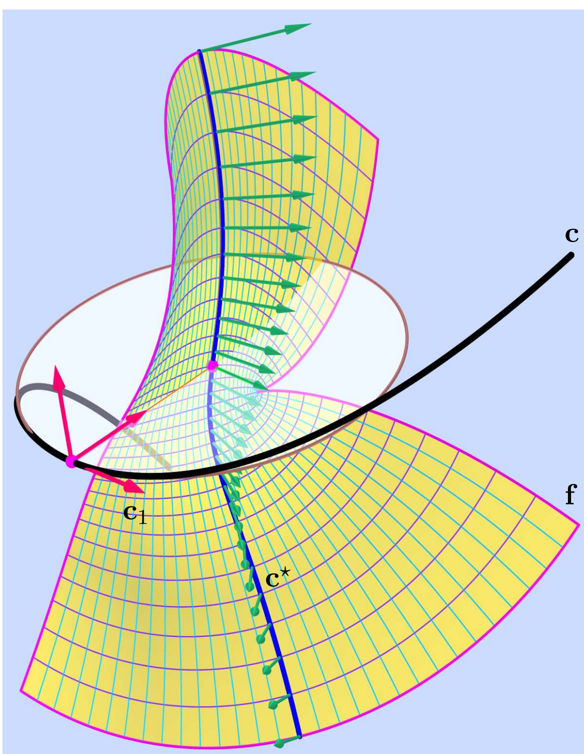

Figure 2.1: The minimal surface on the scroll $(\mathbf{c}^\star, \mathbf{c}_1)$ is derived from the evolute $\mathbf{c}^\star$ of a cubic PH-curve $\mathbf{c}$.

the surface (6) is of degree 16 and the intersection with the ideal plane consists of the ideal line of all planes parallel to $x = 0$ with multiplicity 16. Surprisingly, the class of this minimal surface equals 8 as we can see from the implicit equation of the dual surface:

$$\begin{gathered} 3\omega^2\Omega^2 + (4w_0w_2 - 15w_1^2)\omega\Omega^2 - 2\Omega\omega^3 - \omega^4 \\ +4w_1^2(3w_1^2 - 4w_0 2w_2)\Omega^2 + 4w_1^5(2w_1 + 9w_3)\Omega \\ +w_1(4w_0w_2(2w_1+3w_3) - 9w_1^2(5w_1 - 6w_3))\Omega\omega \\ +2w_1(w_0w_2(w_1+3w_3) - 6w_1^2(w_1+w_3))\omega^2 \\ +(39w_1^2 + 18w_1w_3 - 2w_0w_2)\Omega\omega^2 \\ +(12w_1^2 + 6w_1w_3 - w_0^2 - 2w_0w_2)\omega^3 \\ +w_1^5(w_1 + 6w_3)\omega = 0 \end{gathered} \tag{7}$$

where $\omega := w_1^2 + w_2^2$ and $\Omega := w_1^2 + w_2^2 + w_3^2$.

We can summarize this in

Corollary 2.1. *The minimal surface on the scroll* $(\mathbf{c}^\star, \mathbf{c}_1)$ *with* $\mathbf{c}^\star$ *given in* (5) *(evolute of the polynomial cubic PH-curve* $\mathbf{c}$ *from* (4)*) and with* $\mathbf{c}_1$ *being* $\mathbf{c}$*'s unit tangent vector field is a rational minimal surface of degree 16 and class 8.*

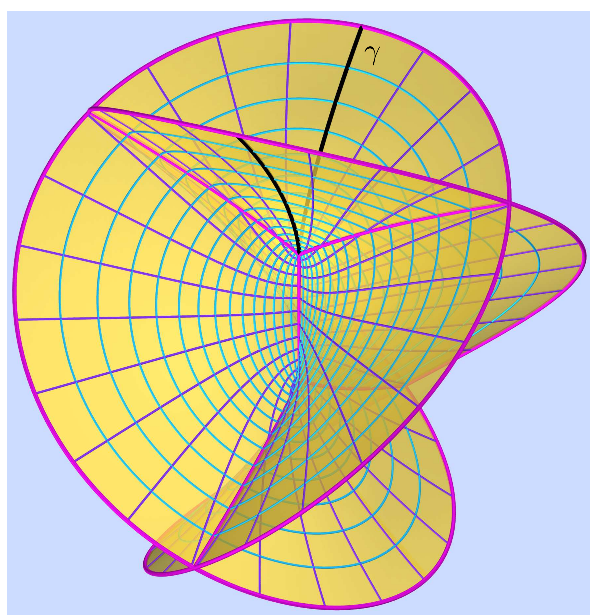

Figure 2.2: HENNEBERG's minimal surface with the geodesic semi-cubic parabola γ.

The cubic curve (4) as well as its evolute (5) are non planar curves. In contrast to that, we can choose the planar PH-cubic (semi-cubi parabola)

$$\gamma(t) = (4t^3, 0, 6t^2 + 3) \tag{8}$$

that lies in the xz-plane. Together with its unit normals

$$\nu(t) = \frac{1}{\sqrt{1+t^2}}(-1, 0, t) \tag{9}$$

a scroll (γ, ν) is defined and (3) yields the isotropic curve

$$\varphi(t) = (4t^3, -4\mathrm{i}\sqrt{(t^2+1)^3}, 6t^2+3). \tag{10}$$

which is subsequently reparametrized by $t = \mathrm{S}_\tau$. Then, $\tau = v + \mathrm{i}u$ (note that the real part equals v). Finally, the extraction of the real part of (10) gives (1). Since the normals ν from (9) along γ from (8) form a developable ruled surface (a plane), γ turns out to be a planar geodesic on HENNEBERG's minimal surface (1). The plane of γ is a plane of symmetry for HENNEBERG's minimal surface, cf. Thm. 1.1. Figure 2.2 shows a part of HENNEBERG's minimal surface with the geodesic semi-cubic parabola (8).

A rational parametrization of HENNEBERG's surface can be obtained in two ways. The usual replacement of trigonometric and hyperbolic functions by their well-known rational equivalents delivers a parametrization involving polynomials of degrees higher than necessary. The substitution $\mathrm{S}_v = V$ yields a parametrization of bi-degree $(6, 3)$, since $\mathrm{C}_{2v} = 1 + 2\mathrm{S}_v{}^2 = 1 + 2V^2$ and $\mathrm{S}_{3v} = 3\mathrm{S}_v + 4\mathrm{S}_v{}^3 = 3V + 4V^3$.

Implicitization shows that HENNEBERG's surface is of algebraic degree 15.

The dual surface, *i.e.*, the set of tangent planes of HENNEBERG's surface, can be given either in parametric form by

$$\mathbf{f}^\star = \begin{pmatrix} \dfrac{2\mathrm{c}_u}{\mathrm{c}_{2u}\mathrm{S}_{3v} + 3\mathrm{c}_{2u}\mathrm{S}_v} \\ \dfrac{2\mathrm{s}_u}{\mathrm{c}_{2u}\mathrm{S}_{3v} + 3\mathrm{c}_{2u}\mathrm{S}_v} \\ \dfrac{-1}{\mathrm{C}_{2v}\mathrm{c}_{2u} + 2\mathrm{c}_{2u}} \end{pmatrix} \tag{11}$$

or by the implicit equation (2).

2.2 WEIERSTRASS's formulae

2.2.1 The integral formula

There are some equivalent formulae which where first given by WEIERSTRASS. These allow us to compute parametrizations of minimal surfaces by prescribing a pair of meromorphic functions: Let $A: D \subset \mathbb{C} \to \mathbb{C}$ and $B: D \subset \mathbb{C} \to \mathbb{C}$ be meromorphic functions, *i.e.*, they are holomorphic except at countably many points $p_i \in D \subset \mathbb{C}$.

From A and B we find a real parametrization of a real minimal surface via

$$\mathbf{f}(u, v) = \mathfrak{Re} \int \begin{pmatrix} A(1 - B^2) \\ \mathrm{i}A(1 + B^2) \\ 2AB \end{pmatrix} \mathrm{d}w. \tag{12}$$

Again, we assume that $w = u + \mathrm{i}v$ is the complex parameter in the domain D. The extraction of the real part of the complex vector valued function gives the real parametrization of the real minimal surface defined by A and B.

There is an alternative, but equivalent form for (12). Let G and H be two meromorphic functions defined over the same domain $D \subset \mathbb{C}$, then

$$\mathbf{f}(u,v) = \mathfrak{Re} \int \begin{pmatrix} G^2 - H^2 \\ \mathrm{i}(G^2 + H^2) \\ 2GH \end{pmatrix} \mathrm{d}w \qquad (13)$$

also yields a real parametrization of a real minimal surface. (13) transforms into (12) by letting $A = G^2$ and $B = HG^{-1}$ provided that $G \not\equiv 0$.

In many textbooks on differential geometry and in a huge amount of publications, a further but equivalent integral representation of minimal surfaces can be found. However, this third version is obtained from (12) by substituting $B(w) = w$ and $A(w)$ is an arbitray meromorphic function. This seems to be a restriction that presumes that $A(w)$ can globally and in a closed form be written as a function $A(B(w))$ depending on $B(w)$.

