A rational minimal Möbius strip

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ABSTRACT: We solve Björlings's problem for minimal surfaces by prescribing data for a ruled Möbius strip and integrating a complex vector valued function. The minimal surface close to the initial data is a surface patch of the topological type of a Möbius strip. Because of the special choice of the initial data, the minimal surface is an algebraic surface of degree 11 and of class 15. We elaborate some special properties of this and some related surfaces, especially the family of associated minimal surfaces. The holomorphic functions corresponding to the Weierstraß-representation of these minimal surfaces are also given. Since the coordinate functions of the surface in question involves only trigonometric and hyperbolic functions, the surface admits a rational parametrization. Moreover, one family of parameter curves appears in a certain orthogonal projection as a family of homofocal cycloidal curves of third order, whereas the other family consists of rational cubic space curves. Finally we show a way to generalize this concept in order to obtain more complicated minimal Möbius strips and introduce the notion of rotoidal minimal surfaces.

Keywords: minimal surface, Björling's problem, Möbius strip, Weierstraß-representation, associate family, rational surface, algebraic surface, cycloidal curve, rational cubic, rotoidal minimal surface.

1. INTRODUCTION

A Möbius strip is a one sided surface that can easily be built from a rectangular piece of paper by gluing together one pair of opposite edges with different orientations, see Fig. 1. Mathematically speaking, we identify the egdes $x = 0$ and $x = 1$ of the unit square by joining points according to $(0, y) \sim (1, 1-y)$. The fact that a Möbius strip can be created from a piece of paper makes it clear that it is possible to create mathematical models of developable Möbius strips. The developable model of a Möbius strip is an embedding with constant Gaussian curvature $K = 0$. An elegant construction of a developable Möbius strip is given in [14]. The result is displayed in Fig. 2. Wunderlich's example is also algebraic. For more information on developable Möbius strips we refer to [7] and the references therein.

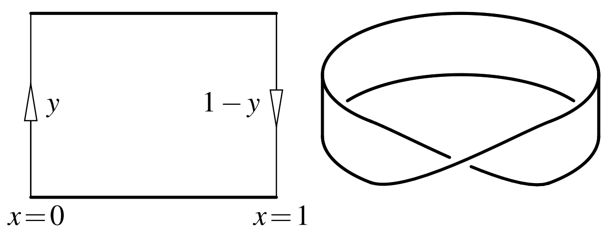

Figure 1: Mathematician's idea of a Möbius strip: joining opposite edges of a rectangle.

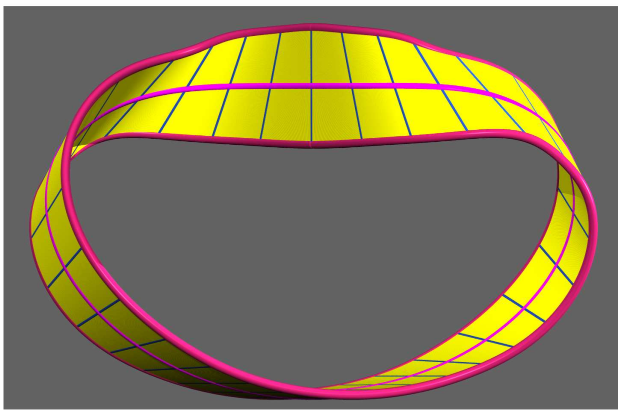

Figure 2: Wunderlich's developable Möbius strip.

In this note we give an embedding of a Möbius strip into Euclidean three-space $\mathbb{R}^3$ with constant

Mean curvature $H = 0$, *i.e.*, a model of a Möbius strip which is part of a minimal surface. Moreover, the presented model is algebraic and admits even rational parametrizations. The most famous example of an algebraic minimal surface is Enneper's surface. It admits a polynomial parametrization with bivariate polynomials of degree three, and therefore, it can be written as a Bézier surface of bidegree $(3,3)$. Implicitization of the polynomial parametrization shows that Enneper's surface is an algebraic minimal surface of degree 9. The dual surface of Enneper's surface, *i.e.*, the manifold of its tangent planes is a surface of degree 6, and thus, Enneper's surface is of class 6. There exist minimal surfaces of degree less than 9 but, unfortunately, they are not real, see *e.g.* [6, 10]. Up to equiform motions there exists only one algebraic minimal surface of degree 12 usually referred to as Richmond's surface, see [9]. In comparison to Wunderlich's developable Möbius strip which is of degree 39 and class 21, the degree and the class of the minimal surface model of the Möbius strip are relativtely low: The degree equals 11, the class equals 22.

We construct this surface by solving Björling's problem: A surface strip or scroll (γ, v), *i.e.*, a smooth curve γ with a smooth unit normal vector field v along γ is prescribed. Then, Björling's formula allows us to compute the uniquely determined minimal surface $\mathscr{M}$ that contains γ and has the unit normal vector field v along γ. In our case, the curve

$$\gamma = (\cos t, \sin t, 0) \tag{1}$$

with $t \in [0, 2\pi[$ is the Euclidean unit circle in the plane $z = 0$. The unit normal vector field

$$v = \left(\cos\frac{t}{2}\cos t, \cos\frac{t}{2}\sin t, \sin\frac{t}{2}\right) \tag{2}$$

with $t \in [0, 2\pi]$ is chosen such that the ruled surface $\mathscr{N}(t,w) = \gamma(t) + w \cdot v(t)$ (with $(t,w) \in [0,2\pi] \times \mathbb{R}$) carries a Möbius strip. The ruled surface $\mathscr{N}$ meets the resulting minimal surface $\mathscr{M}$ along γ at right angles. Slewing the unit vectors of the vector field v about the corresponding tangent of the unit circle through $\pm 90°$, we obtain two ruled surfaces $\mathscr{N}^{\pm}$ which are of the same type as $\mathscr{N}$ but tangent to the minimal surface $\mathscr{M}$, see Fig. 3.

