Permutation Cubics

Boris Odehnal[0000−0002−7265−5132]

University of Applied Arts Vienna, Vienna, Austria
boris.odehnal@uni-ak.ac.at

Abstract. The permutation of the trilinear coordinates of a point yield the six permutation points which are conconic. This idea leads to the following generalization: The six permutations of the trilinears of a point together with the six permutations of the trilinears of the image of that particular point under a certain quadratic Cremona transformation yield twelve points which lie on a single cubic curve. This note is devoted to the study of the thus defined cubic curves, especially those defined by some triangle center and its isogonal or isotomic conjugate.

Keywords: Triangle cubic · permutation points · quadratic Cremona transformation.

1 Introduction

Along with a triangle in the Euclidean plane, there are many known cubic curves: the Neuberg cubic, the Thomson cubic (17-point cubic), the McCay cubic, the Darboux cubic, the Napoleon (or Feuerbach) cubic, the Orthocubic, and approximately 1200 more cubics, cf. the exhaustive collection on B. Gibert's page [1]. These curves are defined by means of algebraic or geometric properties (see [6]) or they are the images of lines and conics under certain algebraic transformations (cf. [1–3]).

In [5], the homogeneous coordinates $p : q : r$ of a point P in the plane of a triangle are permuted which yields in total six points, the so-called *permutation points*. It is of minor importance whether $p : q : r$ are trilinear or barycentric coordinates (or homogeneous coordinates defined by an arbitrarily chosen unit point). In the following, we will use homogeneous trilinear coordinates, *i.e.*, the incenter X_1 of the base triangle is the unit point of the underlying projective frame (cf. [2, 3]). The labeling X_i for the i^{th} triangle center follows the list given in C. Kimberling's book [3] and the Encyclopedia of Triangle Centers [4].

The vertices A, B, C, and the incenter X_1 of the base triangle Δ are described by the following homogeneous trilinear coordinates

$$A = 1 : 0 : 0, \quad B = 0 : 1 : 0, \quad C = 0 : 0 : 1, \quad X_1 = 1 : 1 : 1.$$

Any point P in the plane of the triangle is uniquely defined by a homogeneous coordinate triple $p : q : r \neq 0 : 0 : 0$, and any such triple defines a unique point. Sometimes, we shall prefer the vector notation of coordinates, and then, we simply write $\mathbf{p} = (p, q, r)$.

At first glance, it is surprising that the six *permutation points* are *conconic*, *i.e.*, they lie on a single conic with the equation

$$\circlearrowleft\!\!\!\!\sum p^2 \cdot \circlearrowleft\!\!\!\!\sum \xi\eta = \circlearrowleft\!\!\!\!\sum \xi^2 \cdot \circlearrowleft\!\!\!\!\sum pq,$$

see [5]. Here, we have introduced the symbol $\circlearrowleft\!\!\!\!\sum$ which denotes the *cyclic sum*. This is to be understood literally: For example $\circlearrowleft\!\!\!\!\sum a = a+b+c$, or $\circlearrowleft\!\!\!\!\sum \xi^2 = \xi^2+\eta^2+\zeta^2$, *i.e.*, the argument is replaced twice cyclically in alphabetic order ($a \to b \to c \to a$, $\xi \to \eta \to \zeta \to \xi$, $p \to q \to r \to p, \dots$).

However, the cyclic symmetry of the coefficients of the conic's equation displays the invariance of the conic with respect to the permutation of the coordinates.

In the geometry of the triangle, two special quadratic Cremona transformations play an important role: the *isogonal conjugation* ι and the *isotomic conjugation* τ. These mappings are defined for all points in the projectively extended plane of Δ, except on the side lines of Δ. In terms of homogeneous (trilinear) coordinates, these mappings are given by

$$\iota(\xi,\eta,\zeta) \mapsto (\eta\zeta, \zeta\xi, \xi\eta), \quad \tau(\xi,\eta,\zeta) \mapsto (b^2c^2\eta\zeta, c^2a^2\zeta\xi, a^2b^2\xi\eta). \tag{1}$$

In both cases, the permutation yields 6 permutation points. Thus, an arbitrary point P defines its six permutation points and so does its isogonal conjugate $\iota(P)$ (or its isotomic conjugate $\tau(P)$) which makes 12 points in total.

In the following Section 2, we shall focus on the isogonal conjugation and show that the latter 12 points indeed lie on a single cubic which we will call a *permutation cubic*. Within the family of permutation cubics, only one cubic is known and shows up in Gibert's list [1] as $\mathcal{K}_{228}$. *All other permutation cubics are apparently new triangle cubics.* In Section 3, we look for rational and degenerate cubics among the permutation cubics and study their configuration of real inflection points. In addition, their dual curves and their Hessians are derived. We also consider permutation cubics with triangle centers for their pivot points which yields a relatively small number of cubics containing more than just a pair of isogonal conjugate centers. Finally, in Section 4, we replace the isogonal conjugation with the isotomic conjugation and study the thus defined cubics. The latter cubics differ slightly from those defined with the help of the isogonal conjugate of the pivot point.