2.2.2 Recovering the functions A, B

From the parametrization $\mathbf{f}$ of a minimal surface we can recover the meromorphic functions A and B, see [27, 33, 36]: First, we compute $\mathbf{F} := \partial_u \mathbf{f} - \mathrm{i}\partial_v \mathbf{f}$. Then, we use the coordinate functions $\mathbf{F}^i$ of $\mathbf{F}$ and find

$$A = \frac{1}{2}(\mathbf{F}^1 - \mathrm{i}\mathbf{F}^2) \quad \text{and} \quad \mathrm{B} = \frac{\mathbf{F}^3}{2\mathrm{A}}. \qquad (14)$$

For example, the generating meromorphic functions of the minimal surface given by (6) are

$$A = 3 - 12\mathrm{i}w \quad \text{and} \quad B = 1 + 2\mathrm{i}w.$$

2.2.3 Integral free representation of minimal surfaces

Let $A(w) : D \subset \mathbb{C} \to \mathbb{C}$ be a meromorphic function and let further $A' = \frac{\mathrm{d}A}{\mathrm{d}w}$, $A'' = \frac{\mathrm{d}^2A}{\mathrm{d}w^2}$, and $A''' = \frac{\mathrm{d}^3A}{\mathrm{d}w^3}$ denote its first, second, and third complex derivative. The vector

$$\mathbf{i} = \begin{pmatrix} 1 - w^2 \\ \mathrm{i}(1 + w^2) \\ 2w \end{pmatrix} \qquad (15)$$

is an *isotropic vector* in three-dimensional Euclidean space $\mathbb{R}^3$ since $\langle \mathbf{i}, \mathbf{i} \rangle = 0$. Again, primes $'$ indicate differentiation with respect to the complex variable w. Now, we define

$$\mathbf{j} = A''\mathbf{i} - A'\mathbf{i}' + A\mathbf{i}''. \qquad (16)$$

It is elementary to verify that $\langle \mathbf{j}', \mathbf{j}' \rangle = 0$, and thus, $\mathbf{j}'$ is isotropic. Therefore, $\mathbf{f} = \mathfrak{Re}\,\mathbf{j}$ is a real parametrization of a real minimal surface. This parametrization is usually written as

$$\mathbf{f}(u,v) = \mathfrak{Re} \begin{pmatrix} (1-w^2)A'' + 2wA' - 2A \\ \mathrm{i}(1+w^2)A'' - 2\mathrm{i}wA' + 2\mathrm{i}A \\ 2wA'' - 2A' \end{pmatrix} \qquad (17)$$

where $A''' \not\equiv 0$ in D, see [2, 27, 33]. In case of a quadratic polynomial A, (17) parametrizes a line. A cubic polynomial A delivers an Enneper surface.

The integral free parametrization of minimal surfaces allows us to state:

Theorem 2.2. *Each algebraic function A: $D \subset \mathbb{C} \to \mathbb{C}$ with $A''' \not\equiv 0$ (in the entire domain D) yields an algebraic minimal surface parametrized by* (17).

Moreover, it is clear that polynomials $A \in \mathbb{C}[w]$ deliver polynomial parametrization. Further, each rational function $A = P/Q$ with $P, Q \in \mathbb{C}[w]$ and $\gcd(P,Q) = 1$ yields rational parametrization of minimal surfaces. However, just inserting rational or algebraic functions cannot guarantee that the algebraic degree of the resulting minimal surface is low. Sometimes a reparametrization turns a rational parametrization of a minimal surface into a polynomial one.

2.2.4 The associate family

In any of the above cases, the real parametrization $\mathbf{f}$ of a real minimal surface was found by computing the real part $\mathbf{f} = \mathfrak{Re}\,\varphi(w)$ of some complex vector valued function $\varphi(w)$. The vector valued function $\varphi(w)$ parametrizes an isotropic curve in Euclidean three-space, *i.e.*, a curve with constant slope $\pm\mathrm{i}$. The computation of the real part is equivalent to the addition of the complex conjugate vector function and subsequent multiplication by $\frac{1}{2}$, *i.e.*, $\mathbf{f} = \frac{1}{2}(\varphi + \overline{\varphi}) = \mathfrak{Re}\,\varphi$. This is just the analytical formulation of a fundamental result by Lie (see [27, 30, 33, 36]):

Theorem 2.3. *Translating an isotropic curve φ (curve of constant slope $\pm\mathrm{i}$) along another isotropic curve ψ sweeps a minimal surface. The minimal surface is real if, and only if, φ and ψ are complex conjugate curves.*

The curve $\varphi(w)$ is an isotropic (minimal) curve of Euclidean geometry. This property is not altered if we multiply $\varphi(w)$ by $\mathrm{e}^{\mathrm{i}\tau}$ prior to the extraction of the real part. The latter multiplication by a complex factor of absolute value 1 is, geometrically speaking, just a rotation of the complex curve. The family of real minimal surfaces given by

$$\begin{aligned}\mathbf{f}(\tau) &= \mathfrak{Re}(\mathrm{e}^{\mathrm{i}\tau}\varphi(w)) = \\ &= \mathrm{c}_\tau \mathfrak{Re}\,\varphi(w) + \mathrm{s}_\tau \mathfrak{Im}\,\varphi(w)\end{aligned} \tag{18}$$

is called the *associate family.* Especially, $\mathbf{f}^\perp := \mathbf{f}(\frac{\pi}{2})$ is called the *adjoint minimal surface* to $\mathbf{f}$. The following theorem is obvious:

Theorem 2.4. *The family of minimal surfaces associate to an algebraic minimal surface consists only of algebraic minimal surfaces.*

Proof. From (18) we can see that the parametrizations of the minimal surfaces in the associate family are linear combinations of $\mathfrak{Re}\,\varphi(w)$ and $\mathfrak{Im}\,\varphi(w)$ with coefficients c_τ and s_τ. If $\mathbf{f}$ is obtained via (17), then both $\mathbf{f}(0) = \mathfrak{Re}\,\varphi(w)$ and $\mathbf{f}(\frac{\pi}{2}) = \mathfrak{Im}\,\varphi(w)$ are algebraic and so is any of their linear combinations. □

It is elementary to verify that the meromorphic function A from (12) changes to $\mathrm{e}^{\mathrm{i}\tau}A$ and B does not change during the transition from the minimal surface defined by A and B to the members of its associate family.

We shall have a look at the minimal surface adjoint to Henneberg's surface (1). A parametrization $\mathbf{f}^\perp$ of this adjoint surface is found by multiplying (10) by $\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}$, reparametrizing by $t = \mathrm{S}_\tau$. Then, $\tau = v + \mathrm{i}u$ and we extract the real part which gives

$$\mathbf{f}^\perp(u,v) = \begin{pmatrix} \mathrm{c}_{3u}\mathrm{S}_{3v} - 3\mathrm{S}_v\mathrm{c}_u \\ \mathrm{s}_{3u}\mathrm{S}_{3v} + 3\mathrm{S}_v\mathrm{s}_u \\ 3\mathrm{C}_{2v}\mathrm{c}_{2u} \end{pmatrix}. \tag{19}$$

The surface (19) has more symmetries than Henneberg's surface: It is symetric with respect to the planes

$$x = 0, y = 0, x \pm y = 0, z = 0.$$

The algebraic minimal surface $\mathbf{f}^\perp$ (19) is of degree 26 and a part of it is shown in Fig. 2.3. Its intersection with the ideal plane is the 18-fold ideal line of all planes parallel to $z = 0$ together with the four-fold pair of ideal lines of complex conjugate isotropic planes.

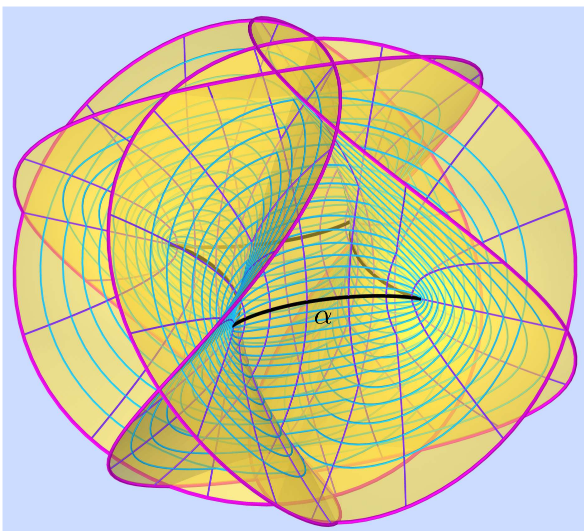

Figure 2.3: The minimal surface adjoint to HENNEBERG's surface is uniquely determined by the geodesic astroid α and its normals.

The surface $\mathbf{f}^\perp$ intersects the plane $z = 0$ along the astroid

$$\alpha(t) = (4\mathrm{c}_t{}^3, 4\mathrm{s}_t{}^3, 0) \tag{20}$$

that turns out to be a geodesic on the surface $\mathbf{f}^\perp$.

