Rectification of an Edgy Photograph

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Abstract. The reconstruction of 3D objects from photographs is well-known and several methods are implemented in various software. The vast majority of reconstruction software uses statistical methods and rather less geometric knowledge. At hand of an example, we shall illustrate that the reconstruction of a 3D object from a single edgy photo can be done with a small portion of geometric knowledge about conics. We first collect the basic facts about central projections, perspective images, and their rectification. Subsequently, we use some basic properties of conics in order to rectify an image and to reconstruct an object without previously undistorting or deskewing the image itself. The constructive approach is preferred because of its simplicity and shall also be understood as a plea for Descriptive Geometry and Constructive Geometry.

Keywords: Perspective image · rectification · reconstruction · edgy photograph · conic

1 Introduction

Usually it is assumed that photographs deliver perspective images, *i.e.*, central projections of the depicted objects. The theory behind Descriptive Geometry and Projective Geometry provides useful results that allow us to reconstruct images, see [1, 2, 7, 16]. In almost all cases, the reconstruction of a 3D object uses two images taken from two separate viewpoints. In principal, the two corresponding images of nine points associated with the object to be reconstructed are sufficient in order to find the relative position of the observer w.r.t. (short hand for with respect/regard to) the image plane and the object itself (cf. [4, 5, 11, 13, 14]). Moreover, even a single image can be used for a reconstruction, provided some additional information is given, such as some measures taken at the depicted object. With a single image, the reconstruction of an object is, in general, only up to scale (see [1, 2, 7, 16]).

Photographs are usually considered to show perspective images if not taken with ultra wide-angle lenses or fisheye lenses. The latter types of lenses do not even cause distortions of depicted objects, they bend and produce curved images of straight lines. Therefore, such images can hardly be rectified by means of constructions. Further, the portion of a scene depicted on a photograph is a (mostly rectangular) part of the total image symmetric w.r.t. the principal vanishing point H, *i.e.*, that point in the image plane right in front of the observer. Objects with straight edges and planar faces (as is mostly the case in architecture) are best suited for reconstruction from perspective images.

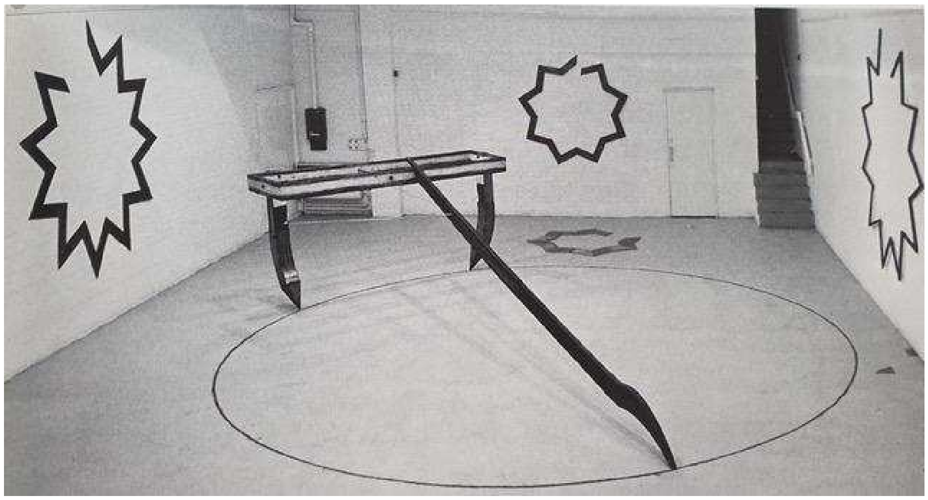

Fig. 1. "Ramp and Hyphen" by Paul Neagu. The picture was taken during an exhibition in Glasgow (Scotland) in 1978 (with kind permission of Toni Neagu).

The image under consideration (see Fig. 1) shows the installation "Ramp and Hyphen" by the British and Romanian sculptor, painter, graphic artist, and poet Paul Neagu (1938 – 2004), cf. [8]. Some metric data of the depicted object was a priori known, but the precise position and size of the claws on which the object is standing were unknown. The rectangular frame measures 94cm×286cm and is 135cm high.

