Rational minimal surfaces tangent to E. Müller's surface

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Abstract

We study a new class of minimal surfaces which are in line contact with a special cubic surface. It turns out that these minimal surfaces admit rational parametrizations and carry a one-parameter family of higher order harmonic oscillation curves. Each of these minimal surfaces defines its own one-parameter family of associated minimal surfaces which in turn are all algebraic. Moreover, they also admit rational parametrizations.

Keywords: *minimal surface, algebraic surface, rational parametrization, polynomial parametrization, Björling formula.*

MSC 2010: 53A10, 53A99, 53C42, 49Q05, 14J26, 14Mxx.

1 Introduction

Algebraic and especially rationally parametrizable minimal surfaces gained less attention in the last years. Though the advantages of rational parametrizations for applications in CAD and CAGD are clearly visible, high degrees and the highly complicated generation of such minimal surfaces may be a reason for the absence of research in this field. Only older literature provides some general results on algebraic minimal surfaces, see [3, 5, 10, 13]. Recently, this topic was picked up in [6] where a rational minimal Möbius strip was studied. In [7], some further new classes of algebraic minimal surfaces were discovered. These surfaces allow rational parametrization, are of relatively low degree, and the a huge variety of algebraic properties which were at least known to S. Lie and H.A. Schwarz (see [3, 10]) could be verified.

Rational or even polynomial parametrizations of minimal surfaces can be found with help of the various Weierstraß-representaions or the Björling formula (also due to Weierstraß), see [1, 2, 5, 8, 13]. In the following, we will use the Björling formula in order to construct rational minimal surfaces on scrolls. The initial (boundary) data shall be taken from a special cubic surface. Until now, useful initial data that yields rational parametrizations of minimal surfaces is found just by chance.

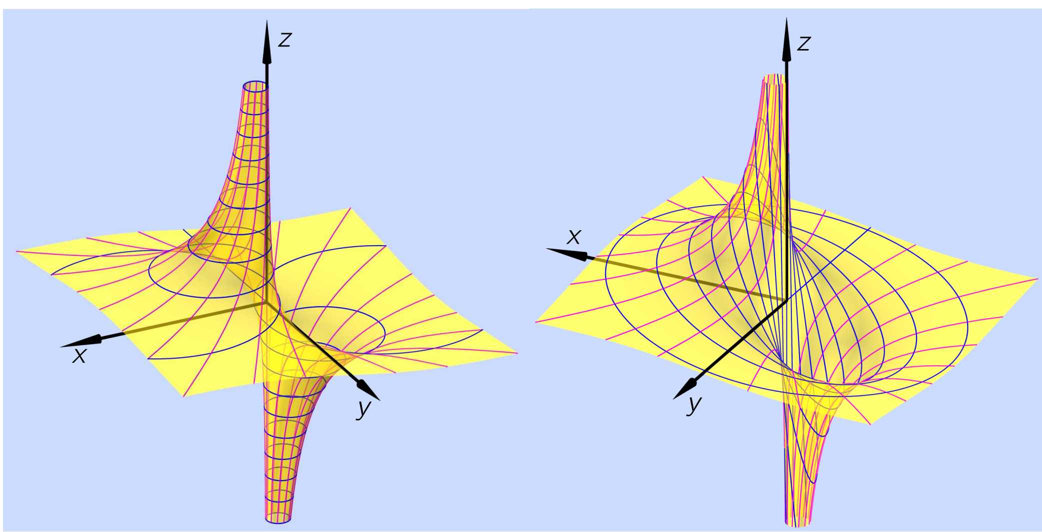

Figure 1.1: Left: Müller's surface with its circles and hyperbolae. Right: Eellipses on Müller's surface.