On the other hand, α can be taken as the spine curve of the scroll (α, ν) with its unit normals

$$\nu(t) = (-\mathrm{s}_t, -\mathrm{c}_t, 0). \tag{21}$$

Inserting (20) and (21) into (3), we obtain a parametrization of $\mathbf{f}^\perp$ that is slightly different from (19) but equivalent to that. Summarizing this, we can state (a known result, see [22, 23, 33]) in

Theorem 2.5. *The adjoint minimal surface to* HENNEBERG*'s minimal surface carries a geodesic astroid α. The adjoint to* HENNEBERG*'s minimal surface is the uniquely determined minimal surface on the scroll (α, ν) with ν being α's unit normal vector field.*

It is noteworthy that the astroid α (20) is a hypocycloid. This will be of importance in Sec. 7.

3 ENNEPER's surfaces

There is not just one Enneper surface even if we don't mention equiform copies of the standard form. The well-known example

$$\mathcal{E}_1(u, v) = \begin{pmatrix} -\frac{1}{3}u^3 + uv^2 + u \\ \frac{1}{3}v^3 - u^2 v - v \\ u^2 - v^2 \end{pmatrix} \tag{22}$$

with its bi-cubic parametrization is one in a one-parameter family of algebraic minimal surfaces that admit even polynomial parametrizations. It can be found with (13) by letting $G = 1$ and $H = w$ or with (17) where $A = \frac{1}{6}z^3$.

The algebraic degree of the classical Enneper surface equals nine since an implicit equation can be given by

$$\begin{aligned} &[9(y^2-x^2)+4z(z^2+3)]^3- \\ &\quad -27z[9(y^2-x^2)- \\ &\quad -z(9(x^2+y^2)+8z^2)+8z]^2=0. \end{aligned} \tag{23}$$

The class of ENNEPER's surface equals six as can be read off from the implicit equation of its family of tangent planes

$$\begin{gathered} w_0^2(w_1^2+w_2^2)^2-3(w_1^2-w_2^2)^2w_3^2- \\ -4(w_1^2-w_2^2)^2(w_1^2+w_2^2)+ \\ +2w_0w_3(w_1^2-w_2^2)\cdot \\ \cdot(3w_1^2+3w_2^2+2w_3^2)=0. \end{gathered} \tag{24}$$

ENNEPER's minimal surface is an example of a non-orientable minimal surface with *even* class, cf. Thm. 1.5.

The term of degree nine in (23) equals z^9 which shows that the ideal line of all planes parallel to $z=0$ comprises the set of ideal points of ENNEPER's surface.

According to LIE [30], the sum of the class and the degree of an algebraic minimal surface is at least 15, and thus, ENNEPER's surface is the confirming example. Its real self-intersection consists of the pair

$$\begin{aligned} s_1 &= (0, \tfrac{3}{8}t(3t^2+8), \tfrac{9}{8}t^2+3), \\ s_2 &= (-\tfrac{3}{8}t(3t^2+8), 0, -\tfrac{9}{8}t^2-3) \end{aligned}$$

of polynomial cubic curves (semi-cubic parabolas) in the symmetry planes $x=0$ and $y=0$.

The more general version of ENNEPER's surface is given by

$$\mathcal{E}_n(u,v)=\mathfrak{Re}\begin{pmatrix} w-\dfrac{w^{2n+1}}{2n+1} \\ \mathrm{i}w+\dfrac{\mathrm{i}w^{2n+1}}{2n+1} \\ \dfrac{2w^{n+1}}{n+1} \end{pmatrix} \tag{25}$$

where $n \in \mathbb{N}\setminus\{0\}$ is usually called the *order of the Enneper surface*. These minimal surfaces are obtained from (13) with

$$G(w)=1 \quad \text{and} \quad \mathrm{H(w)=w^n}.$$

With $n=1$ we obtain the *classical* minimal surface by ENNEPER parametrized by (22) first given in [8].

Dropping the restriction $n \in \mathbb{N}\setminus\{0\}$, we obtain the plane $x=0$, *i.e.*, a flat minimal surface if $n=0$. The case $n=-1$ is still to be excluded if one is interested in algebraic minimal surfaces. However, the case $n=-1$ yields the catenoid

$$2\mathrm{C}_{\frac{z}{2}} = \sqrt{x^2+y^2}.$$

Surprisingly, the case $n=-2$ yields RICHMOND's surface (31), which will be discussed in Sec. 5. The surface $\mathcal{E}_3(u,v)$ is displayed in Fig. 3.4.

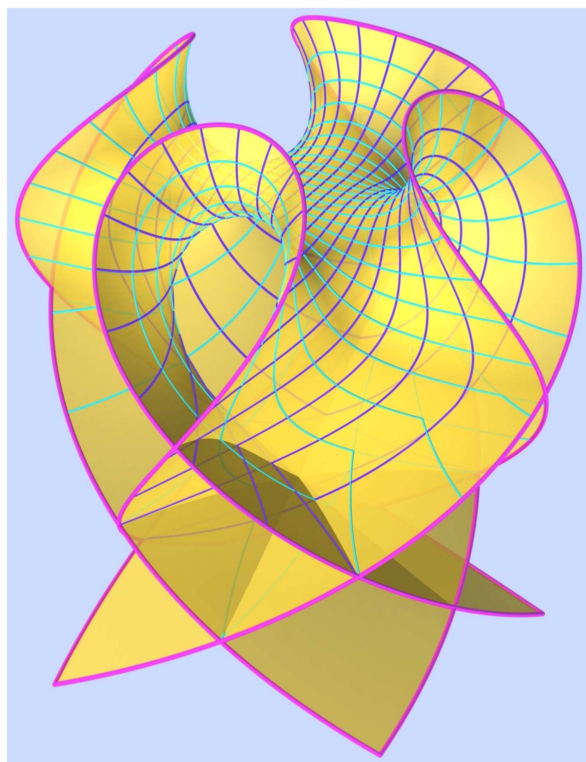

Figure 3.4: The Enneper surface of order 3 is of algebraic degree 49, cf. Thm 3.1.

We can give an upper bound on the algebraic degree and a precise value for the class of the Enneper surfaces of an arbitrary order:

Theorem 3.1. *Enneper surfaces of order $n \in \mathbb{N} \setminus \{0\}$ are algebraic minimal surfaces whose degree is at most $(2n+1)^2$ and the class equals $2n(2n+1)$.*

Proof. The polynomial parametrization of an Enneper surface (25) of order n is of bidegree $(2n+1, 2n+1)$. Elimination of u and v form the coordinate functions means computing resultants with respect to u and v. Thus, the algebraic degree of $\mathcal{E}_n$ is at most $(2n+1)^2$.

In order to show that the class equals $2n(2n+1)$, we use a result by Lie (cf. [30, vol. 1, p. 315]): The rank of the isotropic curve (25) equals $r = 3n+1$ and the multiplicity of the absolute conic as a curve on the tangent developable this particular isotropic curve equals $\mu = n$. According to Lie, the class of the minimal surface generated by the isotropic curve (25) equals $2\mu(r-\mu) = 2n(3n+1-n) = 2n(2n+1)$. □

The computation of the implicit equations of the surfaces $\mathcal{E}_n$ up to $n = 7$ shows that the bound $\deg \mathcal{E}_n = (2n+1)^2$ is sharp at least in these cases.

4 Bour's surfaces

The minimal surfaces by E. Bour (see [4]) are characterized by allowing local isometries to surfaces of revolution. Parametrizations of the surfaces in this one-parameter family are obtained from (12) by inserting

$$A(w) = cw^{m-2}, \; c \in \mathbb{C} \setminus \{0\}, m \in \mathbb{R} \setminus \{0\} \quad (26)$$

and $B(w) = w$. Alternatively, we can use

$$G = \sqrt{c} w^{\frac{m}{2}-1} \quad \text{and} \quad H = \sqrt{c} w^{\frac{m}{2}}$$

together with (13). With (26) and (12) we arrive at the parametrization

$$\mathcal{B}_m(u,v) = \mathfrak{Re}\, c \cdot \begin{pmatrix} \frac{1}{m-1} w^{m-1} - \frac{1}{m+1} w^{m+1} \\ \frac{\mathrm{i}}{m-1} w^{m-1} + \frac{\mathrm{i}}{m+1} w^{m+1} \\ \frac{2}{m} w^m \end{pmatrix}. \quad (27)$$

We call m the *order of the Bour surface* $\mathcal{B}_m$.

It means no restriction to assume $|c| = 1$, *i.e.*, $c = \mathrm{c}_\tau + \mathrm{i}\mathrm{s}_\tau$ since the multiplication of A by c causes only a scaling of the respective minimal surface with the scaling factor $|c|$. On the other hand, the multiplication with any complex number $c = \mathrm{c}_\tau + \mathrm{i}\mathrm{s}_\tau$ (with $\tau \in S^1$) corresponding to a point on the Euclidean unit circle chooses one certain member of the family of minimal surfaces associate to $\mathbf{f}$.

Well-known and non-algebraic minimal surfaces can be found among the surfaces by Bour: $m = 0$, $c = 1$ lead to the catenoid; the choice $m = 0$, $c = \mathrm{i}$ results in the helicoid

$$2 \arctan \frac{x}{y} = z$$

which is adjoint to the catenoid. If $m = \pm 1$ the resulting minimal surfaces are not algebraic independent of c, but they seem to be worth a closer inspection. A part of this non-algebraic minimal surface is displayed in Fig. 4.5.