2 The permutation cubics

In general, the permutation points of a point P lie on one conic, while the permutation points of its isogonal conjugate $\iota(P)$ lie on another conic. However, it is surprising that we have 12 points with the following property:

Theorem 1. *The 12 points obtained from $P = p:q:r$ by permuting P's homogeneous coordinates as well as those of its isogonal image $\iota(P) = qr : rp : pq$ are located on the self-isogonal cubic*

$$\mathcal{C} = \ P\xi\eta\zeta - Q(\xi^2\eta + \xi\eta^2 + \xi^2\zeta + \xi\zeta^2 + \eta^2\zeta + \eta\zeta^2) = 0, \tag{2}$$

where $P = \sum pq(p+q)$ *and* $Q = pqr$.

Proof. The fact that the 12 points lie on the cubic (2) is best shown by inserting the respective coordinates.

The cubic is self-isogonal: Assume $X = \xi : \eta : \zeta$ is a point on $\mathcal{C}$, then its isogonal image $\iota(X) = \eta\zeta : \zeta\xi : \xi\eta$ is also contained in $\mathcal{C}$ regardless of the point X. □

Fig. 1 shows the distribution of the permutation points of a given point P as well as those of the isogonal image $\iota(P)$.

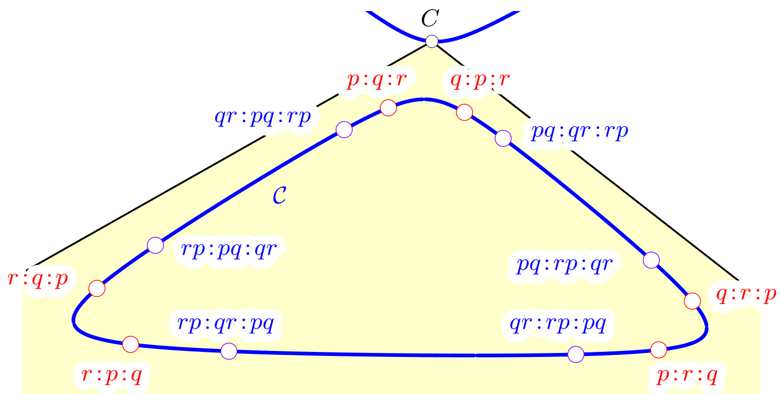

Fig. 1. The permutation cubic $\mathcal{C}$ with pivot point $p{:}q{:}r$ contains 12 a priori known points.

Because of the cyclic symmetry of the equation (2) of the permutation cubics, these cubics are invariant under all six permutations of the homogeneous coordinates. Since these permutations induce collineations in the triangle plane, we can say:

Theorem 2. *Each of the permutation cubics* (2) *is transformed into itself under each element of the discrete group of collineations induced by the permutation of coordinates.*

As can be easily read off from (2), each permutation cubic passes through all three vertices of the base triangle.

3 Properties of permutation cubics

3.1 Rational and singular permutation cubics

At first, we shall determine those curves in the family (2) which degenerate. For that purpose, we derive conditions on p, q, r so that the gradient $\operatorname{grad}\mathcal{C} =$

$(\partial_\xi \mathcal{C}, \partial_\eta \mathcal{C}, \partial_\zeta \mathcal{C})$ of the cubics' equations (2) vanishes. We decide to eliminate the variables ζ and η from $\partial_\xi \mathcal{C} = 0$, $\partial_\eta \mathcal{C} = 0$, $\partial_\zeta \mathcal{C} = 0$ and find

$$\begin{aligned} 0 = 81(pqr)^{15}\xi^{16}(q+r)^6(p+r)^6(p+q)^6(r+q+p)^2(pq+pr+qr)^2 \\ \left(6pqr - \textstyle\sum pq(p+q)\right)^2 \left(5pqr + 2\textstyle\sum pq(p+q)\right) \\ \left(49p^2q^2r^2 + 5\textstyle\sum p^2q(q(p^2+q^2) + 2p(pr+q^2) + 6pr(q+r))\right). \end{aligned} \tag{3}$$

(The choice of variables to be eliminated is not essential. Other combinations than the chosen one lead to products involving sixteenth powers of η or ζ. The other factors (homogeneous polynomials in p, q, and r) remain the same.)

If either of p, q, r equals zero, we obtain the completely degenerate cubic $\xi\eta\zeta = 0$, *i.e.*, the three side lines of the base triangle.