The aim was to verify the validity of known data and to find all missing measures so that the object could be rebuilt. In the beginning, we thought that the naive approach would suffice: We assumed that it is a perspective view with a vertical image plane since the vanishing point of the vertical lines is far out and the images of the vertical lines appear nearly parallel. However, this resulted in a principal point outside the image and the reconstruction of the cuboid bounding the frame (on the left, including the claws) resulted in contradictory measures and the tips of the claws appeared to be outside the rectangular support. Fortunately, the picture shows an ellipse on the floor, which was known to be the image of a circle. This allows us to rectify the image in a proper way and the few known measures fit well with all the extracted ones.

This article is structured as follows: Section 2 gives a brief overview on central projections and perspective images in order to provide the knowledge and techniques used for the reconstruction of a perspective image such as a photograph. Then, in Section 3, we describe the rectification and reconstruction. In this article we shall *emphasize once more that some basic knowledge about geometry can help to solve tasks of practical relevance.* As demonstrated in [6], Constructive Geometry helps in an easy and natural way to understand images without applying a huge computational black box. It is a plea for Constructive and Descriptive Geometry. Moreover, to the best of our knowledge, this is the first time that the rectification of a curved object on an image is used to rectify the entire image and reconstruct the object.

2 Central projections and perspective images

2.1 Mapping and image

A central projection β is the mapping of points X of the projectively closed three-dimensional space of perception onto a plane π (image plane) from a point E (eye point) by joining X with E and intersecting the line $[E, X]$ with the plane π. This yields the central projection $X^c = [E, X] \cap \pi$ of X. The mapping β is clearly not defined for the point E, but still for all points in the plane $\nu \parallel \pi$ through the eye point E. The plane ν is called *vanishing plane* since all points $V \neq E$ in ν are mapped to the ideal points of the image plane β.

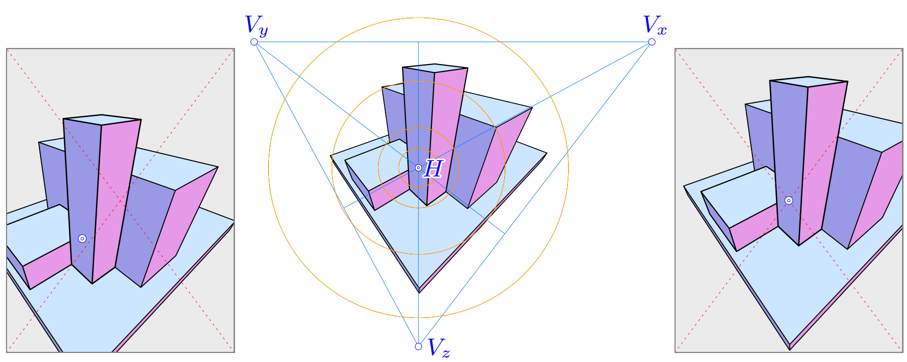

Fig. 2. Improperly chosen portion of the image (left), perspective image with circles marking the traces of the viewing cones with semi apertures 45°, 30°, 15°, 7.5° (middle), and properly chosen display window (right).

Unfortunately, the terms *vanishing point* or *vanishing line* appear with a double meaning:[1] On one hand the vanishing elements are those who are mapped to infinity and, on the other hand, the terms vanishing point / line mean the

[1] This ambiguity occurs only in English texts. In German, *Verschwindungspunkte* are those who are mapped to infinity, while *Fluchtpunkte* are the images of ideal points. In connection with (perspective) collineations, the image and pre-image of ideal points interchange their meaning if we change from the collineation to its inverse.

central projections of an ideal point / ideal line. We call three vanishing points *principal vanishing points* if they are the images of ideal points corresponding to three pairwise orthogonal directions. The result of a central projection is the central image which is usually called a perspective image.[2]

