The Parallelian Quadrics of a Tetrahedron

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Abstract. The quadruple of planes parallel to the faces of a tetrahedron $\mathcal{T} = ABCD$ passing through a pivot point P meet the non-parallel edges of $\mathcal{T}$ in twelve points in total which turn out to lie on a single quadric $\mathcal{P}$. The triple of planes parallel to the pairs of opposite edges of $\mathcal{T}$ intersect the three quadruples of non-parallel edges in twelve points which lie on a single quadric $\mathcal{Q}$. In analogy to the planar parallelian conic of a triangle, we will call these quadrics the parallelian quadrics of the tetrahedron $\mathcal{T}$ with respect to the point P. We shall describe the sets of pivot points P such that $\mathcal{P}$ or $\mathcal{Q}$ are regular or parabolic. Moreover, we determine the surfaces filled by the vertices of singular parallelian quadrics.

Keywords: tetrahedron, parallelian, quadric, Cayley's nodal cubic

1 Introduction

The parallelians of a triangle $\Delta = ABC$ with respect to (shorthand: w.r.t.) a point P are those six intersection points of the three lines parallel to Δ's sides passing through P with the non-parallel sides. The parallelians of Δ w.r.t. P lie on a single conic $\mathcal{P}$ (the parallelian conic) which is regular as long as P is not situated on any of Δ's sides or on the Steiner circumellipse. If P is a point on the Steiner inellipse, then $\mathcal{P}$ is a parabola (cf. [1]). Recently, it was shown that along with the parallelian conic one can find tangent hexagons and pairs of tangent triangles whose vertices lie on conics. Hence, the parallelian conic $\mathcal{P}$ defines triangle and hexagon porisms in a natural way (see also [1]) which do also exist in a projective setting (cf. [9]).

It seems natural to look for the behaviour of parallelians in three-dimensional space. Through a point P, there exist four planes parallel to the sides of a tetrahedron $\mathcal{T} = ABCD$ which intersect the non-parallel edges of $\mathcal{T}$ in twelve parallelian (points). We can show that these twelve parallelians lie on a single quadric $\mathcal{P}$, the parallelian quadric of $\mathcal{T}$ w.r.t. P. However, there is a second kind of parallelian quadric $\mathcal{Q}$. There exist three planes through P which are parallel to the three pairs of opposite edges of $\mathcal{T}$. These planes intersect the four non-parallel edges in twelve points which are located on a quadric $\mathcal{Q}$. We shall study both quadrics (depicted in Figure 1) in detail, discuss regularity and parabolicity. Later, we shall describe the multitude of parallelian quadrics in affine spaces of dimensions higher than three.

This article is organized as follows: In Section 2, we describe the first type of parallelian quadric $\mathcal{P}$ in an affine three-space. We shall discuss where to choose the pivot point P such that $\mathcal{P}$ is singular or parabolic. Then, in Section 3, the second type of parallelian quadric $\mathcal{Q}$ is investigated. Section 4 provides a side glance on a projective version of parallelian quadrics in three-space. A view on parallelian quadrics in spaces of higher dimension is presented in Section 5. Throughout the present paper, we use synthetic and algebraic reasoning, the synthetic approach because of its elegance and simplicity, the algebraic because of its power.

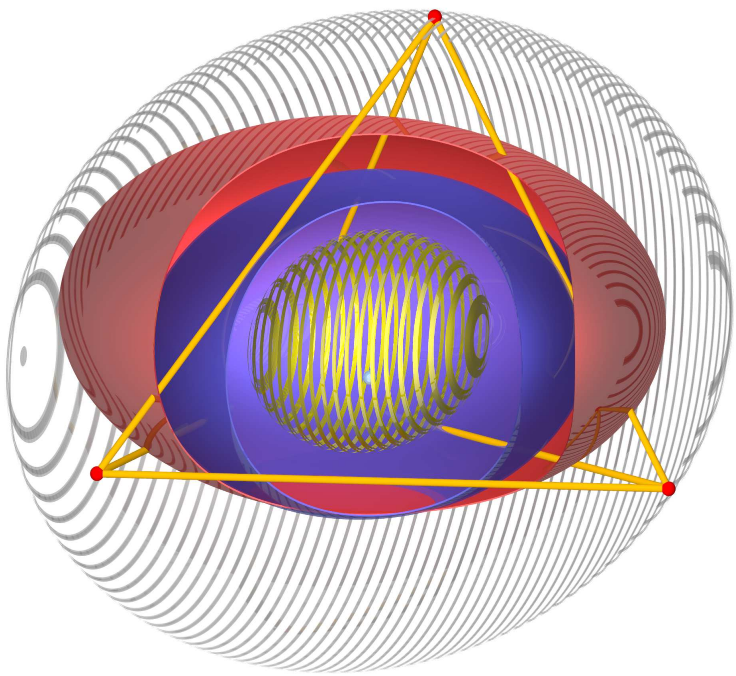

Fig. 1. The two different parallelian quadrics $\mathcal{P}$ (red), $\mathcal{Q}$ (blue) w.r.t. to P (cyan) together with the inscribed and circumscribed Steiner ellipsoids.