If either $p = -q$ or $q = -r$ or $r = -p$, we obtain the singular cubic

$$(\xi + \eta)(\eta + \zeta)(\zeta + \xi) = 0,$$

i.e., the side lines of the excentral triangle Δ_e.

The factor $p + q + r$ set equal to zero is the equation of the line $\mathcal{L}_1$ which is polar to the incenter X_1 with regard to Δ. $\mathcal{L}_1$ is called the *antiorthic axis* of Δ, see [3]. The points on $\mathcal{L}_1$ determine the degenerate cubic

$$\underbrace{(\xi + \eta + \zeta)}_{\mathcal{L}_1} \cdot \underbrace{(\xi\eta + \eta\zeta + \zeta\xi)}_{\text{Steiner circumellipse } e} = 0. \tag{4}$$

It is the union of $\mathcal{L}_1$ and the Steiner circumellipse e. The points on $\mathcal{L}_1$ correspond to points on e and *vice versa*, since $\iota(\mathcal{L}_1) = e$ and $\iota^2 = \mathrm{id}_{\mathbb{P}^{2\star}}$. Thereby, the meaning of the quadratic factor $pq + qr + rp$ is also disclosed.

The first cubic factor yields the equation

$$\mathcal{C}_1 : \ 6\,\xi\eta\zeta - \sum \xi\eta(\xi + \eta) = 0 \tag{5}$$

of the cubic determined by X_1, since $P(1,1,1) = 6$ and $Q(1,1,1) = 1$. In fact $\mathcal{C}_1$ has an acnode at $X_1 = 1 : 1 : 1$ and is therefore rational. In terms of a homogeneous parameter $u_0 : u_1 \neq 0 : 0$, its points can be given as

$$u_0(u_1 - u_0)(2u_0 + u_1) : u_1(u_0 + 2u_1)(u_0 - u_1) : (u_0 + u_1)(u_0 + 2u_1)(2u_0 + u_1).$$

This cubic can be found as cubic $\mathcal{K}_{228}$ in [1] and it is an isogonal circum-conico-pivotal cubic which contains the triangle centers with Kimberling indices 1, 1022, 1023, 23889 – 23894. For a specific triangle, the cubic $\mathcal{C}_1 = \mathcal{K}_{228}$ is shown in Fig. 2.

The second cubic factor corresponds to an elliptic cubic $\mathcal{E}$ whose points do not lead to degenerate or rational permutation cubics.

Finally, the sextic factor is the equation of an elliptic curve $\mathcal{S}$ of degree 6 with three tacnodes at the vertices of the base triangle Δ and ordinary double points at $W_1 = [A,B] \cap \mathcal{L}_1 = 1 : -1 : 0$, $W_2 = [B,C] \cap \mathcal{L}_1 = 0 : 1 : -1$, $W_3 = [C,A] \cap \mathcal{L}_1 = 1 : 0 : -1$. The ellipticity (genus = 1) of the latter curves makes clear that the points on these curves do not yield degenerate or rational permutation cubics. We summarize our results in:

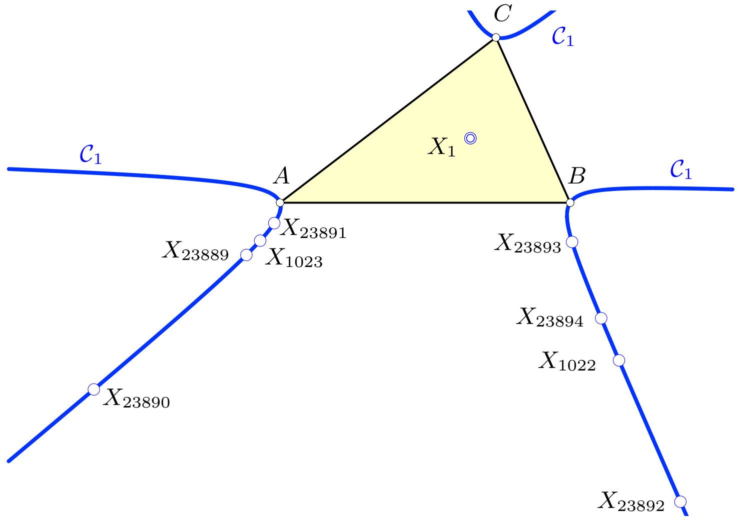

Fig. 2. The only rational cubic $\mathcal{C}_1 = \mathcal{K}_{228}$ with an acnode at X_1 contains eight further triangle centers.