2.2 Image properties, photographs

A central projection maps parallel lines (even if they are not parallel to the image plane) to lines passing through a common point, called *vanishing point*. Thus, a vanishing point is the image of an ideal point (point at infinity) which is common to all lines parallel to a certain fixed direction. We shall call three vanishing points V_x, V_y, V_x a triple of *principal vanishing points* if they are the images of ideal points in three pairwise orthogonal directions (say x, y, z). All triangles $V_xV_yV_z$ of principal vanishing points are acute, see [10, p. 504]. This is important for the central image of cuboids – whose rectification is rather simple – and allows us to decide whether an image is similar to a perspective image or if it is only a collinear image of a perspective image. The *principal point* H of a perspective image is the orthocenter of a triangle of principal vanishing points. Usually, photographs are rectangular domains symmetric with respect to the principal point, cf. Fig. 2 (right). As it is the case with the photo shown in Fig. 1, sometimes it may happen that H does not coincide with the center of the image. This is shown in Fig. 2 (left) and, unfortunately, the photograph displayed in Fig. 1 shows the same phenomenon.

3 Rectification and reconstruction

3.1 Measurement points

In almost all cases, the constructive rectification of a central projection (such as the one shown in Fig. 3) aims at the construction of the principal point H first. The point H can be found as the orthocenter of any triple of principal vanishing points (see $[1, 2, 7, 10, 16]$). Then, the *distance* d is to be determined (as shown in Fig. 3). This enables us to construct the *measurement points* M_1, M_2, ... for planes parallel to the principal planes. The measurement points are the centers of perspective collineations that can be used to rectify planar image contents in the planes parallel to the principal planes. (Indeed, it is possible to find measurement points for each plane and line. For the geometric background we refer to [10, p. 504].) Fig. 3 (left) shows exemplarily how to find the measurement point M_1 for the first principal plane π_1 (parallel to the horizontal $[x, y]$-plane).

The vanishing line for the horizontal planes is spanned by the two principal vanishing points V_x and V_y for the principal directions denoted by x and y. The perspective collineation β_1 with center M_1 and vanishing line $[V_x, V_y]$ sends each planar figure in the horizontal plane π_1 to its *true shape*, *i.e.*, the planar figure is

[2] Here, the German language tends to wipe away the differences between projection and image. Both, the mapping and the image are frequently called *Perspektive*.

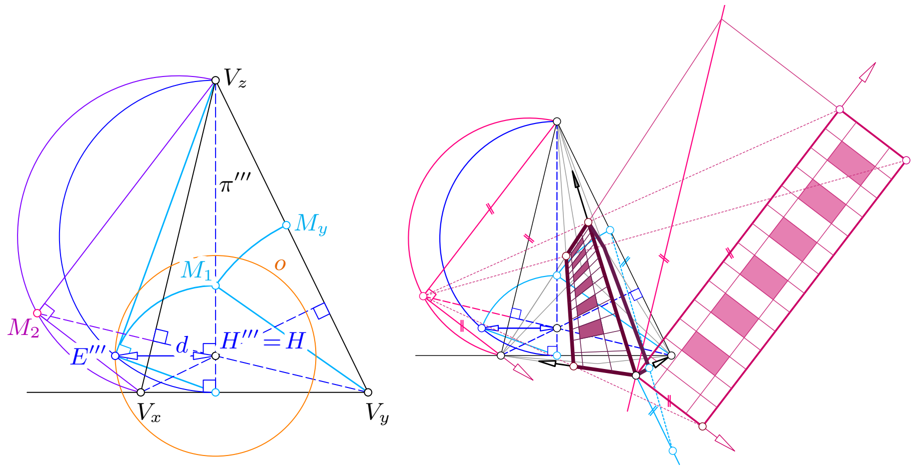

Fig. 3. Left: The orthocenter H of the triangle of principal vanishing points V_x, V_y, V_y is the principal point of the perspective image. It allows for the construction of the *(eye) distance* d and the *measurement points* M_1, M_y, and M_2.
Right: The rectification of a "perfect" image uses perspective collineations with measurement points for their centers and the vanishing lines (image of a plane's ideal line) in the perspective image as their vanishing lines (preimage of the ideal line of π).

rectified. The axis of the collineation can be chosen freely (as long as it is parallel to $[V_x, V_y]$, but different from it) and this causes a scaling of the rectification. Therefore, the rectification is only up to scale. However, the proper choice of the collineation axis (or axes, if there are more different planar figures to be rectified) can be made consistently.