2 The Quadric $\mathcal{P}$

2.1 A Synthetic Approach towards the first Quadric

We assume that A, B, C, D are four non-coplanar points in a three-dimensional affine space. With $\mathcal{T} = ABCD$ we shall denote the tetrahedron with the respective vertices. Further, P be a point which shall not lie in any plane of $\mathcal{T}$. The union of the four planes of the tetrahedron shall be denoted by $\mathcal{T}^\star$.

Let now α, β, γ, δ be those planes through P that are parallel to the planes $[B,C,D]$, $[A,C,D]$, $[A,B,D]$, $[A,B,C]$ (in that particular order). We can define the following twelve points of intersection of $\mathcal{T}$'s edges and the latter planes:

$$\begin{aligned}&[A,D]\cap\delta=P_1,\ [B,D]\cap\delta=P_2,\ [C,D]\cap\delta=P_3,\ [A,C]\cap\gamma=P_4,\\&[B,C]\cap\gamma=P_5,\ [C,D]\cap\gamma=P_6,\ [A,B]\cap\beta=P_7,\ [B,C]\cap\beta=P_8,\\&[B,D]\cap\beta=P_9,\ [A,B]\cap\alpha=P_{10},\ [A,C]\cap\alpha=P_{11},\ [A,D]\cap\alpha=P_{12}.\end{aligned}\tag{1}$$

The six points in each face of $\mathcal{T}$ form a hexagon with three sides parallel to certain edges of $\mathcal{T}$. If all twelve points $P_1, \ldots, P_{12}$ lie on a quadric, then any six coplanar points have to lie on a single conic.

We can prove the following preparatory

Lemma 1. *Let $\Delta = ABC$ be a triangle and let further a', b', c' be three lines parallel to the sides $[B,C]$, $[C,A]$, $[A,B]$ of Δ (not necessarily) passing through a single point and not incident with any vertex of Δ. Then, the six intersections of a, b, c with the non-parallel sides of Δ lie on a single conic.*

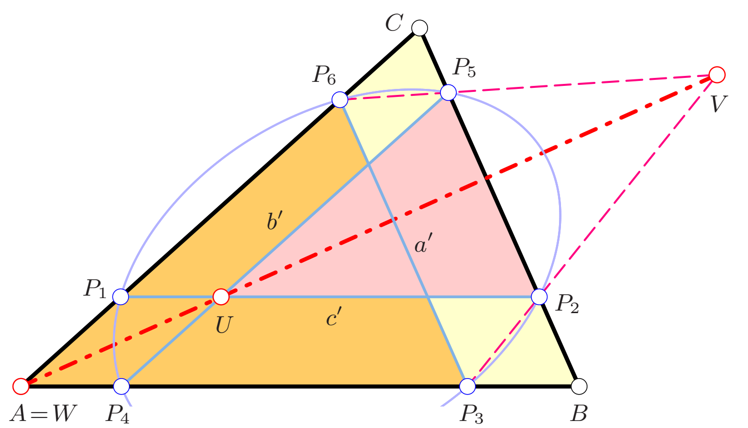

Fig. 2. Generalized parallelians and the conic on them.

The configuration described in Lemma 1 is not that of ordinary parallelians, since the three lines parallel to Δ's sides do not pass through a single point. We may call these points *generalized parallelians*.

Proof. The verification of the above lemma is simplified by properly labelling the points (see Figure 2): $[P_1,P_2] \parallel [A,B]$, $[P_3,P_6] \parallel [B,C]$, $[P_4,P_5] \parallel [C,A]$, and $P_1,P_6 \in [C,A]$, $P_3,P_4 \in [A,B]$, $P_2,P_5 \in [B,C]$.

(Note that these labels do not match the labels of the points in the spatial configuration.) The six points P_i are located on a single conic if they fulfill Pascal's criterion (cf. [6]), *i.e.*, the three points $U := [P_1,P_2] \cap [P_4,P_5]$, $V :=$

$[P_2, P_3] \cap [P_5, P_6]$, $W := [P_3, P_4] \cap [P_6, P_1]$ were to be collinear. In the present setting (see Figure 2), we have $W = A$.

According to the construction (by means of parallel lines), the triangles $\Delta_1 = AP_3P_6$ and $\Delta_2 = UP_2P_5$ are similar. The central similarity $\Delta_1 \to \Delta_2$ relates the vertices in the following way: $W \mapsto U$, $P_3 \mapsto P_2$, and $P_6 \mapsto P_5$. Hence, the center $Z = [P_2, P_3] \cap [P_5, P_6] = V$ of similarity lies on $[U, W]$. This also holds true if, *e.g.*, $P_2 = P_5$.