Theorem 3. *The family of permutation cubics* (2) *contains one rational cubic, the cubic $\mathcal{C}_1 = \mathcal{K}_{228}$ determined by X_1.*
In the family of permutation cubics, there exist only 3 degenerate cubics:
(i) the union of the Steiner circumellipse e and the polar $\mathcal{L}_1$ of X_1,
(ii) the union of the side lines of the excentral triangle Δ_e,
(iii) the union of the side lines of the base triangle Δ.

Figure 3 shows the degenerate curves in the family of permutation curves. The elliptic cubic $\mathcal{E}$ and the elliptic sextic $\mathcal{S}$ are also shown.

3.2 The dual curves

We compute the dual curves of (2) by eliminating ξ, η, and ζ from the system of algebraic equations

$$\partial_\xi \mathcal{C} = \rho u_0, \quad \partial_\eta \mathcal{C} = \rho u_1, \quad \partial_\zeta \mathcal{C} = \rho u_2, \quad \mathcal{C} = 0$$

which yields

$$\begin{aligned} \mathcal{C}^\star :\ & Q^4 \sum u_0^6 + 2Q^3(P+Q) \sum u_0^5(u_1+u_2) + \\ & + Q^2(P^2-4PQ-13Q^2) \sum u_0^4(u_1^2+u_2^2) + \\ & + 2Q^2(2P^2+10PQ+15Q^2)u_0u_1u_2 \sum u_0^3 + \\ & + 2Q(P^3+4P^2Q+PQ^2-8Q^3)u_0u_1u_2 \sum u_0^2(u_1+u_2) + \\ & + (P^4-6P^2Q^2+36PQ^3+90Q^4)u_u^2u_1^2u_3^2 = 0. \end{aligned} \tag{6}$$

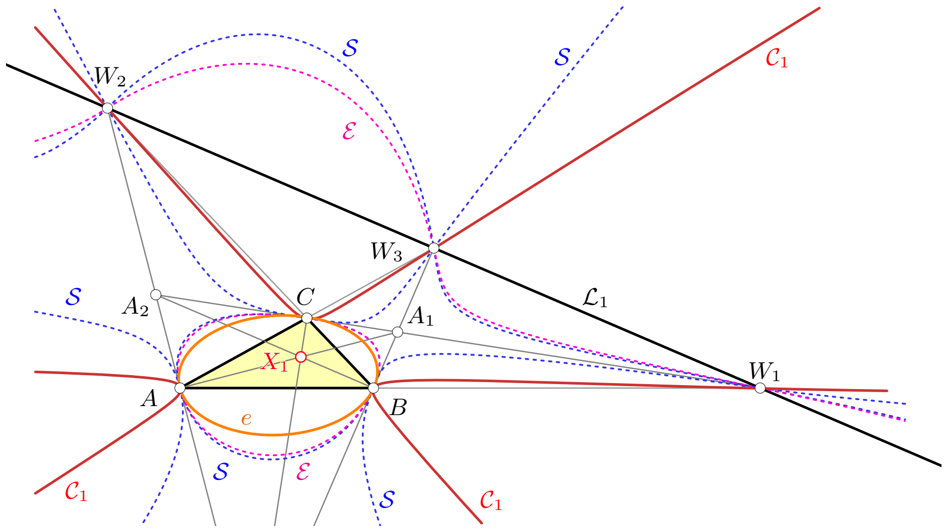

Fig. 3. The locus $\mathcal{C}_{44} = e \cup \mathcal{L}_1$ and $\mathcal{C}_1$ of points in the triangle plane that define singular permutation cubics and the computational artifacts $\mathcal{E}$ and $\mathcal{S}$.

For the curve $\mathcal{C}_1$ (pivot X_1) with equation (5), we have $p : q : r = 1 : 1 : 1$ or $P = 6$ and $Q = 1$. This reduces the equation of the dual curves (6) to the self-isogonal quartic

$$\mathcal{C}_1^{\star}: \ \sum u_0^2(u_0^2 + 12u_0(u_1+u_2) - 26u_1^2 + 244u_1u_2) = 0.$$

To be more precise: With $P = 6$ and $Q = 1$ the factor $u_0+u_1+u_2$ (corresponding to the antiorthic axis) splits off from (6) with multiplicity 2.