The cubic surfaces that deliver the initial data in the present case are spheres in some sence. Together with the planes in the projectively and complex extended Euclidean three-space they form a set of surfaces that is invariant under a group of transformations generated by special birational cubic transformations, called *axial inversions*. In [4], such inversions transforming points with coordinates (x, y, z) to points with coordinates (x', y', z') were studied in the projective extension of Euclidean three-space. A special version reads

$$x' = \frac{x}{x^2 + y^2}, \quad y' = \frac{y}{x^2 + y^2}, \quad z' = z$$

and maps planes $ax + by + cz + d = 0$ with $a, b, c, d \in \mathbb{R}$ and $a : b : c \neq 0 : 0 : 0$ to a special kind of cubic surfaces with the equations

$$ax + by + (cz + d)(x^2 + y^2) = 0,$$

and *vice versa*. (For the sake of simplicity, we have written x, y, and z instead of x', y', and z'.) These cubics carry three one-parameter families of conics which are the intersections with the planes in the pencils $\lambda(ax+by)+\mu(cz+d)=0$, $\lambda z+\mu=0$, and $\lambda x+\mu y=0$ with $(\lambda,\mu)=\in\mathbb{R}^2\setminus\{(0,0)\}$.

The contour lines (horizontal curves on the surface) are circles that appear as the members of a parabolic pencil of circles in a top-view, *i.e.*, in an orthogonal projection in the direction of the lead which is henceforth assumed to be parallel to the z-axis of the underlying Cartesian coordinate system.

The curves of steepest ascent (with respect to the vertical lead) are upright cubic circles, *i.e.*, rational cubic curves on right cylinders. The top-views of the curves of steepest ascent constitute the parabolic pencil of circles that is complementary (and thus, orthogonal) to the one previously mentioned.

In the following, we pay our attention to the special surface that we obtain if we let $a=c=1$ and $b=d=0$. This particular surface is the image of the plane $x-z=0$ under the axial inversion and has the equation

$$x-(x^2+y^2)z=0 \tag{1}$$

and was first studied in [12].

Obviously, this surface allows two different parametrizations over nearly the same parameter domain $D=\mathbb{R}^\star\times[0,2\pi[$

$$\left(\frac{r}{2}(1+\cos u),\frac{r}{2}\sin u,\frac{1}{r}\right),\quad (r,u)\in D, \tag{2}$$

or a seemingly simpler version

$$\left(q\cos t, q\sin t,\frac{\cos t}{q}\right),\quad (q,t)\in D. \tag{3}$$

In (2), the parameter curves are the circles in the pencil of planes $z=c$ (with $c\in\mathbb{R}^\star$) and the hyperbolae in the pencil of planes $\lambda x+\mu y=0$ (with $\lambda:\mu\neq 0:0$), see Figure 1.1 (left). In (3), the parameter curves are the ellipses (different from circles) in the pencil of planes $\lambda x+\mu z=0$ (with $\lambda:\mu\neq 0:0$) and the previously mentioned hyperbolae (see Figure 1.1, right). The ellipses in the pencil of planes $\lambda x+\mu z=0$ through the y-axis are given by $q=$ const. in (3).

Müller's surface (1) has three singular points: the pair of absolute points of Euclidean geometry in the plane $z=0$ with homogeneous coordinates

$(0 : 1 : \pm i : 0)$ and the ideal point of the z-axis with homogeneous coordinates $(0 : 0 : 0 : 1)$.

This particular cubic surface contains three real lines: the y-axis, the z-axis, and the ideal line of all planes parallel to $z = 0$.

For any fixed $q \in \mathbb{R}^\star$, the ellipses parametrized by (3) carry only regular surface points. Along such an ellipse, the surface normals of the cubic surface (1) are parallel to

$$\nu(t) = \frac{1}{\sqrt{1+q^4}}(\cos 2t, \sin 2t, q^2) \tag{4}$$

and determine a regular and non-torsal algebraic ruled surface of degree six; an example of which is shown in Fig. 1.2. For any fixed $q \in \mathbb{R}^\star$, the parametrization (3) gives the parametrization

$$\gamma(t) = \left(q\cos t, q\sin t, \frac{\cos t}{q}\right) \tag{5}$$

of an ellipse. The pair (γ, ν) defines a *scroll* as the envelope of the one-parameter family of planes $\langle \mathbf{x} - \gamma, \nu \rangle = 0$.