In fact, all three ruled surfaces $\mathscr{N}$, $\mathscr{N}^{\pm}$ are the algebraic ruled surface models of a Möbius strip of lowest possible degree which equals 3. Though they are not developable like the one presented in [14], they have a lot of surprising properties, see [3].

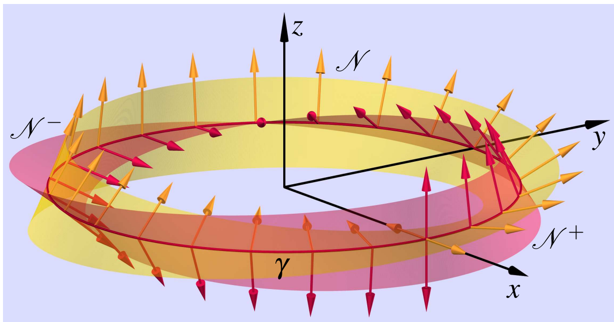

Figure 3: The ruled surfaces $\mathscr{N}$ and $\mathscr{N}^{\pm}$ through γ. All three could serve as initial data.

Due to the nature of the initial data (γ, v), the minimal surface $\mathscr{M}$ along that scroll is algebraic. In Sec. 2, we shall derive a trigonometric parametrization as well as a rational parametrization of the minimal surface $\mathscr{M}$. From that we compute an implicit equation of the minimal Möbius strip. Sec. 3 is dedicated to special curves on the surface and its behaviour at inifinity. Since we are dealing with an algebraic minimal surface, we are able to study its behaviour at infinity. The rational parametrization of the minimal surface $\mathscr{M}$ simplifies the computation of $\mathscr{M}'s$ contour curves with respect to the three principal projections[1]. Then, in Sec. 4, we derive the generating holomorphic functions, *i.e.*, the Weierstraßrepresentation of $\mathscr{M}$. Afterwards,

[1] The three principal projections create the top, front, and (right-)side view. These are the orthogonal projections onto three mutually orthogonal planes, or equivalently, projections in three mutually orthogonal directions. We obtain the top, front, and side view by simply removing the z-, x-, or y-coordinate.

we pay attention to the family of associate minimal surfaces and the behaviour of the surfaces when traversing the associate family in Sec. 5. Finally, in Sec. 6, we show a possible way to create Möbius type minimal surfaces with more windings and introduce the notion of rotoidal minimal surfaces.

2. PARAMETRIZATION OF THE MINIMAL MÖBIUS STRIP

A smooth curve $\gamma:\ I \subset \mathbb{R} \to \mathbb{R}^3$ and a smooth unit vector field $\nu:\ I \to \mathbb{R}^3$ determine a scroll (γ, ν) as the envelope of the one-parameter family of planes $\langle x-\gamma(t), \nu(t)\rangle = 0$ where $\langle \cdot,\cdot\rangle = 0$ is the canonical scalar product of vectors in $\mathbb{R}^3$. In our special case, γ as well as ν are smooth. However, the computation would also work with C^1 curves and C^1 vector fields.

According to Björling (cf. [6]), a parametrization of the minimal surface $\mathscr{M}$ through the scroll (γ, ν), *i.e.* a minimal surface $\mathscr{M}$ through γ with normals ν along γ can be found by computing

$$f(u,v) = \mathfrak{Re}\left(\gamma(t) - i\int_{t_0}^{t} \nu(\theta)\times \mathrm{d}\gamma(\theta)\right) \quad (3)$$

where $\times$ denotes the crossproduct in $\mathbb{R}^3$ induced by the canonical scalar product and $\mathrm{d}\gamma(\theta) = \dot\gamma(\theta)\mathrm{d}\theta$. Subsequent to the integration, the parameter t is considered to be complex, *i.e.*, $t = u + iv$. Then, the real part $\mathfrak{Re}$ of the complex vector function is to be extracted. If we insert γ from (1) and ν from (2) into (3), we obtain

$$f(t) = \begin{pmatrix} \mathrm{c}_{2t} + i\mathrm{c}_t - \frac{1}{3}i\mathrm{c}_{3t} \\ \mathrm{s}_{2t} + i\mathrm{s}_t - \frac{1}{3}i\mathrm{s}_{3t} \\ -2i\mathrm{s}_t \end{pmatrix}, \quad (4)$$

provided that t_0 in (3) is chosen properly and t is replaced with $2t$. The factor 2 causes just a regular change of parameters on $\mathscr{M}$. Here and in the following, we use the abbreviations

$$\mathrm{c}_x := \cos x,\ \mathrm{s}_x := \sin x,\ \mathrm{C}_x := \cosh x,\ \mathrm{S}_x := \sinh x$$

for trigonometric and hyperbolic functions.