The rectification of a perspective image shown in Fig. 3 (right) is applied to a "perfect" image, *i.e.*, a true perspective image. In almost all practical cases, rectifications start with skew and distorted images. This will also be the case when it comes to the rectification of the image in Fig. 1.

3.2 Images of circles

The perspective images of circles are conics (as long as the carrier plane is not projecting, *i.e.*, does not pass through the eye point). The photograph shown in Fig. 1 shows an ellipse from which we knew that it is the image of a circle. Without this information, the image rectification via the rectification of the conic would fail. Each line through the center of a conic is called a *diameter*. A pair of diameters of a conic is called a *conjugate pair* if the two diameters are conjugate with w.r.t. the conic, *i.e.*, each diameter contains the pole of the other w.r.t. the conic (resp. its polar system), see [3, p. 267 ff.]. Pairs of conjugate diameters of a circle are orthogonal (enclose a right angle). Central projections are neither orthogonality preserving nor diameter preserving (except in a few special cases).

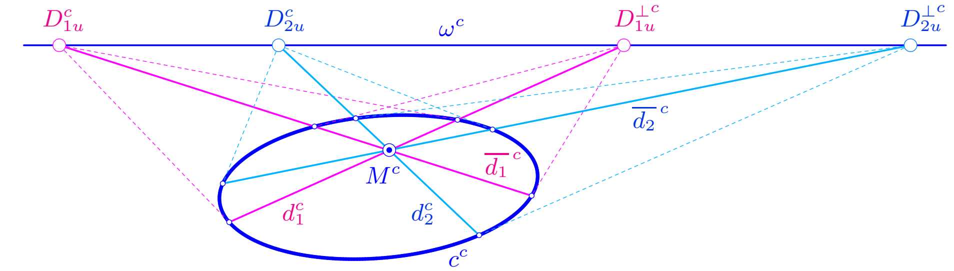

Fig. 4. The perspective images of pairs of orthogonal diameters of a circle c are still conjugate w.r.t. the image curve c^c, but (in general) neither diameters nor orthogonal (except if the carrier plane of c is parallel to π). The intersections with the polar line of the image of the center are pairs of corresponding points in an elliptic involution which is the perspective image of the absolute involution.

Fig. 4 shows the perspective images of two pairs of *conjugate diameters* of an ellipse c^c which is the perspective image of a circle c. Nevertheless, we know that the pairs $\left(D_{1u}^c, D_{1u}^{\perp c}\right)$ and $\left(D_{2u}^c, D_{2u}^{\perp c}\right)$ of vanishing points on the plane's vanishing line ω^c are conjugate pairs. Moreover, they are corresponding pairs of points in the elliptic involution ι on ω^c induced by the orthogonality in the projectively extended three-space of perception (cf. [3, p. 265]). Now, the measurement point M of this particular plane (indeed for all its parallels) is one of the Laguerre points of the elliptic involution ι (see Fig. 5). Therefore, M is one of the two common points of the Thales circles on the segments $D_{1u}^c D_{1u}^{\perp c}$ and $D_{2u}^c D_{2u}^{\perp c}$, see also [3, p. 265]. This guarantees that the rectifications of the diameters d_1 and $\overline{d_1}$ (as well as those of d_2 and $\overline{d_2}$) are again orthogonal, or equivalently, their ideal points are seen at right angles from M. (The choice of either common point only changes the orientation of the rectified planar figure.)