3.3 Hessian curves

The Hessian curves of cubics are again cubics and are obtained as the determinant of the Hessian matrix of the form $\mathcal{C}$. Hence, we have

$$\begin{aligned} \mathrm{H}\mathcal{C}: \ & 8Q^2(P+2Q)(\xi^3+\eta^3+\zeta^3)+2(P^3-12PQ^2-24Q^3)\xi\eta\zeta+ \\ & -2P^2Q(\xi^2\eta+\xi\eta^2+\xi^2\zeta+\xi\zeta^2+\eta^2\zeta+\eta\zeta^2) = 0. \end{aligned} \tag{7}$$

For arbitrary choices of P and Q, *i.e.*, for an arbitrary choice of a pivot point with homogeneous coordinates $p : q : r$, none of the Hessian curves (7) will be a permutation curve. This could only be the case if the coefficient of $\xi^3+\eta^3+\zeta^3$ vanishes: If $Q = 0$ (or equivalently $pqr = 0$), the pivot point is located on a side of the bases triangle and the Hessian curve becomes the union of the side lines of Δ. In the case $P + 2Q = 0$, (7) simplifies to $-8Q^3(\xi+\eta)(\eta+\zeta)(\zeta+\xi) = 0$ and describes the union of the side lines of the excentral triangle Δ_e.

We are able to show the following:

Theorem 4. *All cubics in the family of permutation cubics share their three real points of inflection which lie on the line $\mathcal{L}_1$ and on the side lines of Δ.*

Proof. Possible candidates for points of inflection on a curve $\mathcal{C}$ are located on its Hessian curve H$\mathcal{C}$. The common points of $\mathcal{L}_1$ and the side lines of Δ are

$$W_1 = 1 : -1 : 0, \quad W_2 = 0 : 1 : -1, \quad W_3 = 1 : 0 : -1$$

and it is easily checked that W_1, W_2, W_3 are regular points on $\mathcal{C}$ (independent of the choice of $p : q : r$) and lie also on H$\mathcal{C}$ with equation (7). Therefore, they are inflection points of all curves $\mathcal{C}$ given by (2).

For the regular curves in the family (2), the three inflection tangents are not concurring. Note that the configuration of inflection points on the permutation cubics shows a similar behavior than that of the distance product cubics (cf. [6]).

3.4 Triangle centers on permutation cubics

Each triangle center determines a permutation cubic. Since the cubics (2) are invariant under isogonal transformations, each triangle center X_i shares its permutation cubic $\mathcal{C}_i$ with the isogonal conjugate $\iota(X_i)$.

The cubic $\mathcal{C}_1$ determined by X_1 is a special case: It is the only rational (non-degenerate) permutation cubic and is defined by X_1 (cf. Thm. 3). Since X_1 is self-assigned under the isogonal conjugation, there is no corresponding point on $\mathcal{C}_1$. Earlier, we have mentioned that $\mathcal{C}_1$ contains 9 triangle centers of which 8 can be arranged in 4 pairs of conjugate points. The conjugate pairs on $\mathcal{C}_1$ are:

$$(X_{1022}, X_{1023}), \quad (X_{23889}, X_{23894}), \quad (X_{23890}, X_{23893}), \quad (X_{23891}, X_{23892}).$$

We shall not go through all permutation cubics defined by the approximately 47400 known triangle centers (as to April 2022). There are only a few remarkable examples, which contain more than just one pair of triangle centers.

The centroid X_2 and the Symmedian point X_6 are each others isogonal conjugates. The conjugate pairs on $\mathcal{C}_2$ are (X_2, X_6) and (X_{3570}, X_{3572}). The center X_{3572} is the intersection of $\mathcal{C}_2$'s tangents at X_2 and at X_6. The tangent of $\mathcal{C}_2$ at X_{3570} meets the cubic at X_{3572} (cf. Fig. 4).

The circumcenter X_3 and the orthocenter X_4 form a conjugate pair on $\mathcal{C}_3$. Further, this cubic contains the center X_{1981} whose isogonal conjugate is a yet unknown center with trilinear center function

$$\alpha = a(b-c)(a^2-b^2-c^2)(a^3c + a^2(b^2-2c^2) - ac(b^2-ac^2) - b^4 + b^2c^2)\cdot \\ \cdot(a^3b - a^2(2b^2-c^2) + a(b^3-bc^2) + b^2c^2 - c^4).$$

On the cubic $\mathcal{C}_9$ determined by the Mittenpunkt X_9, we find the following conjugate pairs: (X_9, X_{57}) and (X_{1024}, X_{1025}) (cf. Fig. 5). Similar to the case of $\mathcal{C}_2$, we observe that the tangents to $\mathcal{C}_9$ at X_9 and X_{57} intersect on the cubic at X_{1024}, while the tangents at X_{1024} and X_{1025} meet at a further yet unnamed