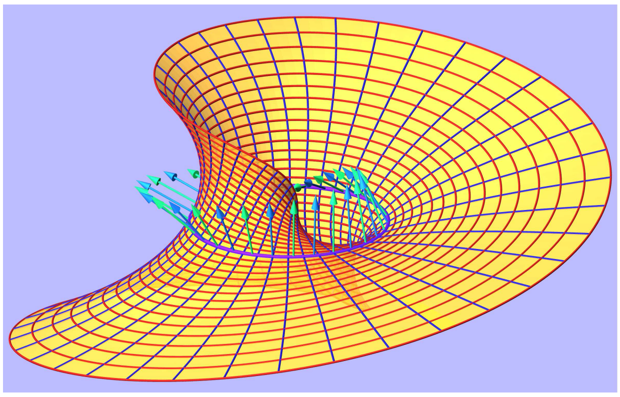

Figure 4: The minimal surface $\mathscr{M}$ according to the parametrization (5) with initial data γ and ν.

We let $t = u + iv$, extract the real part, and use double- and triple-angle formulas for trigonometric and hyperbolic functions which yields

$$f(u,v) = \begin{pmatrix} \mathrm{s}_u\mathrm{S}_v + \mathrm{c}_{2u}\mathrm{C}_{2v} - \frac{1}{3}\mathrm{s}_{3u}\mathrm{S}_{3v} \\ -\mathrm{c}_u\mathrm{S}_v + \mathrm{s}_{2u}\mathrm{C}_{2v} + \frac{1}{3}\mathrm{c}_{3u}\mathrm{S}_{3v} \\ 2\mathrm{c}_u\mathrm{S}_v \end{pmatrix} \quad (5)$$

with $(u,v) \in \mathbb{R}^2$ which is displayed in Fig. 4. This allows us to state

Theorem 2.1. *In a sufficiently close neighbourhood of the circle γ from* (1)*, the minimal surface $\mathscr{M}$ parametrized by* (5) *is a Möbius strip if* $(u,v) \in [0,\pi[\times[-\varepsilon,\varepsilon]$ *and* $\varepsilon \in \mathbb{R}^+$.

In the above theorem, we emphasized that $\mathscr{M}$ is topologically equivalent to a Möbius strip only in a sufficiently small neighbourhood of $\gamma = f(u,0)$, because for increasing v the surface may have self-intersections as is the case with many algebraic minimal surfaces.

Moreover, we can state

Theorem 2.2. *The minimal surface $\mathscr{M}$ given by* (5) *admits a rational parametrization and has a polynomial equation. The minimal surface $\mathscr{M}$ is algebraic, of degree 11, and class 15.*

Proof. In order to prove that $\mathscr{M}$ admits a rational parametrization, we simply use the Weierstrass substitution for trigonometric and hyperbolic functions:

$$\mathrm{c}_u = \frac{1-U^2}{1+U^2}, \mathrm{s}_u = \frac{2U}{1+U^2}, \mathrm{C}_v = \frac{1+V^2}{1-V^2}, \mathrm{S}_v = \frac{2V}{1-V^2}.$$

Note that from the rational parametrization one would expect the surface to be of degree 12.

Now, we show that $\mathscr{M}$ has a polynomial equation. This is easily done with help of a CAS (Maple, Mathematica, ...) by eliminating the parameters U and V. This yields the implicit equation

$$\begin{aligned}\mathscr{M}:\ & 16(x^2+y^2)z^9-72(x^2+y^2)(y-z)z^6 \\ & -27(x^2+y^2)(5y^2-5z^2+2yz)z^3-90yz^4 \\ & -66yz^6-24(yz+x)z^7-243x^2(x^2+y^2)z^3 \\ & -36xyz^4+18(4y^2-3x)z^5+9(z^2+1)^2z^3 \\ & +324x(x^2+y^2)z^3-81y^2(x^2+y^2)-27y^2z \\ & +9(2x+7y^2-12x^2)z^3+108y(x^2+y^2)z^2 \\ & +27(2x-6)xyz^2+108xy^2z \\ & +54(1-x^2-y^2)y^3=0\end{aligned} \tag{6}$$

which is of degree 11. The class of the minimal Möbius strip is found by implicitizing the dual surface. This yields a polynomial of degree 15 in terms of the coordinates of the surfaces tangent planes, see also [5, vol. 1, p. 315]. □

Figure 5 shows an image of the minimal surface $\mathscr{M}$ created from the implicit equation (6).

