Packing COXETER honeycombs with sequences of spheres

Boris Odehnal* and Jenő Szirmai†

May 22, 2012

Abstract

In [16] packings with geometric sequences of spheres to regular polyhedra in the Euclidean d-space with $d \geq 2$ have been introduced. The packing densities depending on the dimension and the type of polyhedron have been determined. Some of these packings can also be performed in d-dimensional Euclidean space. In this paper we generalize the above notion of sphere sequences packed into hyperbolic COXETER honeycombs. We describe a method that determines the data and the density of each considered non-congruent sphere packing to every COXETER tiling. Moreover, we apply our method to some 2- and 3-dimensional totally asymptotic honeycombs.

Keywords: packing, hyperbolic space, honeycomb, sphere, density.
Mathematics Subject Classification (AMS 2010): 52C17, 52C26.

1 Introduction

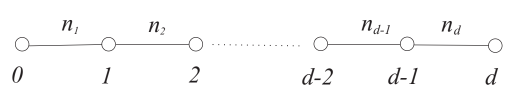
The regular COXETER tilings or regular COXETER honeycombs $\mathcal{P}$ are partitionings of hyperbolic space $\mathbb{H}^d$ with $d \geq 2$ into congruent regular polytopes. A honeycomb with cells congruent to a given regular polyhedron P exists if and only if the dihedral angle of P is a submultiple of 2π. All honeycombs for $d = 3$ with bounded cells were first found by SCHLEGEL in 1883, cf. [18], those with unbounded cells by H.S.M. COXETER in his famous article [5].

*Vienna University of Technology, Institute of Discrete Mathematics and Geometry, A-1040 Vienna, Austria; Email: boris@geometrie.tuwien.ac.at

†Budapest University of Technology and Economics, Institute of Mathematics, Department of Geometry, H-1521 Budapest, Hungary; Email: szirmai@math.bme.hu

Another description of honeycombs involves the study of their symmetry groups. If $\mathcal{P}$ is such a honeycomb, then any hyperbolic motion taking one cell into another takes the whole honeycomb into itself. The symmetry group of a honeycomb is denoted by $\operatorname{sym}\mathcal{P}$. The characteristic simplex $\mathcal{F}$ of any cell $P \subset \mathcal{P}$ is a fundamental domain of the group $\operatorname{sym}\mathcal{P}$ generated by reflections in its facets (hyperfaces).

The scheme of a regular polytope P is a weighted graph, characterizing $P \subset \mathbb{H}^d$ up to congruence, in which the nodes, numbered by $0, 1, \ldots, d$ correspond to the bounding hyperplanes of $\mathcal{F}$. Two nodes are joined by an edge if the corresponding hyperplanes are not orthogonal. The ordered set of weights $\{n_1, \ldots, n_{d-1}\}$ is the Schläfli symbol of P, and n_d is related to the dihedral angle of P, since this angle equals $\frac{2\pi}{n_d}$. $\mathcal{F}$ is the Coxeter simplex with the scheme



.

The ordered set $(n_1, \ldots, n_d)$ is said to be the Schläfli symbol of the honeycomb $\mathcal{P}$. To every scheme there is a corresponding symmetric matrix

$$(a^{ij}) \in \mathbb{R}^{(d+1)\times(d+1)} \quad \text{with} \quad \begin{array}{rcll} a^{ij} &=& 1 & \text{if} \quad i = j, \\ a^{ij} &=& -\cos\frac{\pi}{n_{ij}} & \text{if} \quad i \neq j, \end{array} \tag{1}$$

where $\frac{\pi}{n_{ij}}$ is the angle between the i-th and j-th facet of $\mathcal{F}$. Additionally we define $n_k =: n_{k-1,k}$. Reversing the numbering of nodes in the scheme of $\mathcal{P}$ but keeping the weights, leads to the so-called dual honeycomb $\mathcal{P}^*$ whose symmetry group coincides with $\operatorname{sym}\mathcal{P}$.

In this paper we shall study the sphere packings belonging to the regular hyperbolic Coxeter tilings as a generalization of [16]. We extend the notion of sphere sequences to hyperbolic space such that these are constructed in a geometrically natural way. Further we compute the density of the thereby obtained packings of $\mathbb{H}^d$ and develope a method to determine the global densities of these packings for all regular Coxeter tilings given in [5]. Moreover we introduce the notion of the m-th layer of spheres ($m = 1, 2, \ldots$) and the respective density which give insight into the behaviour of the global density. Finally we apply our method to some 2- and 3-dimensional totally asymptotic honeycombs.