If, *e.g.*, $a' = [B, C]$, then $P_3 = B$ and $P_6 = C$. Consequently, P_2, P_3, P_5, and P_6 are collinear and there is no regular conic on $P_1, \ldots, P_6$. If, *e.g.*, $P_1 = P_4$, we can conclude that all six parallelians fall in pairs into triangle vertices and there is no uniquely defined conic. □

If two parallelians on the same triangle side coincide, *e.g.*, $P_1 = P_6$, there is still a uniquely defined conic touching $[C, A]$ at $P_1 = P_6$.

It is worth mentioning that the triple of triangles Δ, Δ_1, and Δ_2 mentioned in the above proof delivers three very special nested Desarguesian configurations (like those described in [7]) with three collinear perspectors and common perspectrix.

Now, that we know that the generalized parallelians always lie on a single conic, we are able to show the following

Theorem 1. *Let $\mathcal{T} = ABCD$ be a tetrahedron and let P be a point not located in any of the four planes defined by the vertices. Then, the four planes parallel to the planes of the tetrahedron that pass through P intersect the non-parallel edges of $\mathcal{T}$ in twelve points* (1) *that lie on a single quadric.*

In the planar case (studied in [1, 9]), a totality of six points happens to lie on a single conic (which is well-defined by five points where no three are collinear). In three-space, a quadric is uniquely defined by nine independent points[1], *i.e.*, we have an overhang of three points.

Proof. We use the notation of points and planes as given prior to (1). The hexagons $\mathcal{H}_1 = P_4P_7P_{10}P_5P_8P_{11}$ and $\mathcal{H}_2 = P_2P_5P_8P_3P_6P_9$ are the intersections of the star of planes about P (parallel to the planes of $\mathcal{T}$) with $[A, B, C]$ and $[B, C, D]$ (see Figure 3, left). According to Lem. 1, the circumconics c_1 and c_2 of $\mathcal{H}_1$ and $\mathcal{H}_2$ are generalized parallelian conics in the planes $[A, B, C]$ and $[B, C, D]$. They are shown in Figure 3.

The six points $P_4, \ldots, P_{11}$ on c_1 together with P_2, P_3, and P_6 determine a unique quadric $\mathcal{P}'$ which naturally also contains P_9 for it is contained in the planar intersection of $\mathcal{P}'$ with $[B, C, D]$. Since $P_1, P_{12} \in [A, B, D]$, they have to lie in the intersection of $\mathcal{P}'$ and $[A, B, D]$ (*i.e.*, they lie on the generalized parallelian conic through $P_7, \ldots, P_9$.) The same arguments hold true for the points on the generalized parallelian conic in $[A, C, D]$. Hence, $\mathcal{P}'$ houses all twelve points P_1, $\ldots$, P_{12}, and thus, $\mathcal{P}' = \mathcal{P}$. □

[1] Points determining a quadric in n-space are said to be independent if their Veronese images are independent, see [2].

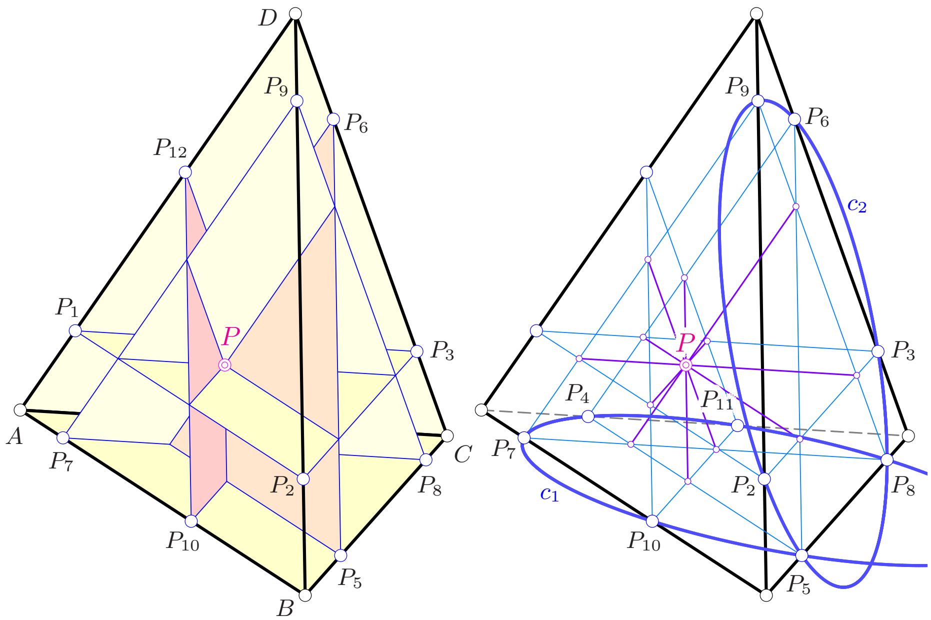

Fig. 3. Left: the four planes parallel to the faces of $\mathcal{T}$ through P. Right: twelve points on a quadric containing four generalized parallelian conics.