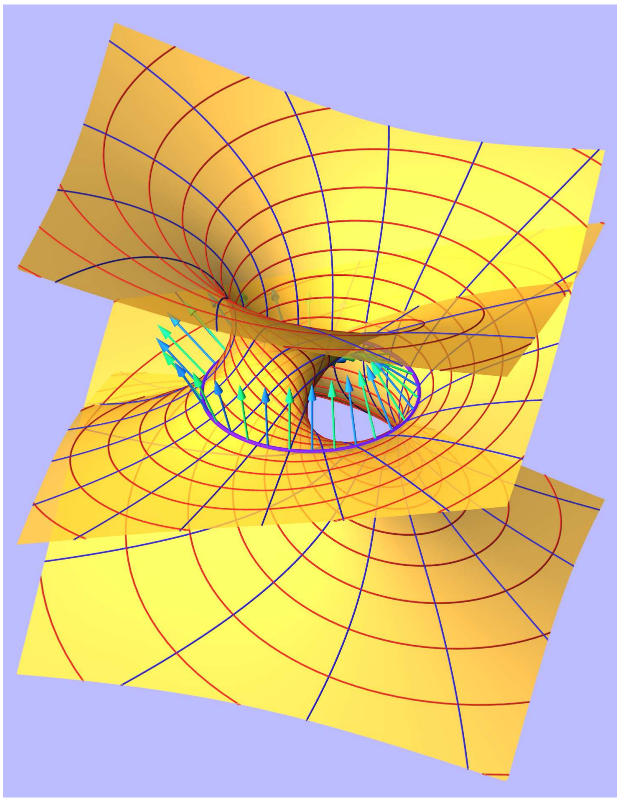

Figure 5: A cuboid part of the minimal surface $\mathscr{M}$ with initial data γ and ν.

3. SOME PROPERTIES OF THE MINIMAL MÖBIUS STRIP

Due to the construction of $\mathscr{M}$, it carries the circle $\gamma:\ x^2+y^2=1$ in the plane $z=0$. Since we have got an implicit equation of $\mathscr{M}$, we are able to study the behaviour of $\mathscr{M}$ with respect to the coordinate planes and at inifinity by stating

Theorem 3.1. *The minimal surface* $\mathscr{M}$ *parametrized by* (5) *(or equivalently given by the implicit equation* (6)*) meets the plane at infinity along a pair of complex conjugate lines and at the common line at infinity of all planes* $z=c$ *(with* $c\in\mathbb{R}$*). The latter line is nine-fold.*

The x-axis of the underlying Cartesian coordinate system is a three-fold line on $\mathscr{M}$*.*

Proof. The first part of the statement is verified by homogenizing $\mathscr{M}$'s equation. Therefore, we set $x=X_1X_0^{-1}$, $y=X_2X_0^{-1}$, $z=X_3X_0^{-1}$ in (6) and multiply by X_0^{11}. Then, we substitute $X_0=0$ which gives an implicit and homogeneous equation (up to a non-zero constant factor)

$$\mu:\ X_3^9(X_1^2+X_2^2)=0 \tag{7}$$

of the set μ of ideal points of $\mathscr{M}$. The factor $X_3^9=0$ gives the equation of the ideal line of all planes $z=c$; the multiplicity equals 9.

In order to verify the second part of the statement, we compute the intersection of $\mathscr{M}$ with all planes of the pencil about the x-axis. Their equations are $py-qz=0$ with $(p,q)\neq(0,0)$. Eliminating either z or y results in a polynomial of degree 11 where either the factor y^3 or the factor

z^3 splits off independent of the choice of $(p,q) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Thus, $\{(x,y,z) | py - qz = y = 0\} = \{(x,y,z) | py - qz = z = 0\} = \{(t,0,0) | t \in \mathbb{R}\}$ is a three-fold line on $\mathscr{M}$.

On the other hand we can easily verify that $\operatorname{grad}\mathscr{M}(t,0,0) = (0,0,0)$ for all $t \in \mathbb{R}$, and thus, the points $(t,0,0)$ with $t \in \mathbb{R}$ are singular on $\mathscr{M}$ and at least of multiplicity 2. Further, the Hessian $\mathrm{H}(\mathscr{M})(t,0,0)$ equals the zero matrix in $\mathbb{R}^{3\times 3}$ for all $t \in \mathbb{R}$. Only the tensor of the third partial derivatives $\frac{\partial^3 \mathscr{M}}{\partial (x,y,z)^3}(t,0,0)$ does not vanish completely for all $t \in \mathbb{R}$, although it is not of full rank in any slice. Thus, the points on the x-axis are at least of multiplicity 3. □

The tensor $\frac{\partial^3 \mathscr{M}}{\partial (x,y,z)^3}(t,0,0)$ is the zero-tensor only for $t = 1$. Thus, the point $(1,0,0)$ can be suspected to be a point of multiplicity higher than 3. It turns out that the point $(1,0,0)$ is the only point of multiplicity 4 on the x-axis since all derivatives of $\mathscr{M}$ up to order three vanish there, but, for example, $\frac{\partial^4 \mathscr{M}}{\partial x \partial y^3}(1,0,0) = 648 \neq 0$.

The fact that the ideal curve μ with equation (7) degenerates completely, *i.e.*, it consists only of lines, fits to an old result given in [5]: The asymptotic cone of an algebraic minimal surface degenerates completely, *i.e.*, it is a collection of finitely many planes.