In the following we use the notation of Coxeter for the d-dimensional honeycombs. In this work we study the following totally asymptotic honeycombs:

1. 2-dimensional totally asymptotic tilings with SCHLÄFLI symbol

$$\{n_1, n_2\} = \{n_1, \infty\}, \quad 3 \leq n_1 < \infty, \quad n_1 \in \mathbb{Z} \tag{2}$$

2. 3-dimensional totally asymptotic tilings with SCHLÄFLI symbol

$$\{3,3,6\}, \quad \{4,3,6\}, \quad \{5,3,6\}, \quad \{3,4,4\}. \tag{3}$$

We note that in four- and five-dimensional hyperbolic space there exist similar tilings with COXETER-SCHLÄFLI notations:

$$\{n_1, n_2, n_3, n_4\} = \{3,4,3,4\}, \quad \{n_1, n_2, n_3, n_4, n_5\} = \{3,3,3,4,3\}.$$

The paper is organized as follows: In Section 2 we summarize some basic facts from projective and non-Euclidean geometry, cf. [7]. Then in Section 3 we shall consider the COXETER honeycombs and introduce the notion of the corresponding sequences of the spheres and compute their densities. This enables us to pack sequences of spheres determined in a geometrically natural way into honeycombs. In Section 4 we derive the packing densities of the afore introduced non-congruent sphere packings and summarize our results for some totally asymptotic honeycombs in tables. Our method applies to all other COXETER honeycombs as well. Numerical computations were carried out by Maple.

2 The projective model of $\mathbb{H}^d$

We use homogeneous coordinates $\mathbf{x} = (x^0 : x^1 : \ldots : x^d)$ and $\mathbf{a} = (a_0 : a_1 : \ldots : a_n)$ in order to represent points X as well as hyperplanes α in projective d-space $\mathbb{P}^d$. Sometimes we write $X = \mathbf{x}\mathbb{R}$ in order to express that the point X is determined by a certain vector $\mathbf{x} \in \mathbb{R}^{d+1}$ and its non-trivial scalar multiples, similarly $\alpha = \mathbf{a}\mathbb{R}$ for hyperplanes. A point $X = \mathbf{x}\mathbb{R}$ and a hyperplane $\alpha = \mathbf{a}\mathbb{R}$ are incident if $\mathbf{x} \cdot \mathbf{a}^T = 0$ with $\cdot$ denoting usual matrix multiplication.

The CAYLEY-KLEIN model of the hyperbolic d-space is realized as the interior of an oval quadric Q, the absolute quadric of the hyperbolic space $\mathbb{H}^d$. Thus $\mathbb{R}^{d+1}$ is equipped with the bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0 y^0 + x^1 y^1 + \cdots + x^d y^d \tag{4}$$

and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ is an equation of Q. The proper points $X = \mathbf{x}\mathbb{R}$ of $\mathbb{H}^d$ are characterized by $\langle \mathbf{x}, \mathbf{x} \rangle < 0$, *i.e.*, they are interior points of Q. The points on the boundary $\partial\mathbb{H}^d = Q \subset \mathbb{P}^d$ are called *points at infinity* or *ideal points* and points X with $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ are lying outside $\partial\mathbb{H}^d$ and are therefore called *outer points* of $\mathbb{H}^d$. Points X and Y with $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ are conjugate with respect to Q. The set of all points Y which are conjugate to X form a hyperplane $H \subset \mathbb{P}^d$. For (4) is non-degenerate, the absolute polarity is too.

Hyperbolic distances and angles can be computed with help of the bilinear form (4), see [7]. The distance $d(X, Y)$ of two points $X = \mathbf{x}\mathbb{R}$ and $Y = \mathbf{y}\mathbb{R}$ is given by

$$\cosh d(X, Y) = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \tag{5}$$

Note that the right hand side of (5) always evaluates to a positive real number, for X and Y are interior points of Q. Further we find the angle $\varphi(\alpha, \beta)$ enclosed by two hyperplanes $\alpha = \mathbf{a}\mathbb{R}$ and $\beta = \mathbf{b}\mathbb{R}$, respectively, with

$$\cos \varphi(\alpha, \beta) = \frac{-\langle \mathbf{a}, \mathbf{b} \rangle}{\sqrt{\langle \mathbf{a}, \mathbf{a} \rangle \langle \mathbf{b}, \mathbf{b} \rangle}}. \tag{6}$$