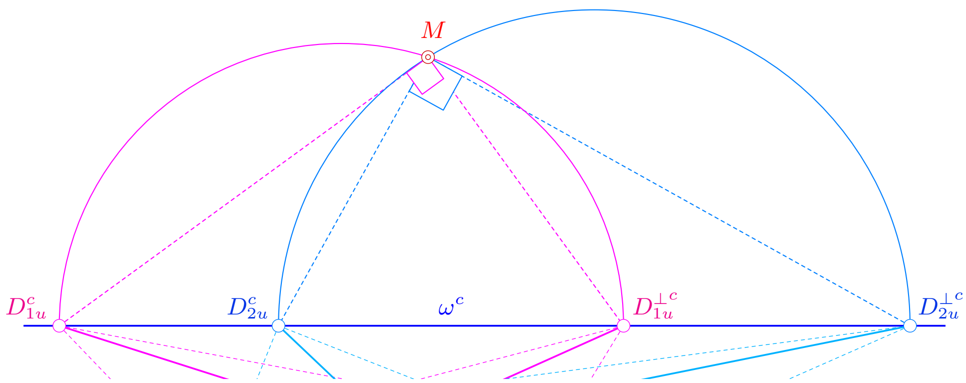

Fig. 5. The Laguerre point M of the elliptic involution acting on ω^c is the measurement point for all (parallel) planes sharing the vanishing line ω^c. M is found as a common point of two Thales circles.

4 Reconstruction of "Ramp and Hyphen"

4.1 The first attempt

Fig. 1 shows the exhibition room with the installation "Ramp and Hyphen". The constructions done in the perspective image are shown in Fig. 6.

We start with the construction of a principal vanishing triangle. It is near to assume that the exhibition room has a rectangular base, that the horizontal lines $[5,6]$ and $[7,8]$ are parallel, and further, that the latter have the common vanishing point F_u^c.

Unfortunately, the base does not deliver information. The rectangular frame $ABCD$ on top of the object yields two more vanishing points V_x and V_y (for orthogonal directions). If we assume that the image plane is vertical, and thus, parallel to the back wall of the room, we needed another pair of vanishing points in orthogonal and horizontal directions. This is not the case here. Moreover, it is a hard task to pick the points $5^c, \ldots, 9^c$ and $A^c, \ldots, D^c$ in the image such that the three vanishing points V_x, V_y, and F_u^c do really lie collinear, gathering on the vanishing line p_{1u}^c (as they should theoretically).

With the information that the rectangular frame $ABCD$ lies approximately 135cm above the base, we started the rectification under the assumption that the image plane π is vertical, to be more precise, that π is parallel to the back wall with the door on the right. These constructions are not shown in Fig. 6 and Fig. 7, since the assumption didn't hold. From another photograph of roughly the same scene (taken from a different standpoint), it was possible to estimate the height of the door (ca. 200cm) since a human person was standing close to it. Unfortunately, the assumption that the image plane is vertical, leads to inconsistent results in the rectification. For example, the perspective collineation rectifying the base plane showed the tips S and T of the claws outside the rectangular area $A'B'C'D'$ below the rectangular frame $ABCD$.

The third of the principal vanishing points is the vanishing point V_z of the vertical lines, such as the edges of the door frame $[T, T']$ and the vertical edges $[9,10]$, $[11,12]$ on the back wall. Besides the fact that V_z was a far out point (which is only an easy to overcome obstacle in the construction), we find the principal point H (as orthocenter of $V_xV_yV_z$) considerably outside the photograph (although it should be in its center). This allows us to construct the distance circle o (centered at H, radius d) as explained in Sec. 3.1 (cf. Fig. 3) and the measurement point M_1 (and also $\overline{M}_1$) for the base plane π_1. (In principle, only one of these is necessary. We have used both in order to keep the constructions clear.) The intersection p_1 of the plane π_1 with the image plane (sometimes called base line) is parallel to π_1's vanishing line P_{1u}^c. The line p_1 is to be inserted such that the projection of the segments $A'^cB'^c$ and $B'^cC'^c$ from the corresponding measurement points M_x and M_y onto p_1 reproduce the (a priori) known lengths 94cm and 157cm for the (top-view/support) of the rectangular frame $ABCD$.