Bour's minimal surfaces are algebraic if, and only if, $m \in \mathbb{Q} \setminus \{-1, 0, -1\}$. The following result makes clear that negative m can be excluded from our considerations:

Lemma 4.1. *For any $m \in \mathbb{Z} \setminus \{-1, 0, 1\}$ we have $\mathcal{B}(m) = \mathbf{S} \cdot \psi(\mathcal{B}(-m))$ where $\mathbf{S} = \mathrm{diag}(1, -1, -1)$ is the matrix describing the reflection in the x-axis and ψ is the reparametrization*

$$u = \frac{U}{U^2 + V^2}, \; v = -\frac{V}{U^2 + V^2}, \quad (28)$$

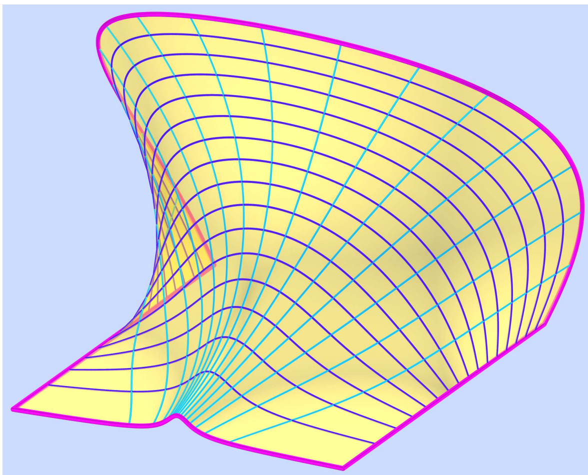

Figure 4.5: The non-algebraic minimal surface $\mathcal{B}_{-1}$.

or equivalently, $\psi : w = u + \mathrm{i}v \mapsto W^{-1}$ *(with* $W = U + \mathrm{i}V$*) which is the inversion in the Euclidean unit circle in the parameter plane.*

Proof. Let $m < -1$. We observe the changes

$$w^{m-1} \to w^{-n-1}, \quad w^{m+1} \to w^{-n+1}$$

with $m = -n$. Then, we reparametrize with ψ according to (28) and the latter powers of w change again:

$$w^{-n-1} \to (W^{-1})^{-n-1} = W^{n+1},$$
$$w^{-n+1} \to (W^{-1})^{-n+1} = W^{n-1},$$

both with positive n. Thus, the second and third coordinate function change their sign and $\mathbf{S} = \mathrm{diag}(1, -1, -1)$. Finally, changing $U \to u$ and $V \to v$ simplifies the comparison of the parametrizations. □

Especially, the surfaces for $m = 2, 3, 4, 5$ are of relatively low degree. Enneper's minimal surface corresponds to $m = \pm 2$ with arbitrary c.

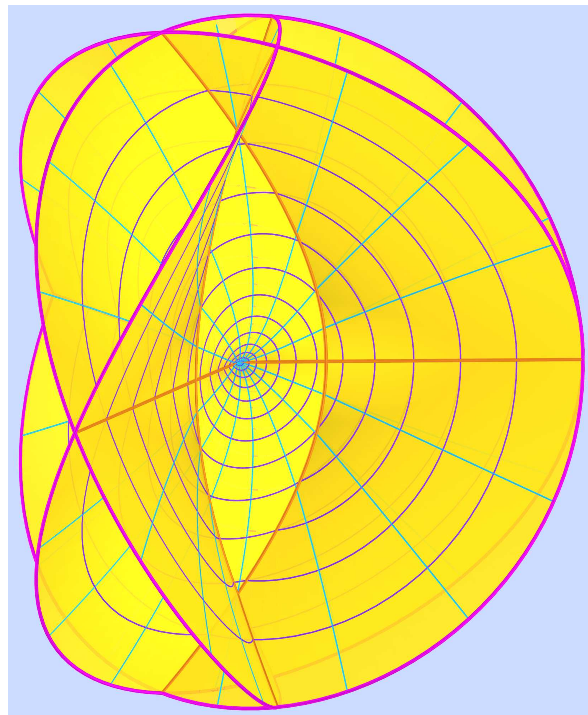

Figure 4.6: Bour's surface of order 3 is a Bézier minimal surface of algebraic degree 16 and of class 8.

With $m = 3$ we find a minimal surface of degree 16 and class 8 which is displayed in Fig. 4.6. The surface has three planes of symmetry: $y = 0$ and $3x^2 = y^2$ whose intersections with the plane $z = 0$ are three straight lines concurrent in the point $(0, 0, 0)$ which lie entirely in the surface. All three lines turn out to be four-fold lines on the surface. The Bour minimal surface of order 3 meets the ideal plane in the ideal line of all planes parallel to $z = 0$ with multiplicity 16. The planar rational (polynomial) quartic PH-curve (cf. [9])

$$\gamma(t) = \left(-\frac{1}{4}t^4 + \frac{1}{2}t^2, 0, \frac{2}{3}t^3\right)$$

together with its normal vectors can be used to construct a parametrization of this minimal surface with the Björling formula (3). Therefore, γ is a geodesic on the surface.

If now $m = \pm 4$, we obtain an algebraic minimal surface of degree 25 and class 10. The four lines

$$(x^2 - 2xy - y^2)(x^2 + 2xy - y^2) = 0$$

are five-fold lines on this minimal surface. With the Björling formula (3) the two planar and congruent PH-curves

$$\gamma_1 = \left(0, -\frac{1}{5}t^5 + \frac{1}{3}t^3, \frac{1}{2}t^4\right),$$
$$\gamma_2 = \left(-\frac{1}{5}t^5 + \frac{1}{3}t^3, 0, \frac{1}{2}t^4\right)$$

in the planes $x = 0$ and $y = 0$ together with their rational normals also define the Bour minimal surface of order 4. Both curves, γ_1 and γ_2 are planar geodesics on the Bour surface of order 4 and the plane $z = 0$ is a plane of symmetry. Again, the intersection with the ideal plane is a line whose multiplicity equals the algebraic degree of the surface.

The above given examples show that BOUR's minimal surfaces can also be obtained as minimal surfaces on PH-scrolls as a solution to Björling's problem. In a more general version, we have

Theorem 4.1. *The minimal surfaces on the scroll* (γ, ν) *with*

$$\gamma(t) = \left(\tfrac{-1}{m+1}t^{m+1} + \tfrac{1}{m-1}t^{m-1}, \tfrac{2}{m}t^m, 0\right) \qquad (29)$$

where $m \geq 2$ *and*

$$\nu(t) = \frac{1}{1+t^2}(-2t, 1-t^2, 0) \qquad (30)$$

are BOUR*'s minimal surfaces of order* m *up to equiform transformations.*

Proof. We insert (29) and (30) into (3) and arrive at (27). Note that ν from (30) satisfies $\nu = w^{m-2}\gamma'^{\perp}$. □

We can give an upper bound on the algebraic degree and class of BOUR's minimal surfaces of order m in

Theorem 4.2. *The algebraic degree and the class of* BOUR*'s minimal surface of order* m *are equal to* $(m+1)^2$ *and* $2(m+1)$ *provided that* $m \geq 2$.

Proof. We use the same arguments as in the proof of Thm. 3.1. □

Like the generalized Enneper surfaces (25), the Bour surfaces (27) are Bézier minimal surfaces (as long as they are algebraic).

5 RICHMOND's surfaces

The original Richmond surface (as shown in Fig. 5.7) comes a long as one special example in a one-parameter family of minimal surfaces. It has the simple parametrization

$$\mathbf{f}(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + \frac{u}{u^2+v^2} \\ \frac{1}{3}v^3 - u^2v - \frac{v}{u^2+v^2} \\ 2u \end{pmatrix}. \qquad (31)$$

RICHMOND's surface is the only real algebraic minimal surface of degree 12 up to equiform transformations, see [33]. The class of RICHMOND's surface equals 12, not 17 as RICHMOND stated in [39] (This was corrected in [40].) The minimal surfaces associated to RICHMOND's surface (31) are just similar copies of that surface, see [39].

When using (12) in order to parametrize the surface, we have to insert

$$A(w) = \frac{1}{w^2}, \quad B(z) = w^2.$$

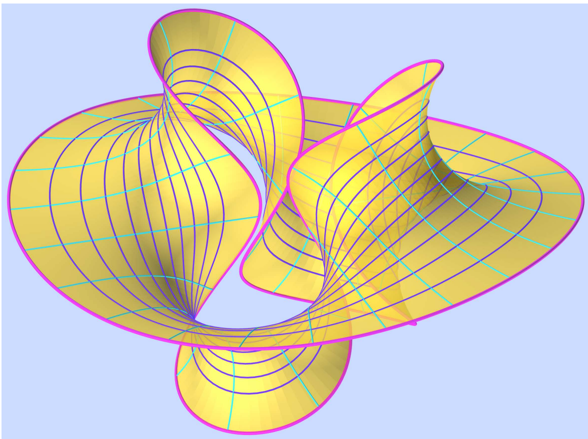

Figure 5.7: Richmond's minimal surface of degree 12 and class 12.