Later we need the footpoint $F = \mathbf{f}\mathbb{R}$ of the perpendicle droppped from a point X to a hyperplane α. In terms of coordinates it can be computed via

$$\mathbf{f} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a}. \tag{7}$$

Let $P \subset \mathbb{H}^d$ be a polyhedron bounded by hyperplanes H^i given by the respective coordinate vectors $\mathbf{b}^i$ with $\langle \mathbf{b}^i, \mathbf{b}^i \rangle = 1$ pointing to the interior of P. Let us further assume that P is an acute-angled polyhedron and the vertices are proper points or at least lie on Q. Then the GRAM matrix

$$G(P) = (\langle \mathbf{b}_i, \mathbf{b}_j \rangle) \quad \text{with} \quad i, j \in \{0, 1, \ldots d\} \tag{8}$$

of the normal vectors of P's faces is an indecomposable symmetric matrix of signature $(-, +, \ldots, +)$ like (4). Since $\langle \mathbf{b}^i, \mathbf{b}^i \rangle = 1$ its entries have the following geometric meaning:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = \begin{cases} 0 & \text{if } H^i \perp H^j, \\ -\cos \alpha^{ij} & \text{if } H^i, H^j \text{ intersect in P at angle } \alpha^{ij}, \\ -1 & \text{if } H^i, H^j \text{ are parallel in hyperbolic sense}, \\ -\cosh l^{ij} & \text{if } H^i, H^j \text{ have a common normal of length } l^{ij}. \end{cases}$$

Frequently we use the following object, cf. [9]:

Definition 2.1 *An orthoscheme $\mathcal{O}$ in $\mathbb{H}^d$ is a simplex bounded by $d+1$ hyperplanes $H^0, \dots, H^d$ such that*

$$H^i \perp H^j, \quad \text{for} \quad j \neq i-1, i, i+1.$$

A planar orthoscheme is a right-angled triangle, whose area can be expressed by the well known defect formula. For an orthoscheme we denote the $(d-1)$-hyperface opposite to the vertex A_i by H^i with $0 \leq i \leq d$. An orthoscheme $\mathcal{O}$ has d dihedral angles which are not right angles. Let α^{ij} denote the dihedral angle between the faces H^i and H^j. Then we have

$$\alpha^{ij} = \frac{\pi}{2}, \quad \text{if} \quad 0 \leq i < j-1 \leq d.$$

The remaining d dihedral angles $\alpha^{i,i+1}$, with $0 \leq i \leq d-1$ are called the *essential angles* of $\mathcal{O}$. The initial and final vertices, A_0 and A_d of the orthogonal edge-path

$$\bigcup_{i=0}^{d-1} A_i A_{i+1}$$

are called *principal vertices* of the orthoscheme.

In order to compute volumes of polyhedra we use a formula, which is due to Lobachevsky, see [8]:

Theorem 2.2 *The volume of a three-dimensional hyperbolic orthoscheme $\mathcal{O} \subset \mathbb{H}^3$ is expressed in terms of the dihedral angles α_{01}, α_{12}, α_{23} by*

$$\begin{aligned}\mathrm{vol}(\mathcal{O}) = \tfrac{1}{4} \big(&\mathfrak{L}(\alpha_{01}+\theta) - \mathfrak{L}(\alpha_{01}-\theta) + \mathfrak{L}(\alpha_{23}+\theta) - \mathfrak{L}(\alpha_{23}-\theta) + \\ &+ \mathfrak{L}(\tfrac{\pi}{2}+\alpha_{12}-\theta) + 2\mathfrak{L}(\tfrac{\pi}{2}-\theta) + \mathfrak{L}(\tfrac{\pi}{2}-\alpha_{12}-\theta)\big),\end{aligned} \tag{9}$$

where $\theta \in (0, \frac{\pi}{2})$ equals

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha_{12} - \sin^2\alpha_{01}\sin^2\alpha_{23}}}{\cos\alpha_{01}\cos\alpha_{23}} \tag{10}$$

and $\mathfrak{L}(x) := -\int\limits_0^x \log|2\sin t| dt$ is the Lobachevsky *function.*

Further results on volumes of orthoschemes and simplices in $\mathbb{H}^d$ can be found in [8, 9, 10, 11] for arbitrary d.

Here and in the following the characteristic simplex $\mathcal{F}$ of any honeycomb $\mathcal{P}$ with Schlegel symbol $\{n_1, \dots, n_d\}$ is an orthoscheme. As defined in

(8), the GRAM matrix $G(P)$ of an orthoscheme with faces $H^i = \mathbf{b}^i\mathbb{R}$ with $i \in \{1,\dots,d+1\}$ is the so-called COXETER-SCHLÄFLI matrix of $\mathcal{F}$ with parameters $n_1,\dots,n_d$. It reads

$$\begin{pmatrix} 1 & -\cos\frac{\pi}{n_1} & 0 & & \cdots & 0 \\ -\cos\frac{\pi}{n_1} & 1 & -\cos\frac{\pi}{n_2} & & \cdots & 0 \\ 0 & -\cos\frac{\pi}{n_2} & 1 & -\cos\frac{\pi}{n_3} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & -\cos\frac{\pi}{n_{d-2}} & 1 & -\cos\frac{\pi}{n_{d-1}} & 0 \\ 0 & \cdots & & -\cos\frac{\pi}{n_{d-1}} & 1 & -\cos\frac{\pi}{n_d} \\ 0 & \cdots & & 0 & -\cos\frac{\pi}{n_d} & 1 \end{pmatrix}. \tag{11}$$