A closer look at the parametrization (5) of $\mathscr{M}$ tells us the following

Theorem 3.2. *The top view (orthogonal projection in the direction of the z-axis) of the u-curves (curves on $\mathscr{M}$ with $v = \mathrm{const.}$) are higher cycloids, in general of third order.*

Proof. From the Cartesian coordinates x, y in the top view, we build complex numbers by letting $w = x + iy$. This gives

$$w = \mathrm{s}_u\mathrm{S}_v + \mathrm{c}_{2u}\mathrm{C}_{2v} - \tfrac{1}{3}\mathrm{s}_{3u}\mathrm{S}_{3v} + \\ + i(-\mathrm{c}_u\mathrm{S}_v + \mathrm{s}_{2u}\mathrm{C}_{2v} + \tfrac{1}{3}\mathrm{c}_{3u}\mathrm{S}_{3v}).$$

We aim at a more compact notation. For that purpose, we apply Euler's formula $\mathrm{e}^{i\phi} = \cos\phi + i\sin\phi$ three times and arrive at

$$w = -i\mathrm{S}_v\mathrm{e}^{iu} + \mathrm{C}_{2v}\mathrm{e}^{2iu} + \frac{i}{3}\mathrm{S}_{3v}\mathrm{e}^{3iu}.$$

Note that v is constant. Figure 6 shows the top view of the parameter curves $v = \mathrm{const}$. Now, we

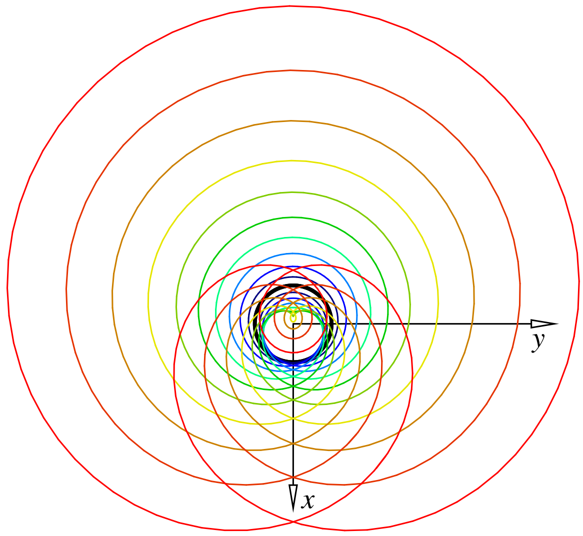

Figure 6: The u-curves on $\mathscr{M}$ appear as cycloidal curves in a top view.

compare with [13, Eq. (6), p. 280] and find that the characteristic of one (out of six possible) generation(s) equals

$$\omega_1 : \omega_2 : \omega_3 = 1 : 2 : 3$$

which gives the ratio of angular velocities of three superponed rotations. These curves can be obtained as the orbits of endpoints of a three-bar mechanism (with open end), see Figure 7. The lenghts of the legs are

$$\overline{A_0A_1} = a_1 = \mathrm{S}_v, \ \overline{A_1A_2} = a_2 = \mathrm{C}_{2v}, \\ \overline{A_2A_3} = a_3 = \tfrac{1}{3}\mathrm{S}_{3v}.$$

According to [13], this motion can also be generated by rolling circles. As shown in Figure 7, there is a fixed circle c_0 (center A_0, radius r_0) and a circle c_2' (center A_1, radius r_2') rolling on it. Attached to the circle c_2' there is a concentric circle

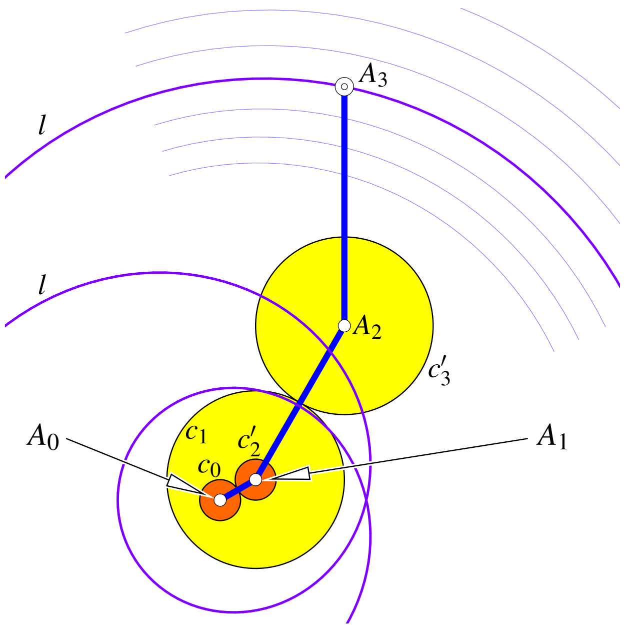

Figure 7: The top views of $\mathcal{M}$'s parameter curves generated either with rolling circles or by an open three-legged kinematic chain.

c_1 (center A_1, radius r_1) on which a further circle c_3' (center A_2, radius r_3') is rolling, taking the point A_3 with it. Following [13, Eq. (6), p. 280], the radii are

$$r_0 = r_2' = \frac{\mathrm{S}_v}{2}, \quad r_1 = r_3' = \frac{\mathrm{C}_{2v}}{2}.$$

These curves are algebraic for $\omega_1 : \omega_2 : \omega_3$ is rational. The algebraic degree equals 6, except for the curve γ which corresponds to $v = 0$. It is of degree two. □

The u-curves ($v =$ const. are closed for any $v \in \mathbb{R}$ since the third coordinate function equals $z = 2\mathrm{c}_u \mathrm{S}_v$. Thus, the u-curves can be seen as generalized harmonic oscillation curves in the sense of [8]. Figure 8 shows one particular u-curve with its three principal views The front and side view of the u-curves on $\mathcal{M}$ can be viewed as generalized Lissajou-curves, cf. the respective parametrizations (5).