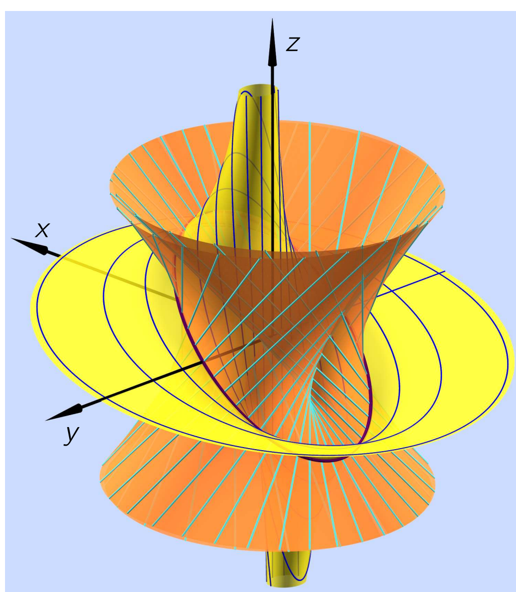

Figure 1.2: The sextic ruled surface of normals along an ellipse.

2 The minimal surfaces

We use the scroll (γ, ν) as the initial (boundary) data for a minimal surface. Following [2, 3, 5, 8, 13], we can use the Björling formula in order to derive a real parametrization of the unique real minimal surface on the scroll (γ, ν), *i.e.*, the uniquely determined real minimal surface through the curve γ with normals parallel to ν along γ. For that we assume that we are given a curve $\gamma : \ I \subset \mathbb{R} \to \mathbb{R}^3$ and a unit vector field $\nu : \ I \to S^2$ along γ. Both are considered to have complex continuations. Then, the Björling formula

$$\varphi(t) = \gamma(t) - \mathrm{i} \int_{t_0}^{t} \nu \times \mathrm{d}\gamma \tag{6}$$

yields a parametrization of an isotropic curve, *i.e.*, a curve of constant slope $\pm\mathrm{i}$ whose tangents are isotropic lines in Euclidean three-space. The existence of the complex continuation of the curve and the unit normal vector field allows us to set $t = u + \mathrm{i}v$. Then, we extract the real part of the vector function $\varphi(t)$ and obtain a real parametrization

$$\mathbf{f}(u, v) = \mathfrak{Re}\, \varphi(t) \tag{7}$$

of the uniquely defined real minimal surface on the scroll (γ, ν).

With the spine curve γ given in (5) and the unit normal vector field ν described by (4), we can derive the parametrization(s) of the minimal surfaces tangent to Müller's cubic surface. We use shorthand

$$\mathrm{c}_x := \cos x, \quad \mathrm{s}_x := \sin x, \ldots, \mathrm{C}_x := \cosh x, \quad \mathrm{S}_x := \sinh x, \ldots$$

together with the abbreviation $p := \sqrt{1 + q^4}$ and state:

Theorem 2.1. *The one-parameter family of minimal surfaces touching Müller's surface* (1) *along the ellipses* (3) *can be parametrized over* $\mathbb{R}^2$ *by*

$$\mathbf{f}(u, v) = \frac{1}{6pq} \begin{pmatrix} 6pq^2\mathrm{c}_u\mathrm{C}_v - 3(p^2 + q^4)\mathrm{c}_u\mathrm{S}_v + \mathrm{c}_{3u}\mathrm{S}_{3v} \\ 6pq^2\mathrm{s}_u\mathrm{C}_v - 3(p^2 + q^4)\mathrm{s}_u\mathrm{S}_v + \mathrm{s}_{3u}\mathrm{S}_{3v} \\ 6\mathrm{c}_u(\mathrm{S}_v q^2 + p\mathrm{C}_v) \end{pmatrix}. \tag{8}$$