2.2 Analytic Representation of $\mathcal{P}$

The geometric objects under consideration are notions of affine geometry. Therefore, we attach an affine coordinate system to the tetrahedron such that the vertices have simple coordinate vectors

$$A = (0,0,0), \quad B = (1,0,0), \quad C = (0,1,0), \quad D = (0,0,1),$$

and the pivot P be given by $P = (\xi, \eta, \zeta)$. In order to avoid the lengthy and barely fruitful discussion of trivial and degenerate cases, we assume that $\xi\eta\zeta(\xi + \eta + \zeta - 1) \neq 0$, *i.e.*, P is not located in any of the four planes of $\mathcal{T}$.

The coordinates of the points P_i defined in (1) are then

$$\begin{aligned}
&P_7 = (\xi,0,0), P_8 = (\xi, 1-\xi, 0), P_9 = (\xi, 0, 1-\xi), P_{10} = (\xi+\eta+\zeta, 0, 0),\\
&P_4 = (0,\eta,0), P_5 = (1-\eta, \eta, 0), P_6 = (0, \eta, 1-\eta), P_{11} = (0, \xi+\eta+\zeta, 0),\\
&P_1 = (0,0,\zeta), P_2 = (1-\zeta, 0, \zeta), P_3 = (0, 1-\zeta, \zeta), P_{12} = (0,0,\xi+\eta+\zeta).
\end{aligned}$$

According to Thm. 1, the points $P_1, \ldots, P_{12}$ lie on the quadric

$$\mathcal{P}: \ \xi\eta\zeta(\xi+\eta+\zeta) + \sum_{\text{cyclic}} \eta\zeta x^2 + \xi(2\eta + \xi + 2\zeta - 1)yz = 0. \qquad (2)$$

Here, and in the following, $\sum\limits_{\text{cyclic}}$ stands for the cyclic sum, *i.e.*, the sum of all three cyclic shifts of the argument. Now, we can show

Theorem 2.
1. The quadric $\mathcal{P}$ is regular, provided that the pivot point P is not chosen in any plane of $\mathcal{T}$ or on the quadric

$$\mathcal{P}_{\text{sing}}: \quad \sum_{\text{cyclic}} x^2 + yz - x = 0. \tag{3}$$

2. The quadric $\mathcal{P}$ is parabolic if P is chosen on the quadric

$$\mathcal{P}_{\text{par}}: \; 1 + 3 \sum_{\text{cyclic}} x^2 + yz - x = 0. \tag{4}$$

Proof.
1. The quadric (3) is regular if, and only if, the coefficient matrix of the homogeneous equation is regular (cf. [8]). The determinant of the coefficient matrix equals $4(\xi\eta\zeta)^2(\xi+\eta+\zeta-1)^2(\xi^2+\eta^2+\zeta^2+\xi\eta+\eta\zeta+\zeta\xi-\xi-\eta-\zeta)$. The first four factors deliver the equations of the planes in $\mathcal{T}^\star$ (if ξ, η, ζ are replaced with x, y, z and then set equal to zero). The last factor confirms the statement. Conversely, if P is chosen on any of the planes in $\mathcal{T}^\star$ or on the quadratic surface described by the last factor, $\mathcal{P}$ becomes singular.
2. $\mathcal{P}$ described by (3) is parabolic if the coefficient matrix of the quadratic form of the inhomogeneous equation is singular (cf. [8]), *i.e.*, if its determinant $2\xi\eta\zeta(\xi+\eta+\zeta-1)(3(\xi^2+\eta^2+\zeta^2+\xi\eta+\eta\zeta+\zeta\xi-\xi-\eta-\zeta)+1)$ is equal to zero. The first four factors are described above, the last factor delivers the equation of $\mathcal{P}_{\text{par}}$. The converse is shown in the like manner. □

The quadrics $\mathcal{P}_{\text{sing}}$ and $\mathcal{P}_{\text{par}}$ are concentric. The common center equals the centroid $G = \frac{1}{4}(1,1,1)$ of $\mathcal{T}$. The quadric $\mathcal{P}_{\text{sing}}$ is circumscribed to $\mathcal{T}$ and the tangent planes at $\mathcal{T}$'s vertices are parallel to the opposite faces. Thus, $\mathcal{P}_{\text{sing}}$ can be viewed as a spatial analog of the Steiner circumellipse (cf. [6]). Therefore, we shall call it a *Steiner circumellipsoid* of $\mathcal{T}$. We shall not claim that $\mathcal{P}_{\text{sing}}$ is *the* Steiner circumellipsoid, since there is a huge variety of ellipsoids circumscribed to $\mathcal{T}$ even if we demand that they are concentric as is shown in [4]. From the same source, we learn that $\mathcal{P}_{\text{sing}}$ shares some *metric* properties with its planar analog. Theorem 2 mirrors the results on parallelian conics (cf. [1, 9]). If the pivot point P is either chosen on the Steiner circum- or inellipse, the parallelian conic is either singular or a parabola.