Further, we can say:

Theorem 3.3. *The v-curves (curves on $\mathcal{M}$ with $u =$ const.) are rational cubic space curves. The*

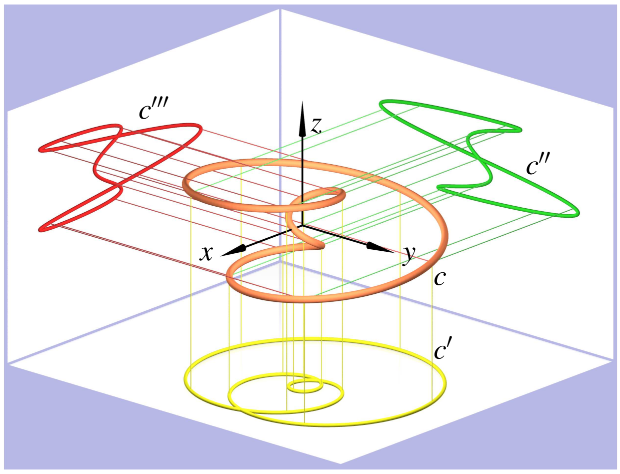

Figure 8: Principal views of a u-curve $c \neq \gamma$.

top views of the v-curves are rational cubic curves with their singularities on a nephroid n.

Proof. Again, we consider the x- and y-coordinate of (5) as the parametrization of curves g in the plane $z = 0$, see Fig. 9. Under the assumption $u =$ const., all the trigonometric expressions in this parametrization are constant and only subject to $\mathrm{c}_a^2 + \mathrm{s}_a^2 = 1$ for all $a \in \mathbb{C}$. The hyperbolic functions C_{2v} and S_{3v} can be expressed in S_v only. Therefore, we have

$$g(t) = (1+2t^2)\begin{pmatrix}\mathrm{c}_{2u}\\ \mathrm{s}_{2u}\end{pmatrix} + t\begin{pmatrix}\mathrm{s}_u - \mathrm{s}_{3u}\\ \mathrm{c}_{3u} - \mathrm{c}_u\end{pmatrix} + \tfrac{4}{3}t^3\begin{pmatrix}-\mathrm{s}_{3u}\\ \mathrm{c}_{3u}\end{pmatrix} \quad (8)$$

with $t \in \mathbb{R}$ and the regular reparametrization $t = \mathrm{S}_v$. If $u \in \mathbb{R}$, (8) is a one-parameter family of cubic Bézier curves with the control points

$$B_0 = \begin{pmatrix}\mathrm{c}_{2u}\\ \mathrm{s}_{2u}\end{pmatrix}, \; B_1 = B_0 + \tfrac{1}{3}\begin{pmatrix}\mathrm{s}_u - \mathrm{s}_{3u}\\ \mathrm{c}_{3u} - \mathrm{c}_u\end{pmatrix},$$
$$B_2 = 2B_1 - \tfrac{1}{3}B_0, \; B_3 = 3B_0 + \tfrac{1}{3}\begin{pmatrix}3\mathrm{s}_u - 7\mathrm{s}_{3u}\\ 7\mathrm{c}_{3u} - 3\mathrm{c}_u\end{pmatrix}.$$

The v-curves on $\mathcal{M}$ are rational cubic space curves, since the third coordinate function can be written as

$$z(t) = 2\mathrm{c}_u t. \quad (9)$$

The locus of singular points of the curves (8) is

$$n(u) = \frac{1}{6}\begin{pmatrix}\mathrm{c}_{6u} + 3\mathrm{c}_{2u} + 2\\ \mathrm{s}_{6u} + 3\mathrm{s}_{2u}\end{pmatrix} \quad (10)$$

which is an algebraic curve of degree 6 known as *nephroid*, see [4, 11] which is also a cycloid since (10) can be rewritten in terms of complex coordinates as

$$w = x + iy = \frac{1}{3} + \frac{1}{2}e^{2it} + \frac{1}{6}e^{6it}.$$

The latter representation allows us to give a kinematic generation of this curve as done in the proof of Thm. 3.2, see also [12, 13].

The singular points of the cubic curves (8) are isolated double points, provided that $u \neq 0, \pi$. The v-curves at $u = 0, \pi$ have ordinary cusps located on n. □

The representation (8) of the v-lines together with the z-coordinate (9) gives rise to a simpler rational parametrization than the one given in the proof of Thm. 2.2: Firstly, we express all hyperbolic functions in (5) in terms of S_v and use the regular reparametrization $t = S_v$. Then, we express c_{2u}, s_{2u}, c_{3u}, s_{3u} in terms of c_u, s_u and replace the latter by their rational equivalents.