The length of the handle and the position of the handle's end R as well as the tips S and T of the claws w.r.t. frame can only be determined if the

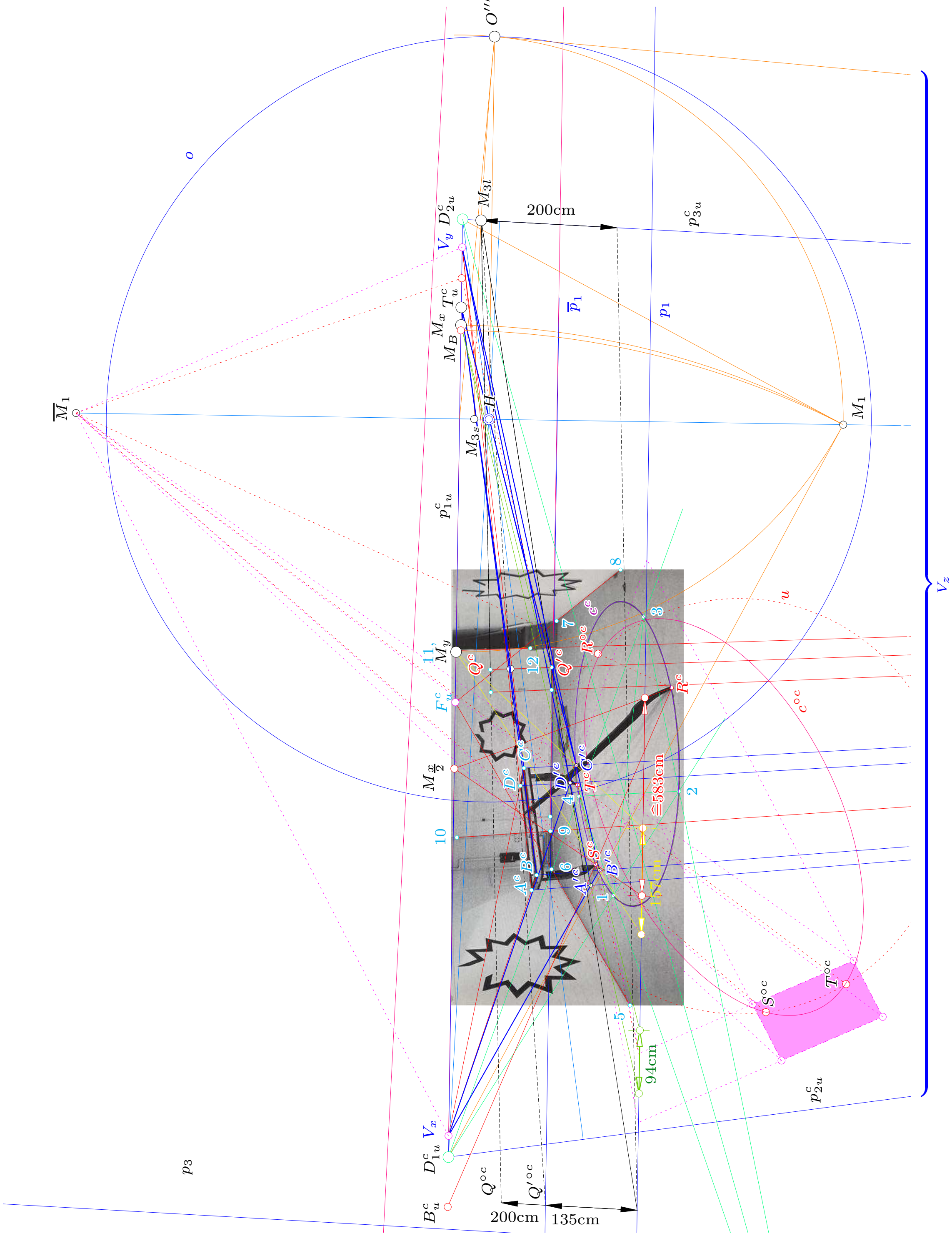

Fig. 6. Rectification, first attempt: with the help of vanishing points of presumably pairwise of orthogonal directions.

top-view (image in the base) is completely rectified. This is done with the help of the perspective collineation with center $\overline{M}_1$, axis p_1, and vanishing line p^c_{1u}. Unfortunately, this collineation sends the ellipse c^c (which is known to be the image of a circle c) to an ellipse $c^{\circ c}$ that deviates so much from a circle that this cannot be explained by mere inaccuracy. Therefore, we had to go another way.