Proof. We insert (5) and (4) into (6) and find the parametrization of the isotropic curve $\varphi : \ \mathbb{R}^3 \to \mathbb{R}^3$

$$\varphi(t) = \frac{1}{6pq} \begin{pmatrix} 6pq^2\mathrm{c}_t + 3\mathrm{i}(p^2 + q^4)\mathrm{s}_t - \mathrm{i}\mathrm{s}_{3t} \\ 6pq^2\mathrm{s}_t - 3\mathrm{i}(p^2 + q^4)\mathrm{c}_t + \mathrm{i}\mathrm{c}_{3t} \\ 6(p\mathrm{c}_t - \mathrm{i}q^2\mathrm{s}_t) \end{pmatrix} \tag{9}$$

depending on the complex parameter t. Then, we replace t by $u + \mathrm{i}v$ and extract the real part and obtain (8). $\square$

Figure 2.3 shows one particular minimal surface mentioned in Thm. 2.1.

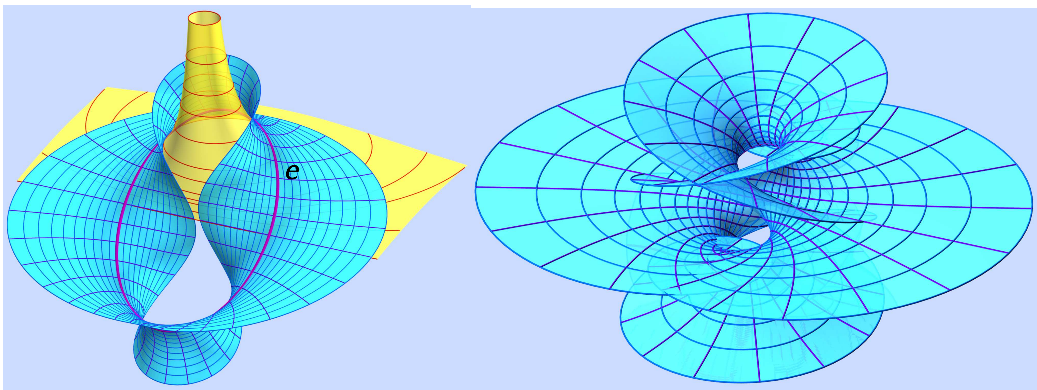

Figure 2.3: A minimal surface tangent to Müller's surface along the ellipse e.

The curves $v = \text{const.}$ on the minimal surface have a very special shape:

Theorem 2.2. *The u-curves (the curves with $v = \text{const.}$) on the minimal surfaces* (8) *are harmonic oscillation curves and appear as cycloidal curves in the top-view (orthogonal projection onto the* $[xy]$*-plane).*

Proof. We use the first and second coordinate function of $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3)$ from (8) and build the complex variable $w(t) = \mathbf{f}^1 + \mathrm{i}\mathbf{f}^2$ which reads in full length

$$w(t) = q\mathrm{c}_u\mathrm{C}_v - \tfrac{1}{2pq}(p^2 + q^4)\mathrm{c}_u\mathrm{S}_v + \tfrac{1}{6pq}\mathrm{c}_{3u}\mathrm{S}_{3v} +$$
$$+\mathrm{i}\left(q\mathrm{s}_u\mathrm{C}_v - \tfrac{1}{2pq}(p^2 + q^4)\mathrm{s}_u\mathrm{S}_v + \tfrac{1}{6pq}\mathrm{s}_{3u}\mathrm{S}_{3v}\right).$$

With Euler's formula, the latter simplifies to

$$w(t) = \left(q\mathrm{C}_v - \frac{p^2 + q^2}{2pq}\mathrm{S}_v\right)\mathrm{e}^{\mathrm{i}u} + \frac{1}{6pq}\mathrm{S}_{3v}\mathrm{e}^{3\mathrm{i}u}$$

which is a parameter representation of family of cycloidal curves according to [14, 15], because $v = \text{const.}$, and thus, S_v, C_v, S_{3v}, and C_{3v} are constant. Since $\mathbf{f}^3 = \mathrm{c}_u\left(\frac{p}{q}\mathrm{S}_v + \frac{1}{q}\mathrm{C}_v\right)$ is a harmonic function as long as $v = \text{const.}$, the u-curves are harmonic oscillation curves as defined in [9]. $\square$