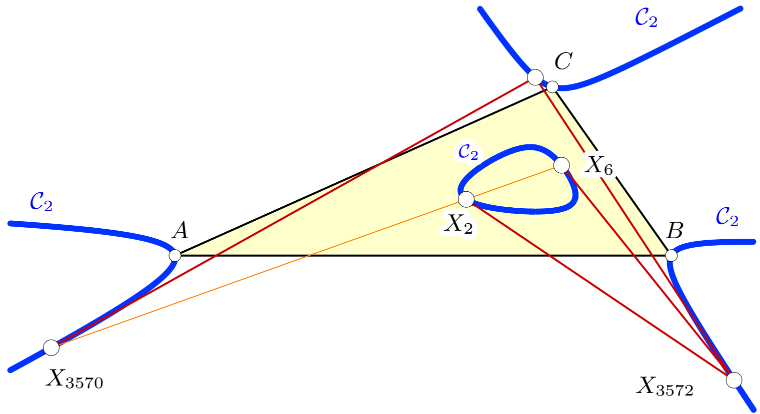

Fig. 4. The permutation cubic $\mathcal{C}_2 = \mathcal{C}_6$.

center with a rather lengthy trilinear representation of algebraic degree 24 in a, b, c (the side lengths of Δ).

The center X_{101} lies on Δ's circumcircle, and therefore, its isogonal conjugate X_{513} is a point on the line at infinity. The tangents at the latter two centers to the cubic $\mathcal{C}_{101}$ intersect in $X_{34906} \in \mathcal{C}_{101}$. The tangents at the points of the isogonal conjugate pair (X_{34905}, X_{34906}) intersect on $\mathcal{C}_{101}$.

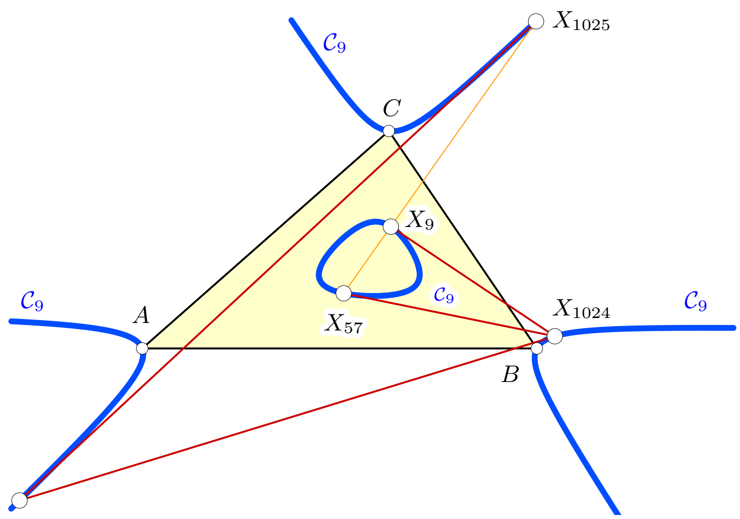

Fig. 5. The permutation cubic $\mathcal{C}_9 = \mathcal{C}_{57}$.

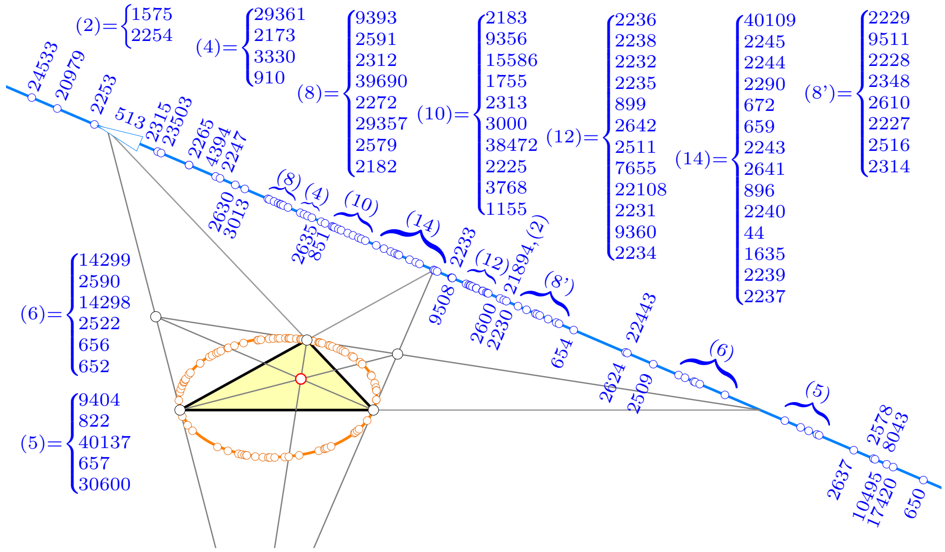

Fig. 6. The distribution of triangle centers on the antiorthic axis $\mathcal{L}_1 \subset \mathcal{C}_{44}$.