2.3 Locus of Vertices of Singular Quadrics $\mathcal{P}$

We can describe the manifold of vertices of all singular quadrics $\mathcal{P}$:

Theorem 3. *The vertices of all singular quadrics fill an algebraic surface $\mathcal{C}$ of degree eight.*

Proof. According to Thm. 1, the pivot P has to be chosen on the quadric $\mathcal{P}_{\text{sing}}$ in order to make $\mathcal{P}$ singular. $\mathcal{P}_{\text{sing}}$ can be parametrized over an affine plane as

$$\mathcal{S}(u,v) = (u^2+uv+v^2-u-v+1)^{-1}\Big(1-u-v, u, v\Big) \quad \text{with} \quad (u,v) \in \mathbb{R}^2. \tag{5}$$

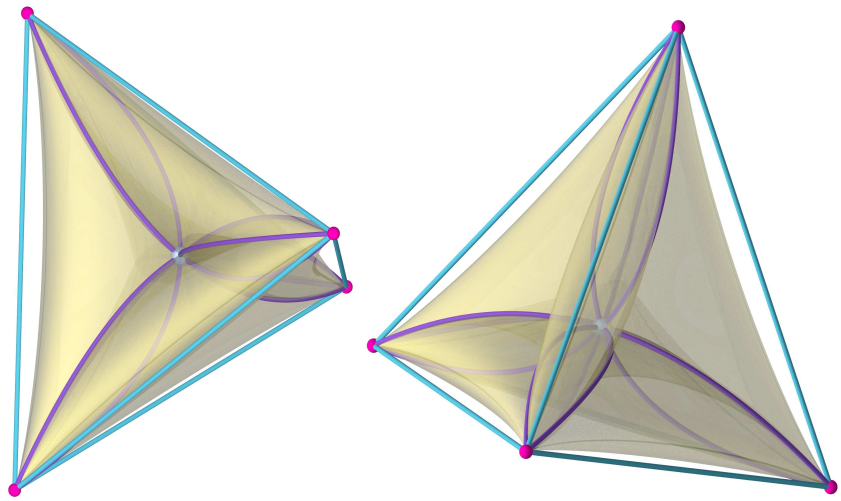

Fig. 4. The vertices of singular quadrics $\mathcal{P}$ fill an algebraic surface $\mathcal{C}$ of degree eight with six parabolas concurrent in the centroid G of $\mathcal{T}$ forming the self-intersections.

We insert (5) into (2), compute the centers C of the quadrics $\mathcal{P}$, and find

$$C(u,v) = (u^2+uv+v^2-u-v+1)^{-2}\Big((1-u-v)^2, u^2, v^2\Big).$$

An implicit equation of the octic surface $\mathcal{C}$ reads

$$\begin{aligned}\mathcal{C}:\ & \Big(\sum_{\text{cyc.}} x^4+2x^3(y+z)+x^2(3y^2+8yz)\Big)^2+\Big(\sum_{\text{cyc.}} x^2-2yz\Big)^2 \\ & -4\Big(\sum_{\text{cyc.}} x^7+2x^6(y+z)+3x^5(y^2+3xy+z^2) \\ & \quad +3x^4(y+z)(3y^2+23yz+3z^2)+3x^3y^2z(13y+25z)\Big) \\ & +2\Big(\sum_{\text{cyc.}} 3x^6+2x^3y(xz+y^2)+48x^3yz(y+z)+41x^2y^2z^2\Big) \\ & -4(x+y+z)\Big(\sum_{\text{cyc.}} x^4-3x^3(y+z)+x^2y(4y-5z)\Big)=0. \qquad \square\end{aligned} \tag{6}$$

Figure 4 shows an example of the octic surface $\mathcal{C}$ for a regular tetrahedron $\mathcal{T}$. The surface $\mathcal{C}$ is the locus of all vertices of singular parallelian quadrics $\mathcal{P}$. This surface is the analog of an elliptic sextic curve (with a very special configuration of six double points) housing the vertices of all singular parallelian conics (cf. [9]). In Figure 4, we can also see the self-intersections of $\mathcal{C}$. These are six parabolas

$$\begin{aligned} p_1(u) &= (u^2, u(1-u), u(1-u)), & p_2(u) &= (u^2, u(1-u), (1-u)^2), \\ p_3(u) &= (u(1-u), u^2, u(1-u)), & p_4(u) &= (u(1-u), u^2, (1-u)^2), \\ p_5(u) &= (u(1-u), u(1-u), u^2), & p_6(u) &= ((1-u)^2, u(1-u), u^2), \end{aligned}$$

with $u \in \mathbb{R}$ joining the edges of $\mathcal{T}$ passing through the centroid $G = \frac{1}{4}(1,1,1)$ of $\mathcal{T}$. The vertices of $\mathcal{T}$ and G are four-fold points on $\mathcal{C}$. At each vertex of $\mathcal{T}$, three parabolas intersect.