If we invert the COXETER-SCHLÄFLI matrix from (11) of an orthoscheme $\mathcal{F}$ we get a symmetric matrix (a_{ij}). According to [13] we can express any distance d_{ij} between two vertices $V_i = \mathbf{b}_i\mathbb{R}$ and $V_j = \mathbf{b}_j\mathbb{R}$ of $\mathcal{F}$ by

$$\cosh d_{ij} = \frac{-a_{ij}}{\sqrt{a_{ii}a_{jj}}}.$$

3 Sequences of spheres for honeycombs

The use of the CAYLEY-KLEIN model of hyperbolic space has a major advantage: Centering a regular polyhedron in the center of the absolute quadric Q it appears as a regular polyhedron from the Euclidean point of view. This simplifies computations.

3.1 Cells with proper centre

COXETER honeycombs exist in $\mathbb{H}^d$ up to five dimensions. Thus we can try to pack COXETER honeycombs in $\mathbb{H}^d$ with $2 \le d \le 5$. Let us consider a d-dimensional COXETER honeycomb with the classical notation $\{n_1,\dots,n_d\}$ which shall henceforth be denoted by $\mathcal{H}_{n_1,\dots,n_d}$. Now choose an arbitrary cell $\mathcal{G}_0$ of the considered honeycomb and assume that the center of this cell coincides with the center of our model, *i.e.*, the point $O = (1:0:\ldots:0)$. So the vertices of the cell form a regular polyhedron even in Euclidean sense. If the honeycomb is totally asymptotic then its vertices lie on the absolute quadric Q. Now we can impose a projective coordinate system on $\mathcal{H}$ such that an arbitrary vertex of the cell has coordinates $E_d = (1:0:\ldots:e_d)$. First we introduce a sequence of spheres corresponding to the vertex E_d and then this notion can be extended in natural way to all other vertices of $\mathcal{H}$. We proceed in the following way:

1. We consider the inscribed sphere B_0 of the cell $\mathcal{G}_0$ and let K_0 be the common point with the half line OE_d. The tangential hyperplane of B_0 at K_0 shall be denoted by Π_0.

2. The common part of Π_0 with $\mathcal{G}_0$ is a regular $(d-1)$-dimensional polyhedron corresponding to the vertex figure of $\mathcal{H}_{n_1 \dots, n_d}$ which together with the vertex E_d forms a d-dimensional pyramid, say $\mathcal{G}_1$.

3.2 Cells with non-proper centres

In this case a cell of the honeycombs is an infinite polyhedron whose center lies on the absolute quadric and its inscribed sphere is a horosphere with the same center.

We choose an arbitrary characteristic simplex of the considered honeycomb which is denoted by $\mathcal{R}_0$. Without loss of generality, we can assume that the centre of this cell is $O = (1 : 1 : \dots : 0)$ and the coordinates of the vertex E_d are $(1 : 0 : \dots : e_d)$. Here we note that the vertex can also lie on the absolute quadric, *i.e.*, $e_d = 1$. Now we go the following way:

1. We consider the inscribed horosphere B_0 of the infinite cell $\mathcal{G}_0$ and let its common point with the half line OE_d be K_0. We denote the tangent hyperplane to B_0 in K_0 by Π_0.

2. The common part of the tangent hyperplane with $\mathcal{G}_0$ is a regular $(d-1)$-dimensional polyhedron P_1 corresponding to the vertex figure of $\mathcal{H}_{n_1 \dots, n_d}$ which together with the vertex E_d forms a d-dimensional pyramid denoted by $\mathcal{G}_1$.

From now on we treat the two types of cells together in a uniform way:

3. We consider the barycentric subdivision of P_1. We choose an arbitrary $(d-1)$-dimensional characteristic simplex $\mathcal{S}^{d-1}$ end erect a pyramid with vertex E_d on it. This yields an orthoscheme $\mathcal{R}_1$, see Figure 1. The d-dimensional pyramid $\mathcal{G}_1$ can be built up by congruent copies of $\mathcal{R}_1$.

   Figure 1 shows two 3-dimensional cases where $\mathcal{G}_i$ is either a simply asymptotic tetrahedron $(H_0^i, H_1^i, H_2^i, E_3)$ or a simply asymptotic pyramid with regular quadrilateral base $(H_0^i, H_1^i, H_2^i, H_3^i, E_3)$. The corresponding orthoschemes R_i with vertices T_0^i, T_1^i, T_2^i, T_3^i are also displayed.