RICHMOND's minimal surface can also be constructed as a minimal surface on a scroll: use the planar curve

$$\gamma(t) = \left(\frac{1}{3}t^3 + \frac{1}{u}, 0, 2t\right) \tag{32}$$

for the spine curve with unit normals

$$\nu(t) = \frac{1}{1+t^2}\left(-2t, 0, t^2-1\right) \tag{33}$$

along γ and insert both into (3). The unit normal vector field of the curve γ from (32) is not precisely that given by (33) but can be transformed by the reparametrization $t \to \sqrt{t}$ into (33). Note that the plane $y = 0$ that contains γ is a plane of symmetry of RICHMOND's minimal surface and γ is a planar geodesic of the surface.

More generally speaking, associated to the family of curves

$$\gamma_a(t) = \left(t^3 + \frac{a^2}{2t}, 0, 2at\right)$$

with $a \in \mathbb{R}\setminus\{0\}$ and the unit normal vector field

$$\nu_a(t) = \left(\frac{-6at^2}{a^2+9t^4}, 0, \frac{a^2-9t^4}{a^2+9t^4}\right)$$

there is a one-parameter family of rational, and thus, algebraic minimal surfaces of Richmond type whose parametrizations read

$$\mathcal{R}(a,u,v) = \begin{pmatrix} u^3 - 3uv^2 + \frac{1}{12}\frac{a^2u}{u^2+v^2} \\ 3u^2v - v^3 + \frac{1}{12}\frac{a^2v}{u^2+v^2} \\ au \end{pmatrix}.$$

The generalization is straight forward. We choose

$$A(w) = \frac{1}{w^2} \quad \text{and} \quad \mathrm{B(w) = w^{m+1}} \tag{34}$$

with $m \in \mathbb{N}\setminus\{0\}$ which yields a one-parameter family of minimal surfaces when inserted into (12). We shall call m the *order of the Richmond surface* $\mathcal{R}_m$. Figure 5.8 shows two Richmond surfaces: one of order 3, the other one of order 4.

Note that A has a pole of degree 2 at $w = 0$. Especially, the surface with $m = 1$ is given by (31). Again, we observe that replacing m by $-m$ results in the same surface. So it is sufficient to consider only positive m.

It is no surprise that the family of generalized Richmond minimal surfaces contains members of other families. For example $\mathcal{R}_1 = \mathcal{B}_1$ with $c = 1$.

Alternatively, we could use the representation (13) with

$$G(w) = \frac{1}{w}, \; H(z) = w^m.$$

The parametrizations of the generalized

Richmond surfaces read

$$\mathcal{R}_m(u,v)=\mathfrak{Re}\begin{pmatrix} -\dfrac{1}{w}-\dfrac{w^{2m+1}}{2m+1} \\ -\dfrac{\mathrm{i}}{w}+\dfrac{\mathrm{i}w^{2m+1}}{2m+1} \\ -\dfrac{2w^m}{m} \end{pmatrix} \tag{35}$$

and they make clear that these are algebraic surfaces that admit even a rational parametrization.

We can give an upper bound for the algebraic degree of the generalized Richmond surfaces:

Theorem 5.1. *The generalized Richmond surfaces of order* $m \in \mathbb{N} \setminus \{0\}$ *are at most of algebraic degree* $2(m+1)(2m+1)$*. The class of the generalized Richmond surfaces equals exactly* $2(m+1)(2m+1)$.

Proof. For the proof of the upper bound of the degree, we use similar arguments as in the proofs of Thm. 3.1 and Thm. 4.2.

In order to verify the formula for the class of the generalized Richmond surfaces, we use the results from [30, vol. 1, p. 315] and compute, like in the proof of Thm. 3.1: $r=3m+2$ and $\mu=m+1$ which yields the class $2\mu(r-\mu)=2(m+1)(3m+2-m-1)=2(m+1)(2m+1)$. □

The regular reparametrization

$$u=r\mathrm{c}_s, \quad v=r\mathrm{s}_s$$

changes (35) to

$$\mathcal{R}_m(r,s)=\begin{pmatrix} -\frac{r^{2m}}{m+1}\mathrm{c}_{(m+1)s}-\frac{1}{(m-1)r}\mathrm{c}_{(m-1)s} \\ -\frac{r^{2m}}{m+1}\mathrm{s}_{(m+1)s}-\frac{1}{(m-1)r}\mathrm{s}_{(m-1)s} \\ 2r\mathrm{c}_s \end{pmatrix} \tag{36}$$

which is not just favorable for plotting the surface. It also enables us to show

Theorem 5.2. *The Richmond minimal surfaces* (35) *with* $m \in \mathbb{Q} \setminus \{-1,0,1\}$ *carry a one-parameter family of harmonic oscillation curves of order two.*

Proof. Let the first and the second coordinate function be the real and the imaginary part of a complex number and build $w=x+\mathrm{i}y$. Then, apply Euler's formula and find

$$w(s)=-\frac{r^{2m}}{m+1}\mathrm{e}^{\mathrm{i}(m+1)s}-\frac{1}{m-1}\mathrm{e}^{\mathrm{i}(m-1)s}.$$

If $r \in \mathbb{R} \setminus \{0\}$ is fixed, then, according to [55, 56], $w(s)$ is a complex parametrization of an ordinary cycloidal curve. Finally, we observe that the third coordinate function $z=2r\cos s$ is periodic for any $r \in \mathbb{R}$. Thus, the s-lines on the surface (36), *i.e.*, the curves with fixed r are higher oscillation curves in the sense of [37]. □

By assumption, $m \in \mathbb{Q}$, and thus, the curves are closed.

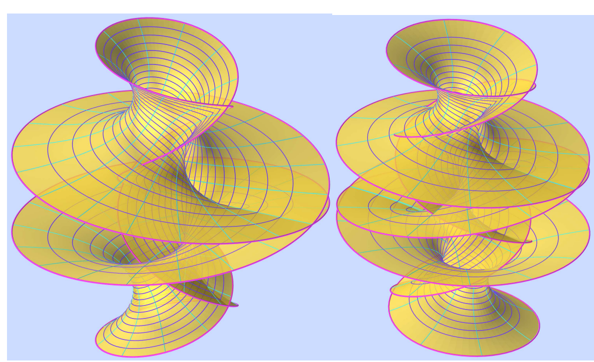

Figure 5.8: Minimal surfaces of Richmond type: Left: $m=3$ of algebraic degree 56; right: $m=4$ of algebraic degree 90.

6 Minimal surfaces tangent to orthogonal hyperbolic paraboloids

We consider the one-parameter family of hyperbolic paraboloids

$$\mathcal{P}:\ (1-b^2)xy = 2bz \tag{37}$$

with $b \in \mathbb{R} \setminus \{-1, 0, 1\}$ and the cylinder of revolution

$$\mathcal{Z}:\ x^2 + y^2 = 1. \tag{38}$$

The cylinder intersects the paraboloids (37) along the rational quartic space curves

$$\gamma(t) = \left(\mathrm{c}_t, \mathrm{s}_t, \frac{1-b^2}{4b}\mathrm{s}_{2t}\right). \tag{39}$$

In the following, we use the abbreviations

$$\beta_1 := 1+b^2,\ \beta_2 := 1-b^2,\ \beta_3 := b^4+6b^2+1.$$

Let now the normal vector field be given by

$$\begin{aligned}\nu(t) &= \mathrm{grad}(P)|_\gamma = \\ &= \frac{1}{\beta_1}(\beta_2\mathrm{s}_t, \beta_2\mathrm{c}_t, -2b).\end{aligned} \tag{40}$$

Then, we insert γ and ν from (39) and (40) into (3) and find the parametrizations of the minimal surfaces in the one-parameter family of minimal surfaces touching the paraboloids (37) along their intersection with $\mathcal{Z}$. From their parametrizations

$$\mathbf{f}(u,v) = \frac{1}{12b\beta_1}\cdot \\ \cdot\begin{pmatrix} \beta_2^2\mathrm{c}_{3u}\mathrm{S}_{3v} + 3\mathrm{c}_u(\beta_3\mathrm{S}_v + 4b\beta_1\mathrm{C}_v) \\ -\beta_2^2\mathrm{s}_{3u}\mathrm{S}_{3v} + 3\mathrm{s}_u(\beta_3\mathrm{S}_v + 4b\beta_1\mathrm{C}_v) \\ 3\beta_2\mathrm{s}_{2u}(\beta_1\mathrm{C}_{2v} + 2b\mathrm{S}_{2v}) \end{pmatrix}, \tag{41}$$

we can immediately see that these surfaces admit rational parametrizations of bidegree (6,6). Figure 6.9 shows the minimal surface parametrized by (41) together with the hyperbolic paraboloid, the curve γ from (39), and the unit normal vector field ν as given in (40). Moreover, Fig. 6.9 gives an idea how the minimal surface tangent to a hyperbolic paraboloid deviates from the paraboloid.

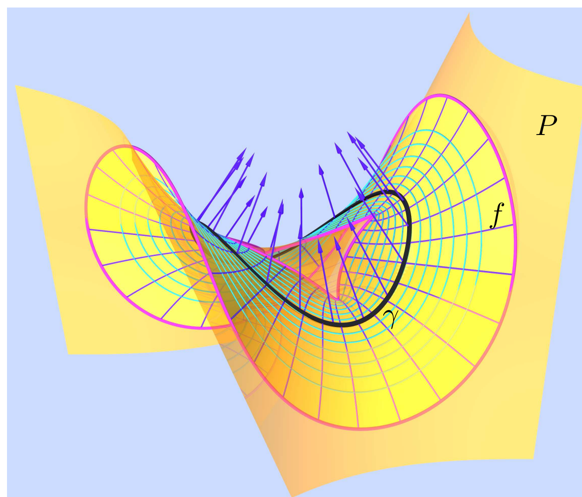

Figure 6.9: The minimal surface (41) on the scroll (γ, ν). The curve γ is a curve along which the Gaussian curvature on the hyperbolic paraboloid $\mathcal{P}$ is constant.