3 The Quadric $\mathcal{Q}$

The second type of parallelian quadric shows up first in three-space, since planes can be parallel to faces and to pairs of opposite edges of $\mathcal{T}$ as well. These two types of parallelian quadrics coincide in the plane. However, we shall see that the quadric $\mathcal{Q}$ described in this section shows a behavior different from that of $\mathcal{P}$ in three-space.

3.1 Synthetic Proof of the Existence

In the star of planes about P, there are three planes which are parallel to pairs of opposite edges of $\mathcal{T}$. These planes are given by

$$\alpha' : y+z=\eta+\zeta \parallel [A,B],[C,D], \quad \beta' : z+x=\xi+\zeta \parallel [A,C],[B,D],$$
$$\gamma' : x+y = \xi + \eta \parallel [A,D],[B,C],$$

and they intersect the non-parallel edges in twelve points

$$\begin{aligned}
&[A,C]\cap\alpha'=Q_1,\ [A,D]\cap\alpha'=Q_2,\ [B,C]\cap\alpha'=Q_3,\ [B,D]\cap\alpha'=Q_4,\\
&[A,B]\cap\beta'=Q_5,\ [A,D]\cap\beta'=Q_6,\ [B,C]\cap\beta'=Q_7,\ [C,D]\cap\beta'=Q_8,\\
&[A,B]\cap\gamma'=Q_9,\ [A,C]\cap\gamma'=Q_{10},\ [B,D]\cap\gamma'=Q_{11},\ [C,D]\cap\gamma'=Q_{12}.
\end{aligned} \tag{7}$$

Figure 5 shows the parallelograms cut out by the planes α', β', γ' in the faces of $\mathcal{T}$. Now, we can show:

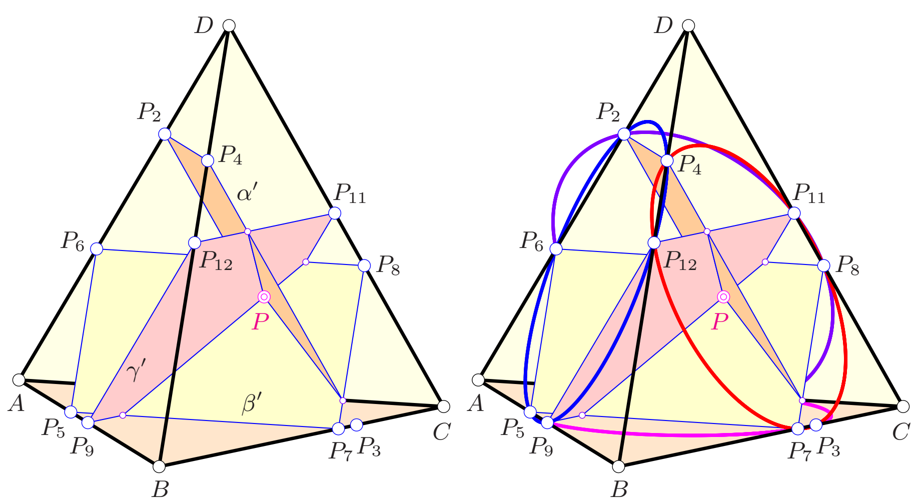

Fig. 5. Left: The planes α', β', γ' intersect the faces of $\mathcal{T}$ along parallelograms. Right: Each face of $\mathcal{T}$ contains a generalized parallelian conic which is the planar intersection of the quadric $\mathcal{Q}$.

Theorem 4. *Let $\mathcal{T} = ABCD$ be a tetrahedron and let P be a point not contained in any plane $\mathcal{T}$. Then, the three planes parallel to pairs of opposite edges of $\mathcal{T}$ intersect $\mathcal{T}$'s edges in twelve points* (7) *that lie on a single quadric.*

Proof. The plane α' parallel to the opposite pair of edges $[A, B]$, $[C, D]$ meets the faces $[B, C, D]$ and $[A, C, D]$ in lines parallel to $[C, D]$, and further, it intersects $[A, B, C]$ and $[A, B, D]$ along lines parallel to $[A, B]$. In the same way, we can characterize the intersections of β' and γ' with the planes of $\mathcal{T}$ (see Figure 5).

Hence, in each face of $\mathcal{T}$, we find three lines parallel to the incident edges which meet the non-parallel edges in six points, the six generalized parallelians in each face. According to Lem. 1, we find a generalized parallelian conic in each face of $\mathcal{T}$.