4. The dihedral angles of orthoschemes $\mathcal{R}_1$ can be determined by the data of the considered honeycombs using its vertex figure and by Eq. (6). Thus we can determine the COXETER -SCHLÄFLI matrix (11) from (b_1^{kl}) and any metric data of $\mathcal{R}_1$. The volume vol$(\mathcal{R}_1)$ of the orthoscheme $\mathcal{R}_1$ can be determined with the formula given in Th. 2.2 in $\mathbb{H}^3$ and in $\mathbb{H}^d$ by formulae given in [9, 10] if $d > 3$.

5. The radius ρ_1 of the insphere of the pyramid $\mathcal{G}_1$ is equal to the radius of the sphere centred on the line segment K_1E_d and touching the faces of $\mathcal{R}_1$ that do not contain the edge K_1E_d. The radius ρ_1 is determined by

$$\sinh \rho_1 = \frac{\langle \mathbf{x}, \mathbf{b}^j \rangle}{\sqrt{-\langle \mathbf{x}, \mathbf{x} \rangle}} = \frac{x^j}{\sqrt{-a_{rs}x^r x^s}} = \frac{1}{\sqrt{-\sum\limits_{r,s \neq 0,1} a_{rs}}}, \quad j \neq 0, 1, \quad (12)$$

where $\mathbf{x}$ is the centre of the sphere lying on the faces 0 and 1, *i.e.*, in the edge K_1E_d of $\mathcal{R}_1$ and (a^1_{rs}) is the inverse of the matrix (b_1^{kl}).

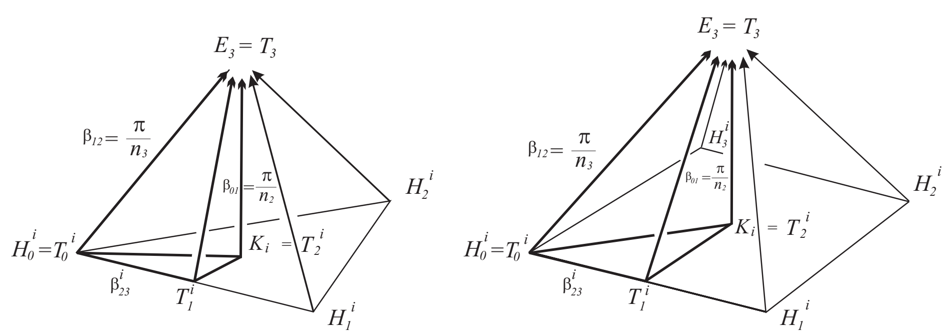


Figure 1: Orthoschemes in pyramids on subsimplices.

6. We consider the inscribed ball B_1 of the cell $\mathcal{G}_1$ of radius ρ_1 with centre O_1 and let K_1 be its common point with hyperbolic half line O_1E_d.

7. Continuing this procedure we get a sphere sequence with centres O_i, where $i = 0, 1, 2, \dots$, first at a certain vertex, then at each vertex of the considered cell, and finally, at every cell of the considered honeycomb. Thus we obtain a non-congruent ball packing $\mathcal{B}_{n_1,\dots,n_d}$ in hyperbolic d-space.

8. We distinguish two cases at the definition of the density of the above ball packing. Definitions are given below.

We use two definitions of densities according to the type of honeycombs we are dealing with:

Definition 3.1 *Let $\{n_1 \dots, n_d\}$ be a d-dimensional honeycomb where the cells have proper centres and the number of vertices of a cell is n. Then the density δ of the packing equals*

$$\delta(\mathcal{B}_{n_1,\dots,n_d}) := \frac{n \sum_{i=0}^{\infty} \mathrm{vol}(B_i)}{\mathrm{vol}(\mathcal{G}_0)}, \tag{13}$$

where B_i are the elements of the sphere sequence.

Definition 3.2 *Let $\{n_1 \dots, n_d\}$ be a d-dimensional honeycomb and let $\mathcal{R}_0$ be its characteristic simplex, where the centres of the cells lie on the absolute quadric Q. Then the density of the packing equals*

$$\delta(\mathcal{B}_{n_1,\dots,n_d}) := \frac{\sum_{i=0}^{\infty} \mathrm{vol}(B_i \cap \mathcal{R}_i)}{\mathrm{vol}(\mathcal{R}_0)}, \tag{14}$$

where B_0 is the inscribed horosphere of $\mathcal{G}_0$ and B_i and $\mathcal{R}_i$ are elements of the sphere sequence and the above defined orthoschemes.

The volumes of our cells and characteristic simplices are finite. So we can formulate the following result:

Theorem 3.3 *The series $\sum_{i=0}^{\infty} \mathrm{vol}(B_i)$ and $\sum_{i=0}^{\infty} \mathrm{vol}(B_i \cap \mathcal{R}_i)$ with non-negative terms converge because their partial sums are bounded.*

Now we have to determine the densities to each regular Coxeter honeycomb. For that purpose we introduce the notion of the m-th layer of spheres with $m = 1, 2, \dots$ and its density:

Definition 3.4 *Let $\{n_1 \dots, n_d\}$ be a d-dimensional honeycomb and $\mathcal{G}_0$ one of its cells. Then*

$$C_m := \{X \in \mathcal{G}_m \setminus \mathcal{G}_{m+1} \quad \text{with} \quad m \geq 1\}$$

is the m-th cell belonging to the vertex E_d.