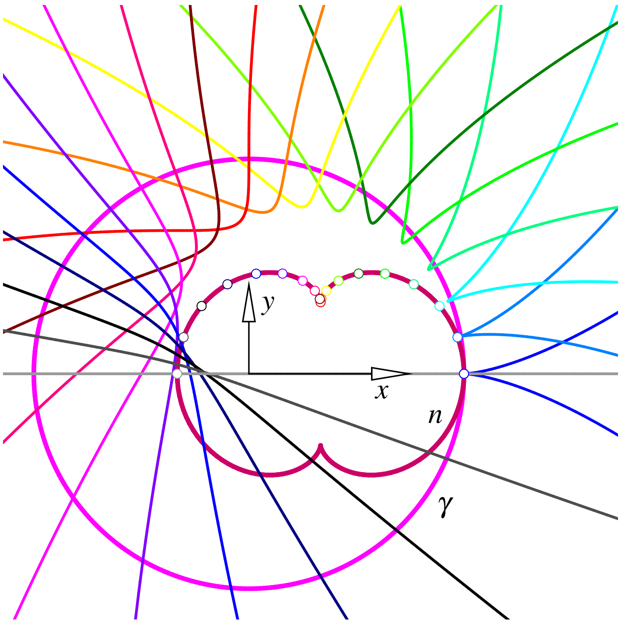

Figure 9: Some v-lines on $\mathscr{M}$ together with the nephroid n carrying the cubics' singular points.

The fact that the v-lines (8) and (9) are (rational) cubic space curves enables us to find the contour lines of $\mathscr{M}$ with respect to the three principal projections. We have

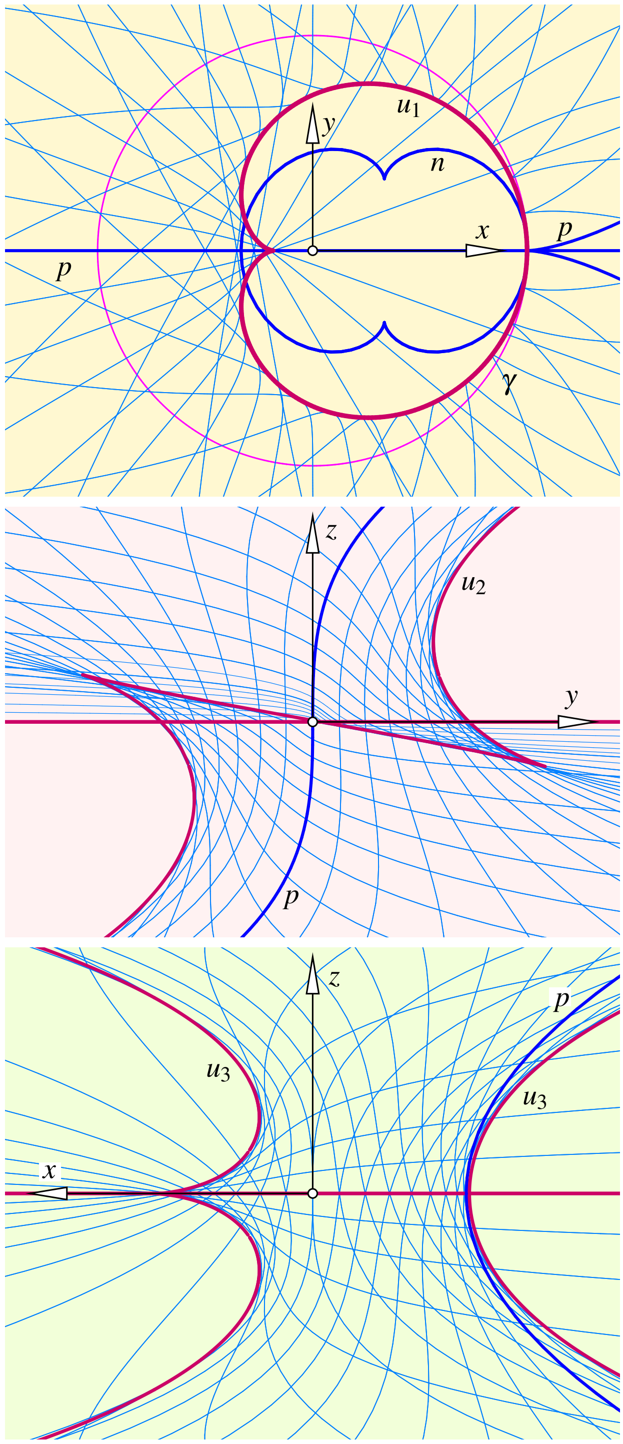

Figure 10: The contours of $\mathscr{M}$ as the envelopes of the cubic v-lines: top view (above), front view (middle), side view (bottom).

Theorem 3.4. *The contour lines u_1, u_2, u_3 of the top, front, and side view of $\mathscr{M}$ are:*
(1) an algebraic curve of degree 12 and genus 1;
(2) the union of a straight line and a rational curve of degree 9;
(3) the union of a straight line and a rational curve of degree 12.

Proof. We use (8) and (9). In order to compute the contours of $\mathscr{M}$ with respect to the three principal views. For a fixed u (or $U = \tan \frac{u}{2}$), any pair of coordinate functions gives a parametrization of a planar cubic curve (which may degenerate for certain but finitely many values of u). This planar cubic curve is the projection of the cubic space curve on $\mathscr{M}$.