Now, we can use the same arguments as in the proof of Thm. 1 to show that the twelve points Q_i given in (7) lie on a single quadric. □

3.2 Equation of $\mathcal{Q}$

According to Thm. 4, we have found another twelve points lying on a single quadric $\mathcal{Q}$. We shall call $\mathcal{Q}$ the *second parallelian quadric* of $\mathcal{T}$ w.r.t. P. Now, we can show

Theorem 5. *The second parallelian quadric $\mathcal{Q}$ has the equation*

$$\begin{aligned}\mathcal{Q}: \sum_{\text{cyclic}} (\eta+\zeta)x^2 - \big(1-2(\xi+\eta+\zeta)\big)yz - (\eta+\zeta)(2\xi+\eta+\zeta)x \\ = -(\xi+\eta)(\eta+\zeta)(\zeta+\xi).\end{aligned} \tag{8}$$

It is regular if P is not chosen in any of the six planes containing an edge of $\mathcal{T}$ parallel to the opposite edge. The second parallelian quadric $\mathcal{Q}$ is parabolic if P is chosen on the cubic surface

$$\mathcal{Q}_{\text{par}}: \ 8xyz + 4\sum_{\text{cyclic}} x^2(y+z-1) - 2yz + x = 1. \tag{9}$$

Proof. The regularity of $\mathcal{Q}$ is equivalent to the regularity of the coefficient matrix of its homogeneous equation (cf. [8]). Besides a constant factor, this yields

$$(\xi+\eta)(\eta+\zeta)(\zeta+\xi)(\xi+\eta-1)(\eta+\zeta-1)(\zeta+\xi-1).$$

Each of the factors of the latter product yields (after ξ, η, ζ are replaced with x, y, z and after setting it equal to 0) the equation of one of the planes mentioned in the theorem.

$\mathcal{Q}$ is parabolic if the coefficient matrix of the quadratic form of its inhomogeneous equation is singular. So, we find the condition

$$8\xi\eta\zeta + 4\sum_{\text{cyclic}} \xi^2(\eta+\zeta-1) - 2\eta\zeta + \xi = 1$$

on the coordinates of the pivot point P, which is turned into the equation of $\mathcal{Q}_{\text{sing}}$ given in (9) by replacing ξ, η, ζ with x, y, z. □

The surface described by (9) is an example of CAYLEY's *nodal cubic* (cf. [3, 5]), an example of which is displayed in Figure 6. It has four nodes at the reflections of $\mathcal{T}$'s vertices in $\mathcal{T}$'s centroid $G = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Further, it contains nine lines: the six reflections of $\mathcal{T}$'s edges (reflections in G) along with three ideal lines.

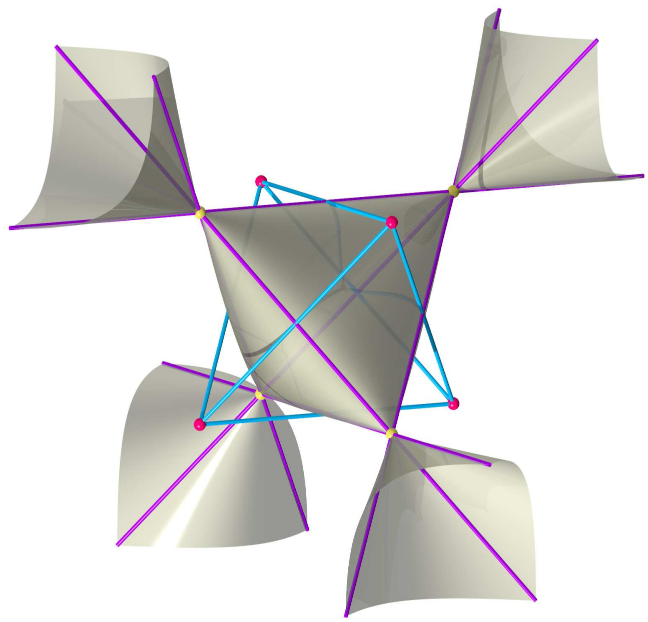

Fig. 6. The second parallelian quadric $\mathcal{Q}$ is singular if P lies on CAYLEY's nodal cubic.

4 Projective Version

Since the planar version of the parallelian conic can be viewed in a superordinate setting, *i.e.*, the projective setting (as shown in [9]), it is near to ask for a projective version of the parallelian quadrics at least in three dimensions. In the projectivized version, the ideal plane is replaced with some plane λ with homogeneous coordinates $l_0 : l_1 : l_2 : l_3$, while it means no restriction to assume that the vertices of $\mathcal{T}$ are described by the canonical basis vectors. Further, the pivot point P is given by $\xi_0 : \xi_1 : \xi_2 : \xi_3$ and we assume that λ does not contain P or any edge of $\mathcal{T}$ and P is also not located in any plane of $\mathcal{T}$. (The non-vanishing algebraic relations fulfilled by P's and λ's coordinates are then easily found and allow for some simplifications of the calculation.)