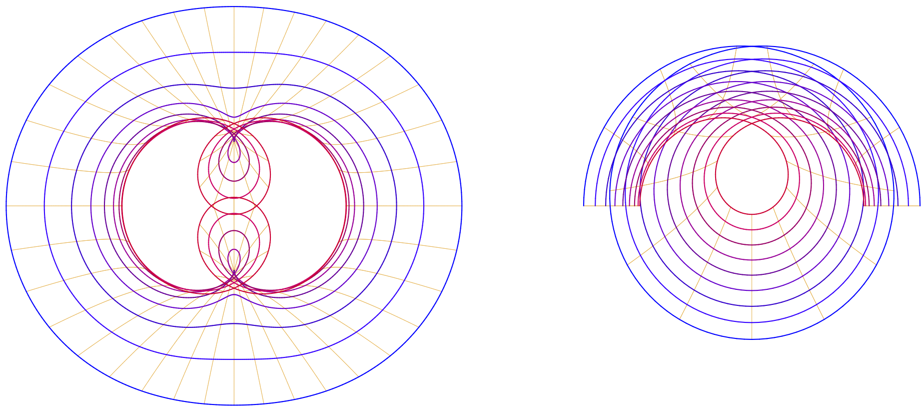

Figure 2.4: The top-views of the u-curves on the minimal surfaces are cycloidal curves: $q \in [0.6, 1.2]$ (left), only one half of the curves for $q \in [1.0, 1.4]$ (right).

Figure 2.4 shows the top-views of some of the harmonic oscillation curves on the minimal surfaces given in (8).

According to Weierstraß, the minimal surface (8) can be generated from two meromorphic functions $A, B: D \subset \mathbb{C} \to \mathbb{C}$ by computing

$$\varphi(t) = \int \begin{pmatrix} A(1-B^2) \\ \mathrm{i}A(1+B^2) \\ 2AB \end{pmatrix} \mathrm{d}t, \tag{10}$$

see, *e.g.*, [1, 2, 3, 5, 8, 13]. Subsequently, we let $t = u + \mathrm{i}v$ and extract the real part $\mathbf{f}(u, v) = \mathfrak{Re}(\varphi)$ in order to obtain a real parametrization of the thus defined real minimal surface. We can state and proof

Theorem 2.3. *The meromorphic functions $A, B: D \subset \mathbb{C} \to \mathbb{C}$ from the representation* (10) *of the minimal surfaces* (8) *tangent to Müller's surface read*

$$A = \frac{\mathrm{i}}{4pq}\left((p-q^2)^2 \mathrm{e}^{-\mathrm{i}w} - \mathrm{e}^{-3\mathrm{i}w}\right), \quad B = \frac{\mathrm{e}^{2\mathrm{i}w}(p-q^2) - (p+q^2)}{(p-q^2)^2 - \mathrm{e}^{-2\mathrm{i}w}}. \tag{11}$$

Proof. Following [1, 2, 3, 5, 8, 13], we compute $\mathbf{F} = (\mathbf{F}^1, \mathbf{F}^2, \mathbf{F}^3) := \partial_u \mathbf{f} - \mathrm{i}\partial_v \mathbf{f}$. Then, $A = \frac{1}{2}(\mathbf{F}^1 - \mathrm{i}\mathbf{F}^2)$ and $B = A/2\mathbf{F}^3$ and we end up with (11). $\square$

The meromorphic functions A and B given in Thm. 2.3 can be replaced by rational functions. This is equivalent to a reparametrization of the isotropic curve φ given in (10). The parametrization of the minimal surface obtained from the reparametrized isotropic curve will in general not be the same, but an equivalent one. We let $\omega = \mathrm{e}^{-iw}$ and arrive at the following rational equivalents to A and B:

$$\widetilde{A} = \frac{i}{4pq}\left((p-q^2)^2-\omega^2\right)\omega, \quad \widetilde{B} = \frac{(p-q^2)-(p+q^2)\omega^2}{\left((p-q^2)^2-\omega^2\right)\omega^2}.$$