The rational representation of these minimal surfaces allows us to compute an implicit equation of each surface in the family. Hereby, we find that all minimal surfaces (41) are algebraic surfaces of degree 30. They all have the cycle $z^{18}(x^2+y^2)^6 = 0$ in the ideal plane in common.

The curves of constant Gaussian curvature K on the hyperbolic paraboloid P given in (37) lie on cylinders of revolution coaxial with the one in (38). For any

$b \in \mathbb{R} \setminus \{-1, 0, 1\}$ these cylinders have the equation

$$x^2 + y^2 = \frac{1}{\beta_2\sqrt{-K}} - \frac{4b^2}{\beta_2^2} \qquad (42)$$

for any admissible value $K < K_{\max} = -\frac{\beta_2^2}{16b^4}$. From

$$r^2 = \frac{1}{\beta_2\sqrt{-K}} - \frac{4b^2}{\beta_2^2}$$

we can determine the cylinder's radius. Conversely, we can choose b such that the radius r corresponds to a certain value K. This gives rise to

Theorem 6.1. *The minimal surfaces that touch an orthogonal hyperbolic paraboloid $\mathcal{P}$ along the curves of constant Gaussian curvature on $\mathcal{P}$ are rational (and thus algebraic) minimal surfaces and can be parametrized by* (41). *These minimal surfaces are of degree 30 and of class 10.*

The parameter curves $v =$ const. *are rational (and thus algebraic and closed) oscillation curves of order two,* i.e., *their orthogonal projections onto $z = 0$ are cycloidal curves of order two and the z-coordinate function is harmonic.*

Proof. From the parametrization (41) we can derive an implicit equation after a rational substitution of the trigonometric and hyperbolic functions. Thus, the rationality is obvious and the degree turns out to be 30. From the parametrization of the set of points (41), we can derive a parametrization of the set of tangent planes. Eliminating the paramters yields a polynomial of degree 10 and so the class of the minimal surface equals 10.

The x- and the y-coordinate can be considered the real and the imaginary part of a complex variable. Thus, for fixed $v \in \mathbb{R}$, we have $w(u) = x + \mathrm{i}y$ which gives a complex representation of the *top view* of the parameter curves:

$$w(u) = \frac{\beta_2^2 \mathrm{S}_{3v}}{12b\beta_1}\mathrm{e}^{-3\mathrm{i}u} + \frac{\beta_3 \mathrm{S}_v + 4b\beta_1 \mathrm{C}_v}{4b\beta_1}\mathrm{e}^{\mathrm{i}u}.$$

Comparing the latter with the formulae given in [55, 56], we can see that these are the path curves of the end points of open two-bar mechanisms. The ratio of the angular velocities of the rotating bars equals $-3 : 1$ and the lengths of the legs are the absolute values of the coefficients of the exponential functions. The z-coordinate is a multiple of $\sin 2u$, and thus, harmonic. $\square$

Finally, we shall mention that meromorphic functions $A, B : \mathbb{C} \to \mathbb{C}$ in the Weierstraß-representation (12) are

$$A = \frac{(1+b)^2}{8\mathrm{i}\, b\beta_1}\left(\mathrm{e}^{3\mathrm{i}w}(1-b)^2 + \mathrm{e}^{-\mathrm{i}w}(1+b)^2\right) \qquad (43)$$

and the simple function

$$B = \mathrm{i}\frac{1-b}{1+b}\mathrm{e}^{-\mathrm{i}w}. \qquad (44)$$

Since $b \in \mathbb{R} \setminus \{-1, 0, 1\}$, the function B can never vanish.

The substitution $t = \mathrm{e}^{\mathrm{i}w}$ in (43) transforms A into a rational function. Together with B from (44) which is linear anyway, and thus, also rational, we can find the minimal surfaces from Thm. 6.1 via (12) with rational generators A and B.

The associate minimal surfaces show a surprising behavior:

Theorem 6.2. *The minimal surfaces associated to* (41) *are congruent to* f. *Traversing the associate family of minimal surfaces means rotating the original one about the* z*-axis.*

Proof. Derive the parametrization or implicit equation of the surfaces in the associate family. The congruence transformation can easily be read of from the parametrization. □

Consequently, all members of the associate family of the minimal surface f given by (41) have the same algebraic properties.

The minimal surfaces (41) intersect the hyperbolic paraboloid $\mathcal{P}$ with equation (37) in the lines $x = z = 0$ and $y = z = 0$, each with multiplicity 6 and along the curve γ with multiplicity two (according to the construction).

7 Minimal surfaces with geodesic cycloids

Thm. 1.1 and Thm. 2.5 give rise to a generalization of HENNEBERG's adjoint surface which was the minimal surface on a scroll with an astroid (20) for its spine curve. Here, we shall recall that there is a notion of cycloid that shall not be of use here: Frequently, the word cycloid is used for a curve that is generated by roling a circle on a straight line, see [29, 32, 51]. The minimal surface with this *straight cycloid* as a planar geodesic is known as CATALAN's minimal surface (see [27, 33, 36] and it is not algebraic.

The cycloidal curves that emerge from roling a circle along another one yields a one-parameter family of rational, and thus, algebraic minimal surfaces. We have

Theorem 7.1. *Let* $r, R \in \mathbb{R} \setminus \{0\}$ *be real constants with* $R + 2r \neq 0$ *and* $R + r \neq 0$. *The minimal surfaces on the scroll* (ζ, ν) *with* $\zeta \subset \pi_3 : z = 0$ *and* $\nu \in \mathrm{S}^1$

$$\begin{aligned} \zeta(t) &= \begin{pmatrix} (R+r)\mathrm{c}_t + r\mathrm{c}_{\frac{(R+r)t}{r}} \\ (R+r)\mathrm{s}_t + r\mathrm{s}_{\frac{(R+r)t}{r}} \\ 0 \end{pmatrix}, \\ \nu(t) &= \frac{1}{2\mathrm{c}_{\frac{Rt}{2r}}} \begin{pmatrix} -\mathrm{c}_t - \mathrm{c}_{\frac{(R+r)t}{r}} \\ -\mathrm{s}_t - \mathrm{s}_{\frac{(R+r)t}{r}} \\ 0 \end{pmatrix} \end{aligned} \tag{45}$$

can be parametrized by

$$\mathbf{f}(u,v) = \begin{pmatrix} (R+r)\mathrm{c}_u\mathrm{C}_v + r\mathrm{c}_{\frac{(R+r)u}{r}}\mathrm{C}_{\frac{(R+r)v}{r}} \\ (R+r)\mathrm{s}_u\mathrm{C}_v + r\mathrm{s}_{\frac{(R+r)u}{r}}\mathrm{C}_{\frac{(R+r)v}{r}} \\ -\frac{4r(R+r)}{R}\mathrm{c}_{\frac{Ru}{2r}}\mathrm{S}_{\frac{Rv}{2r}} \end{pmatrix}. \tag{46}$$

These minimal surfaces are algebraic, rational, and closed if, and only if, $R, r \in \mathbb{Q} \setminus \{0\}$.

In any case, the cycloid $\zeta \subset \pi_3$ *is a geodesic on the minimal surface.*

The surfaces with $R, r \in \mathbb{Q} \setminus \{0\}$ *contain at least one straight line.*

Proof. Insert γ and ν from (45) into (3). This gives (46).

The geodesic property of the cycloidal spine curves is a direct consequence of Thm. 1.1.

The straight lines are part of the double curves in symmetry planes. □

In the case $R + 2r = 0$, the cycloid ζ from (45) collapses to a diameter of the circle $(R\mathrm{c}_t, R\mathrm{s}_t, 0)$. If $R + r = 0$, the polhodes of ζ are not just congruent, they are identical and no rolling takes place.

The cycloids ζ parametrized by (45) are closed, rational, and thus, algebraic, if, and only if, $r : R \in \mathbb{Q} \setminus \{0\}$. They have cusps of the first kind at

$$\cos \frac{tR}{r} = -1 \iff t = (2k+1)\pi \frac{r}{R},$$

i.e., finitely many if $r : R \in \mathbb{Q} \setminus \{0\}$, provided the admissible choice of r and R. Consequently, the minimal surfaces (46) have branch points exactly at the cusps of the cycloids ζ given by (45).