There are 224 triangle centers on the degenerate cubic $\mathcal{C}_{44} = e \cup \mathcal{L}_1$ given by (4). On the Steiner circumellipse, the 97 centers (cf. Fig. 7) with the Kimberling indices

$$88, 100, 162, 190, 651, 653, 655, 660, 662, 673, 771, 799, 823, 897, 1156, 1492, 1821, 2349, 2580, 2581, 3257, 4598, 4599, 4604, 4606, 4607, 8052, 20332, 23707, 27834, 29059, 32680, 34085, 34234, 36083 - 36102, 37128 - 37143, 37202 - 37223, 38340, 40110, 43069, 43192$$

can be found. The remaining 127 centers with the Kimberling numbers

$$44, 649, 650, 652, 654, 656, 657, 659, 661, 672, 770, 798, 822, 851, 896, 899, 910, 1155, 1491, 1575, 1635, 1755, 2173, 2182, 2183, 2225, 2227 - 2240, 2243 - 2247, 2252 - 2265, 2272, 2290, 2312 - 2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591, 2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, 4724, 4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 8061, 9356, 9360, 9393, 9404, 9508, 9511, 10495, 13401, 14298 - 14300, 15586, 17410, 17418, 17420, 18116, 20331, 20979, 21127, 21894, 22108, 22443, 23503, 24533, 29357, 29361, 30600, 38472, 39690, 40109, 40137, 40338$$

are located on the line $\mathcal{L}_1$ (see Fig. 6). The ideal point X_{513} is the only real improper center on the singular permutation cubic $\mathcal{C}_{44}$.

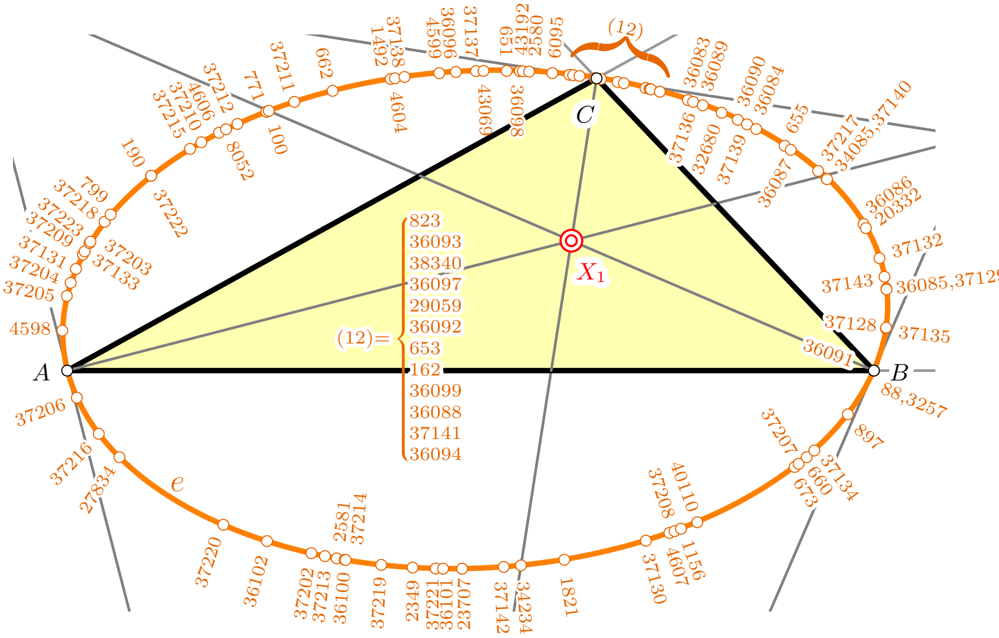

Fig. 7. The distribution of triangle centers on the Steiner ellipse $e \subset \mathcal{C}_{44}$.

4 Isotomic instead of isogonal conjugation

The cubics mentioned in Thm. 1 contain the permutation points of the isogonal image of the pivot point $P = p : q : r$. The isogonal transformation can be replaced with any other quadratic Cremona transformation.

However, the *isotomic conjugation* is the second prominent quadratic Cremona transformation closely related to the geometry of a triangle. In this case, a point $P = p : q : r$ is mapped to a point

$$\tau(P) = b^2c^2qr : c^2a^2rp : a^2b^2pq.$$

Again, there are 12 points obtained from P (its permutation points and the permutation points of its isotomic conjugate) which lie on cubics with the equations

$$\begin{aligned} \mathcal{T}:\ & pqr\left(\sum a^6b^6p^3q^3 - a^4b^4c^4pqr\sum p^3\right)\sum \xi\eta(\xi+\eta) \\ & -a^2b^2c^2p^2q^2r^2\left(\sum a^2b^2pq(a^2p+b^2q) - a^2b^2c^2\sum pq(p+q)\right)\sum \xi^3 \\ & +\xi\eta\zeta\left(a^2b^2c^2pqr\sum a^2b^2pq(p^4+q^4) + \sum a^6b^6p^4q^4(p+q)\right. \\ & +\sum a^4b^4p^3q^3r(b^2(c^2-b^2)q^2 + b^2(c^2-a^2)p^2) \\ & \left.+\sum a^2b^2p^3q^3r^2(b^4(c^4-a^4)p + a^4(c^4-b^4)q)\right) = 0. \end{aligned} \quad (8)$$

Compared to the cubics (2), the latter cubics do not pass through the vertices of the base triangle.