A closer look at the parametrization (8) makes clear that the parameter u appears only as argument of trigonometric functions, whereas v shows up only as argument of hyperbolic functions. Thus, it is obvious that a reparametrization can turn (8) into a rational parametrization. Moreover, we can show

Theorem 2.4. *The minimal surfaces* (8) *that touch Müller's surface along ellipses admit rational parametrizations and are algebraic minimal surfaces of degree 24 and class 42.*

Proof. The existence of a rational parametrization is confirmed by simply substituting the rational equivalents

$$\mathrm{c}_u = \frac{1-U^2}{1+U^2}, \ \mathrm{s}_u = \frac{2U}{1+U^2}, \ \mathrm{C}_v = \frac{1+V^2}{1-V^2}, \ \mathrm{S}_v = \frac{2V}{1-V^2}$$

of the trigonometric and hyperbolic functions into (8). This yields a rational parametrization of bi-degree (6,6) in U and V which can be rewritten in terms of rational Bézier functions.

From the rational parametrization, the implicit algebraic equation can be found (more or less) easily by eliminating U and V. This results in a polynomial of degree 24.

The class of these surfaces has to be even, for these algebraic minimal surfaces are orientable, cf. [3]. The algebraic degree of $\varphi(t)$ equals 6 and its rank equals $r = 10$. The absolute conic of Euclidean geometry is a three-fold curve on the tangent developable of φ, *i.e.*, its multiplicity on the developable is $m = 3$. According to [3], the class of the minimal surface swept by φ and its conjugate equals $c = 2m(r-m) = 42$. □

The rational parametrization mentioned in Thm. 2.4 can easily be converted into a rational Bézier representation. Figure 2.5 shows a part of a minimal surface described in Thm. (2.1) with its control structure.

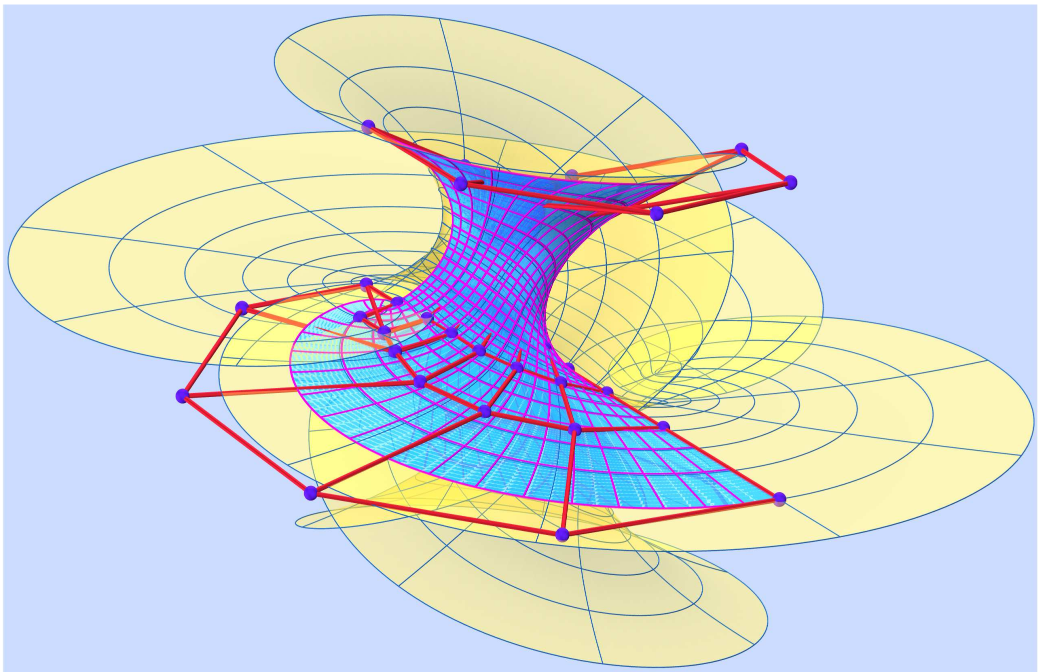

Figure 2.5: A part of a minimal surface described in Thm. 2.1 is defined by its control structure.