Extending the above notion to every vertex of $\mathcal{G}_0$ we obtain the m-th layer and spheres, *i.e.*, the union of all spheres lying in the cells of the m-th layer. We study the density of the ball packing in the m-th layer and use the following definition:

Definition 3.5 *The layer density δ_m of the packing in the m-th layer is computed as*

$$\delta_m := \frac{\mathrm{vol}(B_m)}{\mathrm{vol}(C_m)}, \quad \text{with} \quad m \geq 1, \tag{15}$$

where B_m is the sphere of radius ρ_m about O_m defined in points 6 and 7.

4 Packings in special tilings

4.1 {3,3,6}, tetrahedron tiling

In this section we apply our method to the {3,3,6} regular COXETER honeycomb, see Figure 2. In this case the cells of the regular tiling have ideal vertices and proper centers. Figure 2 shows a cell (E_0, E_1, E_2, E_3) of this totally asymptotic regular COXETER tiling.

We impose a projective coordinate system on the tetrahedron such that the origin coincides with the centre of the tetrahedron and the coordinates of the vertices are

$$E_0 = (3 : -\sqrt{6} : -\sqrt{2} : -1), \quad E_1 = (3 : \sqrt{6} : -\sqrt{2} : -1),$$
$$E_2 = (3 : 0 : 2\sqrt{3} : -1), \quad E_3 = (1 : 0 : 0 : 1).$$

We consider the sphere B_0 inscribed to the cell $\mathcal{G}_0$ and let K_0 be its common point with the half line OE_3. Again Π_0 denotes the tangent plane of B_0 in K_0. The common part of Π_0 with $\mathcal{G}_0$ is a regular triangle corresponding to the vertex figure of $\mathcal{H}_{336}$. Together with the vertex E_3 it forms a 3-dimensional pyramid $\mathcal{G}_1$ (see Figure 1 and Figure 2).

It is clear that the coordinates of the hyperplane carrrying the face (E_0, E_1, E_3) are $(1 : 0 : 2\sqrt{2} : -1)^T$. The radius of the insphere of $\mathcal{G}_0$ is equal to the hyperbolic distance between O and the face (E_0, E_1, E_3) and can thus be calculated with formulae given in Eq. (7) and Eq. (5) and we find $\rho_0 = \operatorname{artanh} \frac{1}{3}$.

The coordinates of the plane Π_0 are $(1 : 0 : 0 : -3)^T$. The common part of Π_0 with $\mathcal{G}_0$ is a regular triangle P_1 corresponding to the vertex figure of $\mathcal{H}_{336}$ which forms a pyramid $\mathcal{G}_1$ with vertex E_3.

We consider the barycentric subdivision of P_1 and we choose an arbitrary characteristic simplex which forms an orthoscheme $\mathcal{R}_1$ with E_3. In Figure 1 the general case is illustrated. $\mathcal{G}_1$ is a simply asymptotic tetrahedron $(H_0^1, H_1^1, H_2^1, E_3)$ and the corresponding orthoscheme is $\mathcal{R}_1 = (T_0^1, T_1^1, T_2^1, T_3)$. The dihedral angles of $\mathcal{R}_1$ at the edges $T_3 \vee T_2^1$ and $T_3 \vee T_0^1$ are $\beta_{01} = \frac{\pi}{3}$ and $\beta_{12} = \frac{\pi}{6}$. The dihedral angle β_{23} can be calculated via coordinates of the planes Π_0 and $E_0 \vee E_1 \vee E_3$ with using Eq. (6). All metric data of $\mathcal{R}_1$ can be extracted from the COXETER-SCHLÄFLI matrix of $\mathcal{R}_1$

$$(b_1^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{3} & 0 & 0 \\ -\cos\frac{\pi}{3} & 1 & -\cos\frac{\pi}{6} & 0 \\ 0 & -\cos\frac{\pi}{6} & 1 & -\cos\beta_{23}^1 \\ 0 & 0 & -\cos\beta_{23}^1 & 1 \end{pmatrix}, \qquad (16)$$

where $\beta^1_{23} \approx 1.31811607$. We obtain the volume of $\mathcal{R}_1$ and $\mathcal{G}_1$ according to Th. 2.2 as

$$\mathrm{vol}(\mathcal{R}_1) \approx 0.02894980 \quad \text{and} \quad \mathrm{vol}(\mathcal{G}_1) \approx 0.17369879.$$

With Eq. (12) we compute the radius ρ_1 of the insphere of $\mathcal{G}_1$. Together with the volume formula for the hyperbolic sphere this yields

$$\rho_1 \approx 0.20273255 \quad \text{and} \quad \mathrm{vol}(\mathcal{B}_1) \approx 0.03519073.$$

The coordinates of K_1 are $(1:0:0:\tanh(\rho_0+2\rho_1)) \approx (1:0:0:0.63636363)$ and so the coordinates of Π_1 are $(1:0:0:-1/\tanh(\rho_0+2\rho_1))^T$.