4.2 Second attempt: rectification of the conic

The constructions for the second attempt towards the rectification are shown in Fig. 7. Only very important points and lines are labeled therein and the "upper" measurement point (now labeled M_1) is used as the center of the rectifying collineation.

This time, the measurement point M_1 is determined as a Laguerre point of the elliptic involution on the horizontal vanishing line p^c_{1u} (as explained in Sec. 3.2). The corresponding vanishing points of pairwise orthogonal horizontal directions can be found by choosing two of c^c's chords $[1,2]$, $[3,4]$ through D^c_{1u} on p^c_{1u}. Besides D^c_{1u}, the quadrilateral 1234 has two further diagonal points $[1,3] \cap [2,4]$ and $[1,4] \cap [2,3]$ whose join is the polar line of D^c_{1u} w.r.t. the ellipse c^c and whose intersection with p^c_{1u} yields the vanishing point $D^{\perp^c}_{1u}$ (of the lines conjugate w.r.t. D^c_{1u}), *i.e.*, the vanishing point of the orthogonal direction. A second pair $\left(D^c_{2u}, D^{\perp^c}_{2u}\right)$ of conjugate vanishing points can easily be found: D^c_{2u} is the vanishing point of the frontal lines (orthogonal to π) joining the front most and back most point of c^c (tangents parallel to p^c_{1u}, p_1). Then, the corresponding vanishing point $D^{\perp^c}_{2u}$ is the ideal point of p_{1u} which makes D^c_{2u} the central point of the involution on p^c_{1u} and the Thales circle on $D^c_{2u}D^{\perp^c}_{2u}$ becomes the normal to p^c_{1u} through D^c_{2u}.

Clearly, this time the rectification of c^c becomes a circle $c^{\circ c}$. The fact that the tips S and T of the claws and the tip R of the handle are located on c is exactly displayed in the rectification. With the measures $\overline{AB} = 94cm$, $\overline{BC} = 286cm$, we can determine all missing metric data needed for the reconstruction of the object. The top-view of the object (given in Fig. 8) shows the result. The missing lengths and the angle between the handle and the frame can now be read off from the top-view. This is also the case with the length of the handle. (Obviously, a scale has to be taken into account.)

4.3 Concluding remarks

The rectification and the reconstruction depend on the choice of points picked in the image to be rectified. This can only be done with the accuracy of a pixel and one has to clarify the actual size of that portion of the scene covered by one pixel. Obviously, pixels interpreted as points in the background are more sensitive in that respect. The constructive approach vividly demonstrates the inaccuracies caused by marginal deviations in the choice of points/pixels. For example, in the beginning the vanishing point of horizontal lines did not gather on one line (horizon, vanishing line p^c_{1u}). This can be achieved by an adapted choice.