The following fact is worth to be noted and elementary to verify:

Lemma 2.1. *The curves of constant Gaussian curvature on Müller's surface are the ellipses given in* (3) *with* $q \in \mathbb{R}^\star$.

Proof. From (3) we compute the Gaussian curvature function on Müller's surface and arive at

$$K(q,t) = -\frac{4q^2}{(1+q^4)^2}$$

which is obviously independent of t, and thus, constant along each ellipse for the corresponding fixed $q \in \mathbb{R}^\star$. □

As a consequence of Lemma 2.1 and Theorem 2.1, we can formulate

Theorem 2.5. *The minimal surfaces* (8) *touching Müller's surface* (1) *along the ellipses* (3) *agree with the minimal surfaces that touch Müller's surface* (1) *along the curves of constant Gaussian curvature.*

A similar result holds true for minimal surfaces tangent to orthogonal hyperbolic paraboloids, see [7]: In this case the curves of constant Gaussian curvature are harmonic oscillation curves and the minimal surfaces are of algebraic degree 30, of class 10, and admit rational parametrizations.

3 The associate family

The parametrizations (8) of the minimal surfaces tangent to Müller's surface (1) are obtained by extracting the real part of the isotropic curve (9). The imaginary part of $\varphi(t)$ (with $t = u + \mathrm{i}v$) is given by

$$\mathbf{f}^{\perp}(u, v) = \frac{1}{6pq}\begin{pmatrix} -6pq^2\mathrm{s}_u\mathrm{S}_v - 3(p^2+q^4)\mathrm{s}_u\mathrm{C}_v - \mathrm{s}_{3u}\mathrm{C}_{3v} \\ 6pq^2\mathrm{c}_u\mathrm{S}_v - 3(p^2+q^4)\mathrm{c}_u\mathrm{C}_v + \mathrm{c}_{3u}\mathrm{C}_{3v} \\ -6(q^2\mathrm{s}_u\mathrm{C}_v + p\mathrm{s}_u\mathrm{S}_v) \end{pmatrix}. \tag{12}$$

Ofcourse, the real algebraic, and indeed rational surfaces parametrized by (12) are minimal surfaces. They are the *adjoint* minimal surfaces to (8). The family of *associate minimal surface* containing (8) and (12) is obtained as the real part of the one-parameter family of isotropic curves

$$\mathbf{f}(u, v, \tau) = \mathfrak{Re}\left(\mathrm{e}^{\mathrm{i}\tau}\varphi(t)\right) = \mathrm{c}_\tau\mathbf{f}(u, v) + \mathrm{s}_\tau\mathbf{f}^{\perp}(u, v), \quad \tau \in \mathrm{S}^1. \tag{13}$$

Now, we are able to prove the following

Theorem 3.1. *For any $q \in \mathbb{R}^\star$, the one-parameter family of associate minimal surfaces described by* (13) *consists of rational minimal surfaces of algebraic degree 24 and class 42.*

Proof. Independent of the choice of q, we can say: The minimal surfaces parametrized by (12) are rational, since the trigonometric and the hyperbolic functions showing up in the parametrization can be replaced by their rational equivalents. Subsequent to the reparametrization, we can eliminate the parameters and find an algebraic equation of degree 24. The class of the adjoint surfaces is 42 as it is the case with (8), since the isotropic curves corresponding to (12) has the same algebraic properties. From (13) it is clear that the degree and the class of all surfaces in the family of associate minimal surfaces agrees with that of $\mathbf{f}$ and $\mathbf{f}^{\perp}$, because for any $\tau \in \mathrm{S}^1$, (13) is just a linear combination of both. ☐

Actually, there are two one-parameter families of rational minimal surfaces associated to Müller's surface (1), *i.e.*, a one-parameter family of minimal surfaces related to each ellipse (from the one-parameter family of ellipses) on Müller's surface. It is not worth to mention that this construction of rational minimal surfaces works well for each of the cubic surfaces mentioned in the very beginning of the present paper. So, there is a five-parameter family of rational minimal surfaces sharing the algebraic properties with (8).