The dihedral angles of $\mathcal{R}_2$ are $\beta^2_{01} = \frac{\pi}{3}$, $\beta^2_{12} = \frac{\pi}{6}$ and $\beta^2_{23} \approx 1.40334825$. We obtain the volumes of $\mathcal{R}_2$ and $\mathcal{G}_2$ in a way similar to the above case:

$$\mathrm{vol}(\mathcal{R}_2) \approx 0.01237801 \quad \text{and} \quad \mathrm{vol}(\mathcal{G}_2) \approx 0.07426807.$$

The radius ρ_2 of the insphere $\mathcal{B}_2$ of $\mathcal{G}_2$ and the respective volume evaluate to

$$\rho_2 \approx 0.14384104 \quad \text{and} \quad \mathrm{vol}(\mathcal{B}_2) \approx 0.01251797.$$

Applying Def. 3.5 to the sphere packing in the first layer we arrive at

$$\delta_1 := \frac{\mathrm{vol}(B_1)}{\mathrm{vol}(C_1)} \approx 0.35392216. \tag{17}$$

Repeating this procedure we can determine the densities in the m-th layer and the global density of the above sphere packing.

We use Def. 3.1 and the density of the considered sphere packing is

$$\delta(\mathcal{B}_{336}) := \frac{4\sum_{i=0}^{\infty} \mathrm{vol}(B_i)}{\mathrm{vol}(\mathcal{G}_0)} \approx 0.42809897.$$

Similar to the above case we can apply the method to all types of honeycombs and we have summarized the results for some 2- and 3-dimensional honeycombs in Table 1.

Our computations produce the following result:

Corollary 4.1 *The radii ρ_i of the spheres $\mathcal{B}_i$ in the $\{3,3,6\}$ tiling are given by*

$$1/\sinh\rho_i = \sqrt{4(i+1)(i+2)}.$$

	$\{3,\infty\}$				$\{4,\infty\}$			
m	ρ_m	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{3,\infty})$	ρ_m	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{4,\infty})$
0	0.549	0.9720	—	0.309	0.881	0.2603	—	0.414
1	0.255	0.2061	0.7440	0.506	0.301	0.2882	0.7257	0.598
2	0.168	0.0891	0.7682	0.591	0.187	0.1010	0.7640	0.668
10	0.045	0.0065	0.7842	0.740	0.047	0.0069	0.7841	0.779
20	0.024	0.0018	0.7851	0.770	0.024	0.0018	0.7851	0.799
30	0.016	0.0008	0.7852	0.781	0.016	0.0008	0.7852	0.807
50	0.009	0.0003	0.7853	0.791	0.009	0.0003	0.7853	0.814
∞	0	0	0.7854	0.794	0	0	0.7854	0.817
	$\{3,3,6\}$				$\{5,3,6\}$			
m	$\sinh^2\rho_m$	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{336})$	ρ_m	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{536})$
1	1/24	0.03519	0.3539	0.315	0.317	0.13629	0.2645	0.4599
2	1/48	0.01252	0.3791	0.364	0.193	0.03011	0.3590	0.4891
10	1/528	0.00035	0.4009	0.421	0.047	0.00044	0.4006	0.5128
20	1/1848	0.00005	0.4025	0.426	0.024	0.00006	0.4024	0.5143
30	1/3968	0.00002	0.4028	0.427	0.016	0.00002	0.4028	0.5146
∞	0	0	0.4031	0.428	0	0	0.4030	0.5148
	$\{4,3,6\}$				$\{3,4,4\}$			
m	ρ_m	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{436})$	ρ_m	$\mathrm{vol}(B_m)$	δ_m	$\delta(\mathcal{B}_{344})$
0	0.659	1.30405	—	0.257	0.659	1.30405	—	0.3559
1	0.275	0.08810	0.3067	0.396	0.275	0.08810	0.4333	0.5002
2	0.176	0.02308	0.3665	0.432	0.176	0.02308	0.4879	0.5379
10	0.046	0.00041	0.4007	0.465	0.024	0.00007	0.5229	0.5745
20	0.024	0.00006	0.4024	0.467	0.024	0.00006	0.5231	0.5745
30	0.016	0.00002	0.4027	0.468	0.016	0.00002	0.5233	0.5750
∞	0	0	0.4030	0.468	0	0	0.5235	0.5754

Table 1: Radii, volumes, and densities of the $\{3,\infty\}$, $\{4,\infty\}$, $\{3,3,6\}$, $\{5,3,6\}$, $\{4,3,6\}$, and $\{3,4,4\}$ packing.