Now, the points P_i from (1) and Q_i from (7) are not defined by means of planes parallel to faces or edges of $\mathcal{T}$. The ideal points of edges and the ideal lines of faces are replaced with the respective intersections with λ. With these assumptions, we can show:

Theorem 6. *The twelve projectivized parallelians of a tetrahedron $\mathcal{T}$ w.r.t. a pivot point P of the first and second kind lie on two quadrics $\mathcal{P}$ and $\mathcal{Q}$ with the equations*

$$\mathcal{P}: \sum_{i=0}^{3} \frac{l_i}{\xi_i}(\sigma - l_i\xi_i)x_i^2 - \sum_{i<j} \frac{x_i x_j}{\xi_i \xi_j}\,\theta = 0,$$
$$\mathcal{Q}: \sum_{i=0}^{3} l_i^2 x_i^2 \prod_{\substack{j<k \\ j,k\neq i}} (l_j\xi_k + l_k\xi_j)x_i^2 - \sum_{i<j} l_i l_j x_i x_j \left(\sum_{k\neq i,j} l_k\xi_k\right)\cdot\theta,$$

where $\sigma := \sum_{i=0}^{3} l_i\xi_i$, $\tau := \sum_{i<j} l_i l_j \xi_i \xi_j$, *and* $\theta := \tau + l_i l_j \xi_i \xi_j + \sum_{k,l\neq i,j} l_k l_l \xi_k \xi_l$.

The discussion of parabolic quadrics can now replaced with the search for λ-parallelian quadrics touching the plane λ. The discussion of singular λ-parallelian quadrics and the corresponding set of vertices will not deliver substantially new results. Finally, we remark that the two (regular) parallelian quadrics $\mathcal{P}$, $\mathcal{Q}$ are always different, except in the plane, where they always coincide.

5 A View in Higher Dimensions

In an n-dimensional affine space, we consider a simplex $\mathcal{S}_n = A_0A_1\ldots A_n$. The pivot point $P = (\xi_1, \ldots, \xi_n)$ shall not be incident with any of the hyperplanes $\mathcal{H}_j = [A_0, A_1, \ldots, \not{A_j}, \ldots, A_n]$, or any k-dimensional subspace (k-dimensional face of $\mathcal{S}_n$) spanned by any $(k+1)$-tuple of points taken from $A_0, \ldots, A_n$.[2]

Now, we can distinguish between several different types of parallelian quadrics. The first type $\mathcal{P}_1$ agrees with the planar version and the quadric $\mathcal{P}$ from Sec. 2 and houses the $n(n+1)$ intersections of $\mathcal{H}_k(P)$ (hyperplane parallel to $\mathcal{H}_k$ passing through P) with the non-parallel (non-incident) edges of $\mathcal{S}_n$. Since a quadric in n-space is determined by $\frac{1}{2}n(n+3)$ independent points, it is surprising that these $n(n+1)$ points (almost twice the necessary amount) always lie on a single quadric. Now, we can show that the first type $\mathcal{P}_1$ of parallelian quadric has the equation

$$\mathcal{P}_1: \sum \frac{x_i^2}{\xi_i} + \sum_{i<j} \frac{x_i x_j}{\xi_i \xi_j}(\xi_i + \xi_j - 1 + S) - \sum x_i\xi_i(\xi_i + S) + S \cdot \prod \xi_k = 0$$

with $S = \sum \xi_k$ and regardless of the dimension and is always an ellipsoid and centered at the centroid $G = \frac{1}{n+1}\underbrace{(1, 1, \ldots, 1)}_{n}$ of the simplex $\mathcal{S}_n$.

We can give a more general result in:

Theorem 7. *In an n-dimensional affine space, we can distinguish $t = \lfloor \frac{n}{2} \rfloor$ different types of parallelian quadrics w.r.t. a simplex and a pivot point P (in*

[2] With $[X_1, \ldots, \not{X_j}, \ldots, X_k]$ we denote a list from which the element X_j is canceled.

admissible position). The number of points lying on the edges of $\mathcal{S}^n$ and on the parallelian quadric equals $N(n,k) = \binom{n}{k}(n+1)(n-k)$ with $k = 0, 1, \ldots, t$ and has to be halved in case $n-1 = 2k$.

Proof. It takes $k+1$ vertices of $\mathcal{S}^n$ in order to span a k-face of $\mathcal{S}^k$. Then, $n-k-1$ vertices span the complementary face. There are $C_k^n = \binom{n+1}{k+1}$ combinations of vertices serving as base for k-face, and thus, $(n-k-1)(k+1)$ (non-parallel, non-incident) edges are to be intersected with C_k^n hyperplanes through P. If $n-1 = 2k$, then each k-face (and its complementary) shows up twice. It takes nested induction on k and n in order to show that $N(n,k)$ points always lie on a single quadric. □

6 Final Remarks

So far, we have discussed affine properties of a tetrahedron and quadrics related to it. This is to show that it is still possible to find some results of the affine geometry of quadrics, although this geometry is not very rich. The discussion of higher-dimensional versions of parallelian quadrics shows more diversification.

The search for spheres among (at least) $\mathcal{P}$ and $\mathcal{Q}$ makes sense only if the tetrahedron $\mathcal{T}$ fulfills special metric relations and will, thus, not lead to results valid for a generic tetrahedron. The only Euclidean specialities that deserve mention are:
If the pivot point P equals the centroid G of $\mathcal{T}$, then the centers of $\mathcal{P}$ and $\mathcal{Q}$ are equal to G. If the pivot point P equals the circumcenter U of $\mathcal{T}$, then the center of $\mathcal{P}$ equals the Monge point of $\mathcal{T}$.

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