4 Some algebraic properties

4.1 Self-intersections

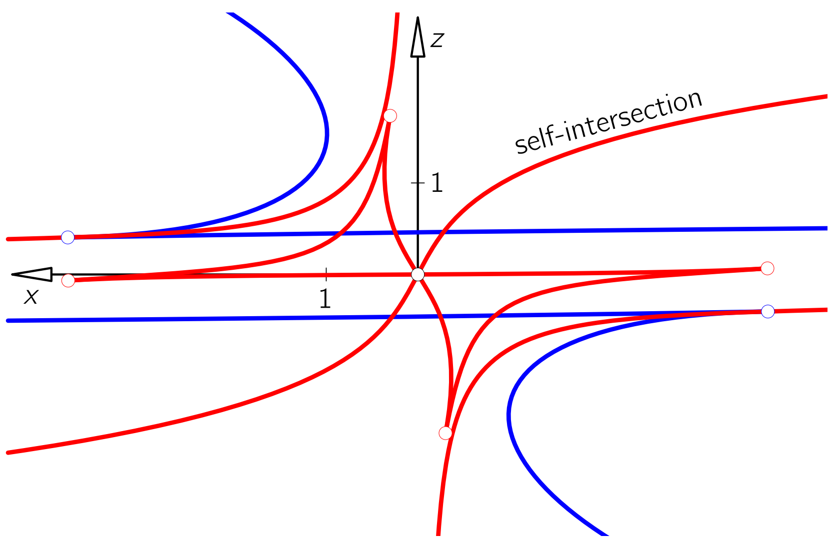

Figure 4.6: The self-intersection (red) of the minimal surface (with $q = 4/3$) is a part of the intersection with the plane $y = 0$.

Since the initial data (γ, ν) is symmertic with respect to the plane $y = 0$, the minimal surfaces (8) show the same symmetries. Thus, the plane $y = 0$ carries one part of the self-intersection of the minimal surfaces. Fig. 4.6 shows the intersection of the minimal surface (8) (with $q = 4/3$) with the plane $y = 0$. This planar cycle is of algebraic degree 24 and consists of a sextic (with multiplicity one, shown in blue) which is not part of the self-intersection. The curve shown in red is of degree nine and has multiplicity two as the planar intersection of (8). Therefore, it is part of the self-intersection of the minimal surface. This nonic has triple-points on the x-axis and the z-axis (at infinity) and a four-fold point at the origin. Further there are four ordinary cusps on the curve, two of which are located on the (non-singular) sextic and are branch points of the minmal surface. Since the nonic is a planar intersection of the minimal surface, it is no surprose that it is a rational curve.

In the plane $x = 0$, we can only find the y-axis as a part of the self-intersection which is also contained in the plane $z = 0$.

4.2 The curve at infinity

The minimal surfaces (8) intersect the ideal plane along the cycle

$$z^{18}(x^2 + y^2)^3 = 0$$

independent of the choice of $q \in \mathbb{R}^\star$. This tells us that the ideal line of all planes parallel to $z = 0$ is of multiplicity 18 and the pair of complex conjugate ideal lines $y = \pm ix$ through the ideal point of the z-axis is three-fold. The fact that the curve at infinity of these minimal surfaces degenerates completely, *i.e.*, it splits off into a finite number of lines, is in accordance with an old result by Lie, see [3].

Because of the high multiplicities of the components in the plane at infinity, these lines also contribute to the self-intersection(s).

4.3 The intersection of Müller's surface

The intersection of Müller's surface (1) with the minimal surfaces (8) along its ellipses contains:

1. the six-fold y-axis (of the underlying Cartesian coordinate system),
2. the two-fold ellipse γ (3) (for any fixed $q \in \mathbb{R}^\star$),
3. the three-fold pair of ideal lines of the complex conjugate pair of planes $x^2 + y^2 = 0$, and
4. the eighteen-fold ideal line of all horizontal planes (parallel to $z = 0$).

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