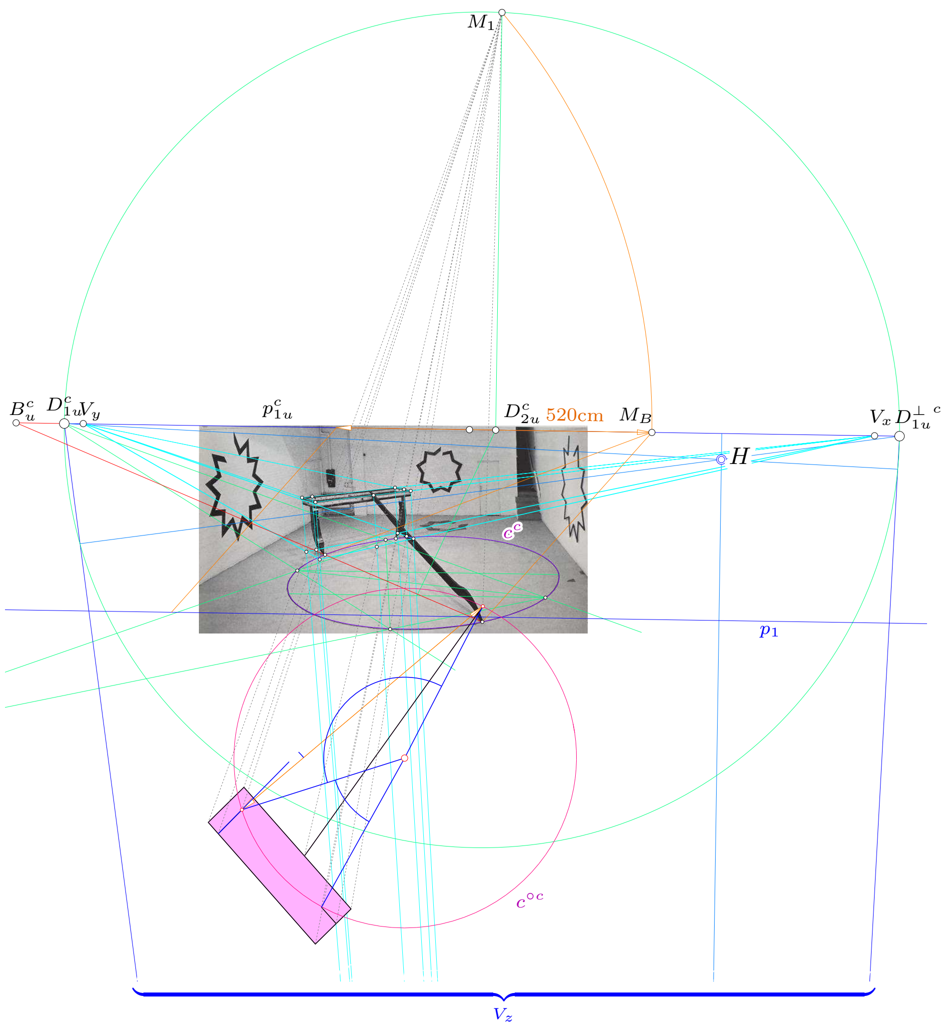

Fig. 7. Rectification, second attempt: with the help of the perspective collineation that maps the elliptic image c^c of the circle to a circle $c^{\circ c}$.

Nevertheless, we prefer the constructive approach because of its simplicity, elegance, and the geometric considerations in behind. It is rather doubtful, whether any software package can choose between proper and improper methods for the rectification/reconstruction, provided that it is aware of geometric techniques.

In principle, no computations are necessary in order to extract data from the image depicted in Fig. 1, maybe except for the determination of the natural size of the object.

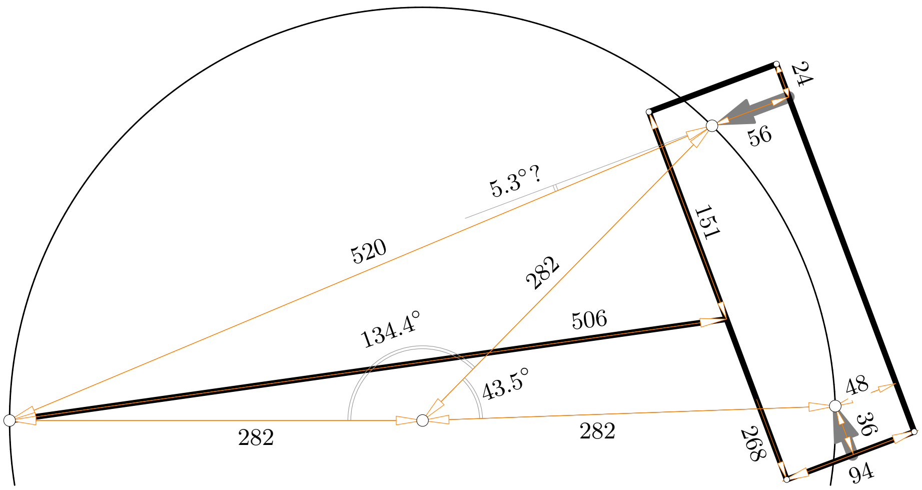

Fig. 8. Top-view of "Ramp and Hyphen", measures in cm.

Acknowledgements

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