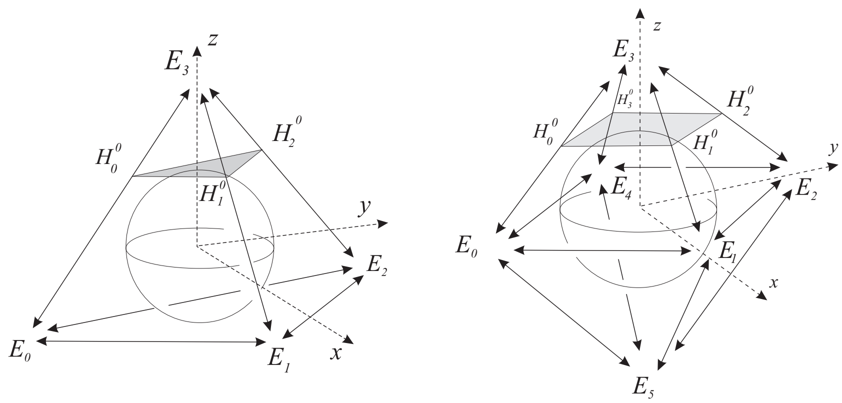


Figure 2: Totally asymptotic tetrahedron and octahedron with coordinate systems imposed on them.

References

[1] K. Bezdek: *Sphere packings revisited.* Eur. J. Comb. **27**/6 (2006), 864–883.

[2] J. Böhm, E. Hertel: *Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung.* Birkhäuser, Basel, 1981.

[3] K. Böröczky: *Packing of spheres in spaces of constant curvature.* Acta Math. Acad. Sci. Hung. **32** (1978), 243–261.

[4] K. Böröczky, A. Florian: *Über die dichteste Kugelpackung im hyperbolischen Raum.* Acta Math. Acad. Sci. Hung. **15** (1964), 237–245.

[5] H.S.M. Coxeter: *Regular honeycombs in hyperbolic space.* Proc. internat. Congr. Math. 1954 Amsterdam 3 (1956), 155–169.

[6] G. Fejes Tóth, G. Kuperberg, W. Kuperberg: *Highly saturated packings and reduced coverings.* Monatsh. Math. **125**/2 (1998), 127–145.

[7] O. Giering: *Vorlesungen über höhere Geometrie.* Vieweg, Stuttgart, 1981.

[8] R. Kellerhals: *On the volume of hyperbolic polyhedra.* Math. Ann. **285**/4 (1989), 541–569.

[9] R. Kellerhals: *On Schläfli's reduction formula.* Math. Z. **206**/2 (1991), 193–210.

[10] R. Kellerhals: *Volumes in hyperbolic 5-space.* Geom. Funct. Anal. **5**/4 (1995), 640–667.

[11] R. Kellerhals: *Ball packings in spaces of constant curvature and the simplicial density function.* J. Reine Angew. Math. **494** (1998), 189–203.

[12] T.R. Kozma, J. Szirmai: *Optimally dense packings for fully asymptotic* Coxeter *tilings by horoballs of different types.* Submitted to Monatshefte für Mathematik, 2011.

[13] E. Molnár: *Projective metrics and hyperbolic volume.* Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math. **32** (1989), 127–157.

[14] E. Molnár: *The projective interpretation of the eight 3-dimensional homogeneous geometries.* Beitr. Algebra Geom. **38**/2 (1997), 261–288.

[15] J. Milnor: *Geometry.* Collected papers, Publish or Perish, Vol **1**, 1994.

[16] B. Odehnal: *Packing with geometric sequences of spheres.* Technical report No. 218, Geometry Preprint series, Vienna University of Technology, June, 2011.

[17] C.A. Rogers: *Packing and Covering.* Cambridge University Press, Cambridge, 1964.

[18] V. Schlegel: *Theorie der homogen zusammengesetzten Raumgebilde.* Nova Acta Leop. Carol. **44** (1883), 343–459.

[19] J. Szirmai: *Horoball packings for the* Lambert*-cube tilings in the hyperbolic 3-space.* Beitr. Algebra Geom. **46**/ 1(2005), 43–60.

[20] J. Szirmai: *The optimal ball and horoball packings to the* Coxeter *honeycombs in the hyperbolic d-space.* Beitr. Algebra Geom. **48**/1 (2007), 35–47.

[21] J. Szirmai: *The densest geodesic ball packing by a type of Nil lattices.* Beitr. Algebra Geom. **48**/2 (2007), 383–397.