From the parametrization (46) it is clear that the u-lines (curves with $v = \text{const.}$) on the cycloidal minimal surfaces have a very special shape. We have

Theorem 7.2. *The u-lines on the cycloidal minimal surfaces given by* (46) *are generalized oscillation curves. Their orthogonal projections onto the planes $z = c$ (with $c \in \mathbb{R}$) are cycloidal curves.*

Proof. A closer look at the first and second coordinate function of the parametrization (46) tells us that, for fixed $v \in \mathbb{R}$, we have the parameterization of cycloidal curves. These curves can also be written in terms of complex coordinates by letting $w(u) = x + \mathrm{i}y$ and applying Euler's formula as

$$w(u) = (R+r)\mathrm{C}_v \mathrm{e}^{\mathrm{i}u} + r\mathrm{C}_{\frac{(R+r)t}{r}} \mathrm{e}^{\mathrm{i}u\frac{R+r}{r}}.$$

Comparing with [56], we find the lengths

$$A_1 = (R+r)\mathrm{C}_v, \quad A_2 = r\mathrm{C}_{\frac{(R+r)v}{r}}$$

of the legs of a generating two-bar mechanism and the (ratio of the) angular velocities of the bars are

$$\omega_1 : \omega_2 = 1 : \frac{R+r}{r}.$$

From that we can compute the radii of the polhodes of the motion that generates the orthogonal projections of u-lines as path curves, see [55, 56]. □

The meromorphic functions $A, B : D \subset \mathbb{C} \to \mathbb{C}$ from (13) can also be given:

Lemma 7.1. *The cycloidal minimal surfaces can be obtained from the Weierstraß-representation* (13)*. Therein, the meromorphic functions A and B are:*

$$\begin{aligned} A(w) &= -\tfrac{\mathrm{i}}{2}(R+r)\left(\mathrm{e}^{-\mathrm{i}w} + \mathrm{e}^{-\mathrm{i}\frac{R+r}{r}w}\right), \\ A(w)\cdot B(w) &= \mathrm{i}(R+r)\mathrm{c}_{\frac{(R+r)w}{2r}}. \end{aligned} \tag{47}$$

Proof. In order to find A and B from (46), we use (14). □

More ore less surprisingly, there is a connection to the curves of constant slope on quadrics of revolution and the curves $\gamma(u) = \mathbf{f}(u, 0)$ on the cycloidal minimal surfaces. The family of minimal surfaces associated to (46) can be given with (18) as

$$\mathbf{f}(u, v, \tau) = \mathrm{c}_\tau \cdot \mathbf{f} + \mathrm{s}_\tau \cdot \mathbf{f}^\perp \tag{48}$$

where $\mathbf{f}$ is the parametrization (46) and $\mathbf{f}^\perp$ reads

$$\mathbf{f}^\perp = \begin{pmatrix} (R+r)\mathrm{S}_v \mathrm{s}_u + r\mathrm{S}_{\frac{(R+r)v}{r}} \mathrm{s}_{\frac{(R+r)u}{r}} \\ -(R+r)\mathrm{S}_v \mathrm{c}_u - r\mathrm{S}_{\frac{(R+r)v}{r}} \mathrm{c}_{\frac{(R+r)u}{r}} \\ -\frac{4r(R+r)}{R} \mathrm{s}_{\frac{Ru}{2r}} \mathrm{C}_{\frac{Rv}{2r}} \end{pmatrix}. \tag{49}$$

The spine curves of the scrolls are obtained by substituting $v = 0$ in (48). These spine curves can be taken as the spine curve γ of a scroll on which, according to the Björling formula (3), minimal surfaces can be errected. Now, we have the following

Theorem 7.3. *The one-parameter family of curves $\mathbf{f}(u,0,\tau)$ with parametrization* (48) *and* (49) *are curves of constant slope on quadrics of revolution. These curves are closed, rational, and thus, algebraic space-curves provided that $r : R \in \mathbb{Q} \setminus \{0\}$, $R+2r \neq 0$, and $R+r \neq 0$. The slope angle σ is independent of R and r and is related to τ (modulo 2π) by*

$$\mathrm{c}_\sigma = -\mathrm{s}_\tau \iff \sigma = \tau + \frac{\pi}{2}. \qquad (50)$$

Proof. The top views of the curves $\mathbf{b} = f(u,0,\tau)$, *i.e.*, the orthogonal projections of the curves $f(u,0,\tau)$ onto planes parallel to $z=0$ are cycloids (with cusps). It is well-known (see, *e.g.*, [3, 7]) that the curves of constant slope on quadrics of revolution appear as epi-, hypo-, hyper, and paracycloids in a top view (in the direction of the lead). The case of paraboloids of revolution differs a little bit: In the corresponding top views, we can see the involutes of circles, cf. [26].

We compute $\mathbf{b}' = \frac{\mathrm{d}}{\mathrm{du}}\mathbf{b}$. The lead is given by the unit vector $\mathbf{l} = (0,0,1)$. Now, it is elementary to verify that

$$\mathrm{c}_\sigma = \frac{\langle \mathbf{b}', \mathbf{l}\rangle}{\|\mathbf{b}'\|} = -\mathrm{s}_\tau$$

which makes clear that the slope of the spine curves $\mathbf{b} = f(u,0,\tau)$ is constant and independent of the choice of R and r and (50) is valid. It is easily verified that the coordinate functions of $\mathbf{b}$ satisfy

$$Q : x^2 + y^2 + \frac{k^2 R^2 z^2}{4r(r+R)} = (2r+R)^2 {\mathrm{c}_\tau}^2 \qquad (51)$$

with $k = \cot\tau$ which is the equation of quadrics Q of revolution.

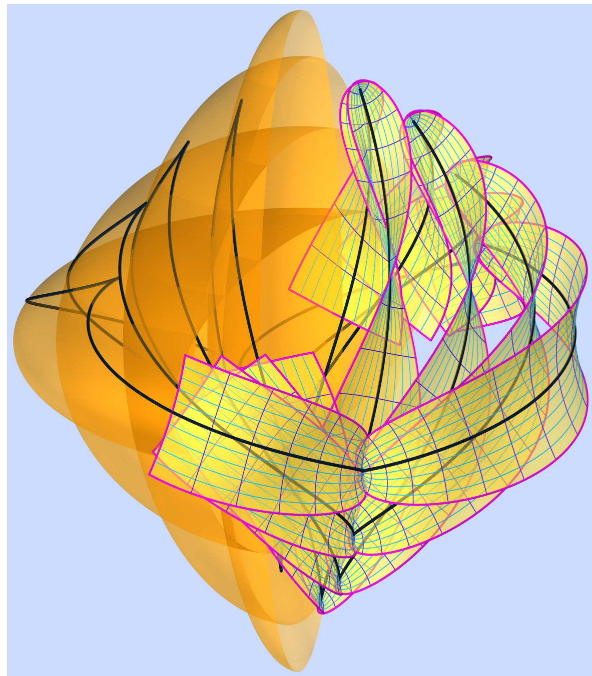

Figure 7.10: The spine curves of the cycloidal minimal surfaces are bent smoothly into curves of constant slope on quadrics of revolution.

The rationality is clear if $r : R \in \mathbb{Q} \setminus \{0\}$ since then $\cos nu$ and $\sin nu$ can be expressed in $\cos u$ and $\sin u$ which can subsequently be replaced with their rational equivalents provided that $(R+r)/r = n$ is an integer. If $(R+r)/r = m/n$ with $\gcd(m,n) = 1$, we reparametrize by letting $u' = ru$, expand $\cos mu, \ldots$ in $\sin u$ and $\cos u$ followed by the rational reparametrization. Since cycloids are closed if $r : R \in \mathbb{Q} \setminus \{0\}$, the curves of constant slope on the quadrics (51) are also closed. $\square$

Figure 7.10 illustrates the contents of Thm. 7.3.

We shall note (51) can be the equation of an ellipsoid or a one-sheeted hyperboloid as well. The latter appears if $r < 0$. Two-sheeted hyperboloids will not be described by (51) since then the coefficient of z^2 as well as the right-hand side of (51) have to be negative. This is not possible since the right-hand side is a full square.

On the other hand, the top-views of the curves of constant slope on a two-sheeted hyperboloid of revolution are *paracycloids*, *i.e.*, curves that belong to the class of spiraloids and are transcendental independent of r and R are, cf. [29, 32, 51, 54]. In the case that (51) describes a one-sheeted hyperboloid, k, and thus, the slope of the curves $\mathbf{b}$ is always larger than that of the quadrics' asymptotic cone. Otherwise the curves of constant slope appear as *hypercycloids* in the top-view. These curves are closely related to paracycloids, and like these, they are always transcendental and belong to the class of spiraloids, see [29, 32, 51, 54].

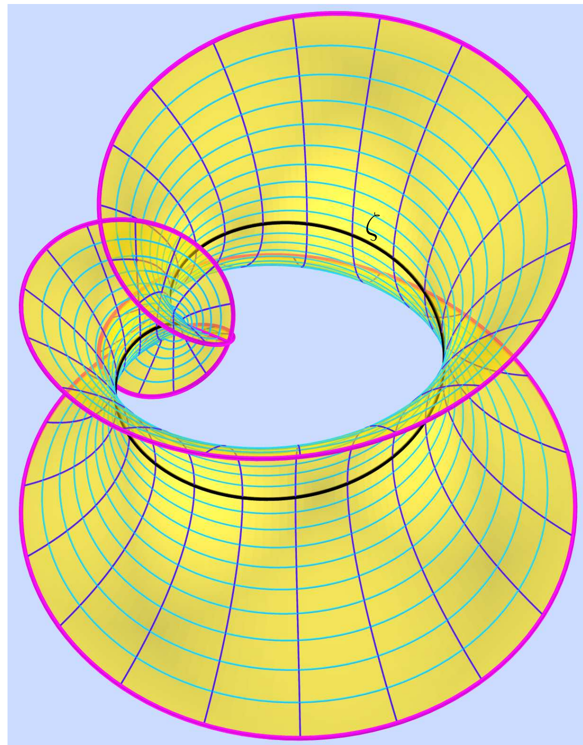

Figure 7.11: The cycloidal minimal surface with $R = r = 1$ and its geodesic cardiod ζ.