With the help of a CAS it is a rather simple task to show that the following holds true:

Theorem 5. *The six permutation points of a point $P = p : q : r$ and the six permutation points of its isotomic conjugate lie on a triangle cubic with equation* (8)*. All cubics in the family share the inflection points which agree with the inflection points of the self-isogonal permutation cubics* (2)*.*

The cubic with pivot X_1 is rational and has an acnode at X_1.

The cubic with pivot point X_2 degenerates completely, i.e.*, its equation is the zero form.*

The cubics (8) *whose pivot points are the triangle centers with Kimberling numbers*

$$649, 650, 652, 654, 656, 657, 659, 661, 672, 770, 798, 799, 822, 851, 896, 899, 910, 1155, 1491, 1575, 1635, 1755, 2173, 2182, 2183, 2225, 2227 - 2240, 2243 - 2247, 2252 - 2254, 2265, 2272, 2290, 2312 - 2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591, 2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, 4724, 4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 8061, 9356, 9360, 9393, 9404, 9508, 9511, 10495, 13401, 14298 - 14300, 15586, 17410, 17418, 17420, 18116, 20331, 20979, 21127, 21894, 22108, 22443, 23503, 24533, 29357, 29361, 30600, 38472, 39690, 40109, 40137$$

split into the line $\mathcal{L}_1$ and a further conic. Especially, if $i = 661$ and $i = 799$, the degenerate cubics are equal to that given in (4)*.*

Note that the permutation cubics that carry the permutation points of the isotomic conjugate of the pivot point are not self-isotomic.

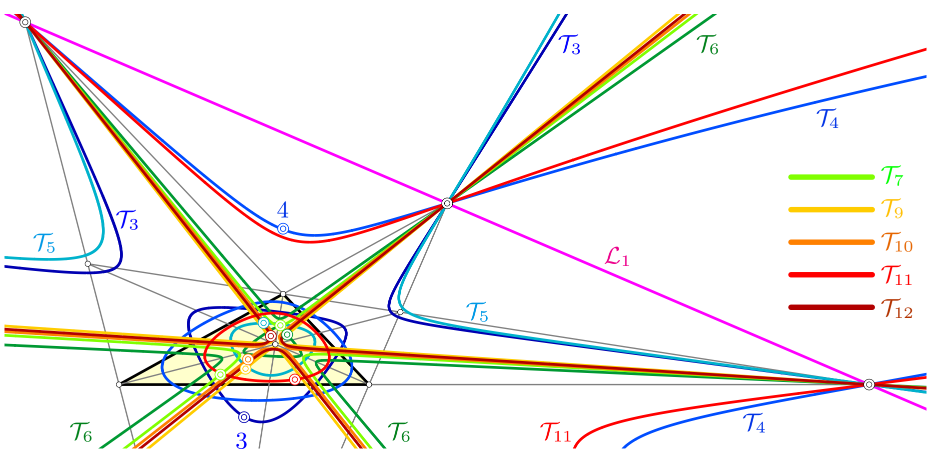

Fig. 8. Some permutation cubics (8) with low-indexed centers for their pivot points.

Fig. 8 shows some of the cubics $\mathcal{T}_i$ defined by triangle centers with low Kimberling indices $i \in \{1, \ldots, 12\}$. Note that $\mathcal{T}_7 = \mathcal{T}_8$ since $X_8 = \tau(X_7)$.

References

1. B. Gibert: *Cubics in the Triangle Plane.* Available at: `https://bernard-gibert.pagesperso-orange.fr/index.html`. Last accessed on April 1, 2022.
2. G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Concis.* From the ancient Greeks to 21st century developments. Springer-Spektrum, Springer-Verlag, Heidelberg, 2016.
3. C. Kimberling: *Triangle Centers and Central Triangles.* (Congressus Numerantium Vol. 129) Utilitas Mathematica Publishing, Winnipeg, 1998.
4. C. Kimberling: *Encyclopedia of Triangle Centers.* Available at: `http://faculty.evansville.edu/ck6/encyclopedia`. Last accessed on April 1, 2022.
5. C. Kimberling: *Permutation Ellipses.* J. Geom. Graphics **24**/2 (2020), 233–247.
6. B. Odehnal: *Distance Product Cubics.* KoG **24** (2020), 29–40.