7.1 A cardioid as a geodesic curve

The low degree minimal surfaces of cycloidal type can be found by choosing small values for the radii R and r of the polhodes of the cycloid ζ. The case of an astroid which occurs with $r : R = -1 : 4$ is described in Sec. 2, especially in Thm. 2.5.

Figure 7.11 shows the algebraic minimal surface along the cardiod ζ. This surface occurs with $r : R = 1 : 1$. The algebraic degree of the *cardioidal minimal surface* is 20 and the class equals 36. The intersection μ of the minimal surface with the plane at infinity has the equation $z^{16}(x^2 + y^2)^2 = 0$ which tells us that the ideal line of all planes parallel to $z = 0$ is the only real part of μ (with multiplicity 16). The second factor corresponds to a pair of complex conjugate ideal lines with multiplicity 2.

The x-axis of the underlying Cartesian coordinate frame is a four-fold line on the surface and together with the cardiod ζ and the six-fold isotropic pair of lines through the origin of the underlying coordinate frame it completes the surface's intersection with $z = 0$. A rational parametrization can be achieved by substituting

$$\mathrm{c}_u = \frac{1 - U^2}{1 + U^2}, \quad \mathrm{s}_u = \frac{2U}{1 + U^2} \tag{52}$$

and, surprisingly, with

$$\mathrm{S}_v = V, \quad \mathrm{C}_v = \sqrt{1 + V^2} \tag{53}$$

since the hyperbolic functions showing up in the coordinate functions can be exressed in

$\sinh v$ exclusively. Thus, the cardioidal minimal surface admits a rational Bézier representation of bi-degree $(8, 4)$.

The adjoint surface looks like a *compressed helicoid*, see Fig. 7.12. Note that this surface cannot be a ruled surface, because the transcendental helicoid is the only ruled minimal surface. It is of algebraic degree 38. The intersection with the ideal plane is the cycle $z^{32}(x^2 + y^2)^3$. The surface carries the two eight-fold straight lines $x = z = 0$ and $x = y = 0$.

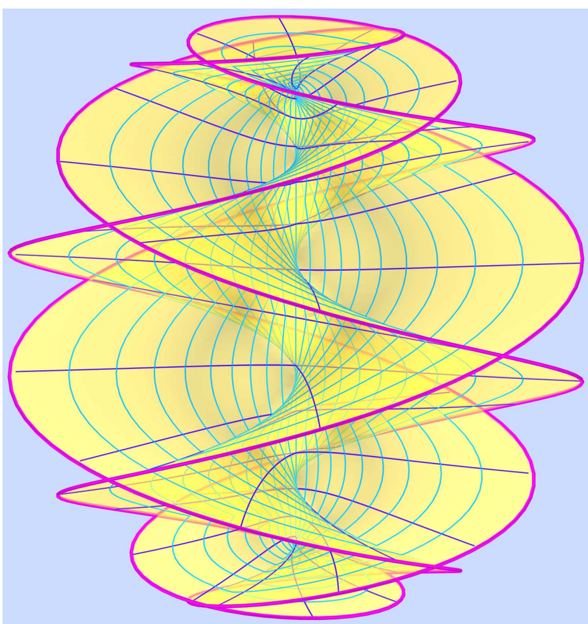

Figure 7.12: A *compressed helicoid* as the adjoint to the cardiodal minimal surface.

7.2 Steiner's hypocyloid

Steiner's hypocycloid appears in geometry in many ways. However, it is also a cycloidal curve and we can obtain it by choosing $R = 3$ and $r = -1$ in (45). The corresponding cycloidal minimal surface (46) turns out to be of algebraic degree 28 and of class 16. From the construction it is clear, that the horizontal cross-section with the plane $z = 0$ consists of the three-cusped hypocycloid. Moreover, the lines of symmetry $y = 0$ and $3x^2 = y^2$ (all three with multiplicity four) are part of the cross-section. Sine $y = z = 0$ annihilates the equation of this minimal surface, the x-axis of the underlying coordinate frame is entirely contained in this minimal surface.

The intersection of the *hypocycloidal minimal surface* with the ideal plane is given by the equation $z^{16}(x^2 + y^2)^6 = 0$. Thus, the ideal line of all planes parallel to $z = 0$ is a 16-fold line on this surface. As is the case with any algebraic minimal surface, the ideal curve degenerates completely and splits into a finite number of lines. A ratio-

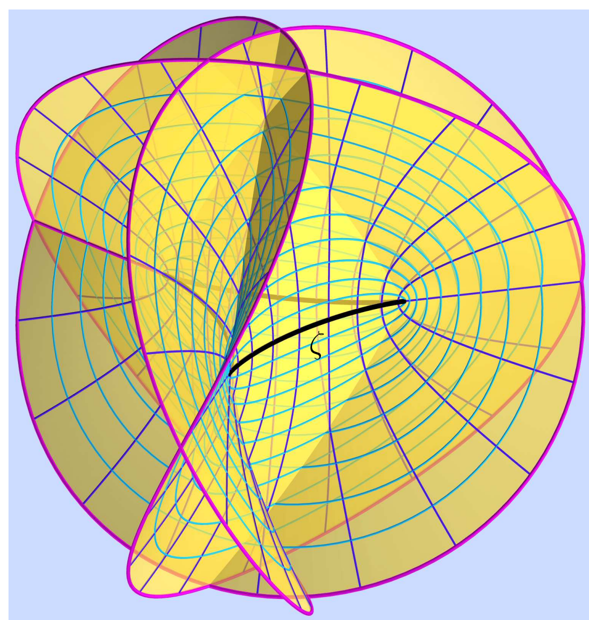

Figure 7.13: The minimal surface on a geodesic hypocycloid ζ with three cusps.

nal Bézier representation of bi-degree (8,4) can be found by substituting (52) and (53). Figure 7.13 shows a part of the surface with a geodesic hypocycloid.

7.3 A geodesic nephroid

A final low degree example shall be discussed: We choose $R = 2$ and $r = 1$. This results in a minimal surface with a geodesic nephroid. The surface is of algebraic degree 24 and of class 72. The intersection with the ideal plane is the 18-fold ideal line of all planes parallel to $z = 0$ together with a three-fold pair of complex conjugate lines. Figure 7.14 shows the minimal surface with a geodesic nephroid. The *nephroidal minimal surface* admits a rational Bézier representation of bi-degree (6,6) since we have to substitute

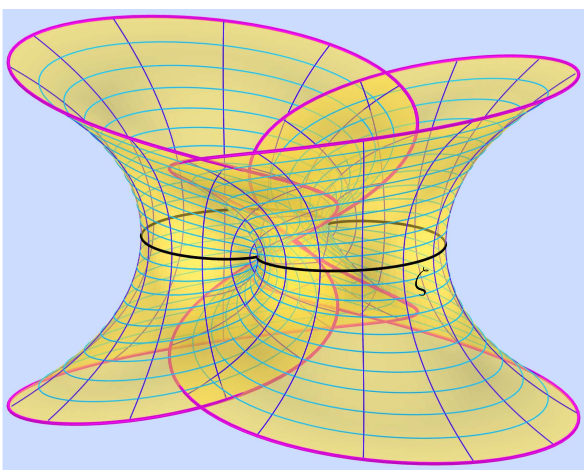

Figure 7.14: The rational minimal surface with a geodesic nephroid ζ.

$$\mathrm{C}_v = \frac{1+V^2}{1-V^2}, \quad \mathrm{S}_v = \frac{2V}{1-V^2}.$$

The z- and the y-axis are contained in the surface.

8 Final remarks

The curves of constant slope mentioned in Thm. 7.3 can also be used as spine curves of scrolls on which minimal surfaces can be errected. Unfortunately, the minimal surface that touch the quadrics of revolution along curves of constant slope are, in general, not algebraic. With Thm. 1.2 the following theorem is a natural consequence:

Theorem 8.1. *The minimal surfaces that touch the vertical cylinders (generators parallel to the lead) along the curves of constant slope on quadrics of revolution are algebraic if the curves of constant slope are algebraic too.*

Note that the curves of constant slope on quadrics of revolution are algebraic if they are closed. Thus, the minimal surfaces mentioned in Thm. 8.1 are algebraic if the spine curves of the scrolls are closed curves of constant slope. Since the normals of all minimal surfaces described in Thm. 7.1 stay horizontal while the surfaces traverse the associate family, and furthermore, since the vertical cylinders' (horizontal) normals are always orthogonal to the tangents of the curves of constant slope, we can state

Theorem 8.2. *The algebraic minimal surfaces that touch the vertical cylinders along the curves of constant slopes on quadrics of revolution are precisely the algebraic minimal surfaces mentioned in Thm. 7.3.*

The algebraic degrees are growing rapidly and there will hardly be some low degree examples among the minimal surfaces described in Thm. 8.1.

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