However, eliminating t delivers an equation $q(U) = 0$ of a planar cubic curve whose equation has coeffcients that are polynomials in U. (Like in many cases, the rational parametrization of $\mathscr{M}$ again turns out to be favorable.) The contour curve of $\mathscr{M}$ with respect to that particular projection is the envelope of the images $q(U) = 0$ of the cubic curves. In order to compute this envelope, we differentiate the cubics' equations $q(U) = 0$ with respect to the parameter U. Then, we eliminate U from $q(U) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}U} q(U) = 0$ and find the equation of the envelope, *i.e.*, the contour curve.

In any of the three cases, the elimination process results in an algebraic equation with many factors. Only a small number of factors really contributes to the contour.

Table 1 collects all components of the final resultants in the computation of the contour curves with respect to the three principal projections. There, d gives the degree of the component and m its multiplicity. Parasitic branches are labelled with p. Further abbreviations are explained in the caption of Table 1. ☐

The contours of $\mathscr{M}$ with respect to the three principal views are displayed in Fig. 10. Parasitic branches are shown in blue only if they do not consist exclusively of isolated real points.

top view						
d	1	2	3	6	6	**12**
m	12	9	1	2	4	2
shape	p	i	p	c	n, p	$\mathbf{u_1}$
front view						
d		**1**	2	3	6	**9**
m		6	1	1	2	2
shape		$\mathbf{u_2}$	i	p	c	$\mathbf{u_2}$
right-side view						
d		1	2	2	6	12
m		12	1	1	2	2
shape		$\mathbf{u_3}$	c	p	c	$\mathbf{u_3}$

Table 1: The components of the contour curves: p indicates a parasitic branch, c indicates a curve with real equation and only finitely many real points which are then isolated and singular, i stands for a pair of isotropic lines, u_i are the real components of the i^{th} contour curve, and n is the nephroid mentioned in Thm. 3.3.

4. AN ALTERNATIVE GENERATION

We have found a parametrization of $\mathscr{M}$ while solving the Björling problem. It is well known, see for example [6] that for each minimal surface a parametrization can be found via

$$f = \mathfrak{Re}\left(\frac{1}{2}\int \begin{pmatrix} A^2 - B^2 \\ i(A^2 + B^2) \\ 2AB \end{pmatrix} \mathrm{d}z\right) \qquad (11)$$

with meromorphic functions A and B. This representation of minimal surfaces is due to Weierstraß. The meromorphic functions A and B can found by first computing

$$F = \partial_u f - i\partial_v f.$$

Then, from $F = (F^1, F^2, F^3)$ we obtain

$$A = F^1 - iF^2, \;\; B = F^3 A^{-1},$$

provided that $A \neq 0$. This will not be the case here and so it needs no further discussion. The parametrization of the minimal Möbius strip (5) leads to

$$A = F^1 - iF^2 = \cos w - i \sin w$$
$$-2i\cos 2w - 2\sin 2w - \cos 3w + i\sin 3w,$$

and

$$F^3 = -2i\cos w$$

provided that $w = u + iv$. In terms of the natural exponential function we have

$$A = \mathrm{e}^{-iw} - 2i\mathrm{e}^{-2iw} - \mathrm{e}^{-3iw}, \quad B = \frac{-i(\mathrm{e}^{iw} + \mathrm{e}^{-iw})}{A}.$$

The change of parameters $t = \mathrm{e}^{-iw}$ shows that A and B can be replaced with rational functions:

$$A' = -t(t+i)^2, \quad B' = \frac{1+it}{t^2(t+i)}.$$

Unfortunately, the functions A' and B' together with (11) will not reproduce the rational parametrization mentioned in the proof of Thm. 2.2, but an equivalent one.

5. THE ASSOCIATE FAMILY

Each minimal surface defines a smooth one-parameter family of *associated minimal surfaces*, see [6]. Here, we obtain a parametrization of this family by modifying (3) to

$$f^\star(\tau) = \mathfrak{Re}\left(\mathrm{e}^{i\tau}\left(c(t) - i\int_{t_0}^{t} \nu(\theta) \times \mathrm{d}\gamma(\theta)\right)\right) \quad (12)$$

where each $\tau \in [0, 2\pi]$ corresponds to a minimal surface of the family. With $\mathrm{e}^{i\tau} = \cos\tau + i\sin\tau$ and $f = \mathfrak{Re}(\Phi)$ we find

$$\begin{aligned} f^\star(\tau) &= \mathfrak{Re}(\mathrm{e}^{i\tau}(\mathfrak{Re}\,\Phi(\tau) + i\,\mathfrak{Im}\,\Phi(\tau))) = \\ &= \cos\tau \cdot \mathfrak{Re}\,\Phi(\tau) - \sin\tau \cdot \mathfrak{Im}\,\Phi(\tau) \end{aligned} \quad (13)$$

with f from (4). The parametrization $\mathfrak{Im} f$ is the imaginary part of f from (3) after the substitution $t = 2(u+iv)$. Note that the surface $\mathscr{M}(\frac{\pi}{2})$ parametrized by $\mathfrak{Im} f = f^\star(\frac{\pi}{2})$ is itself a minimal surface. We have

Lemma 5.1. *The minimal surface $\mathscr{M}(\frac{\pi}{2})$ with the parametrization $\mathfrak{Im} f = f^\star(\frac{\pi}{2})$ is an algebraic minimal surface of degree 22.*

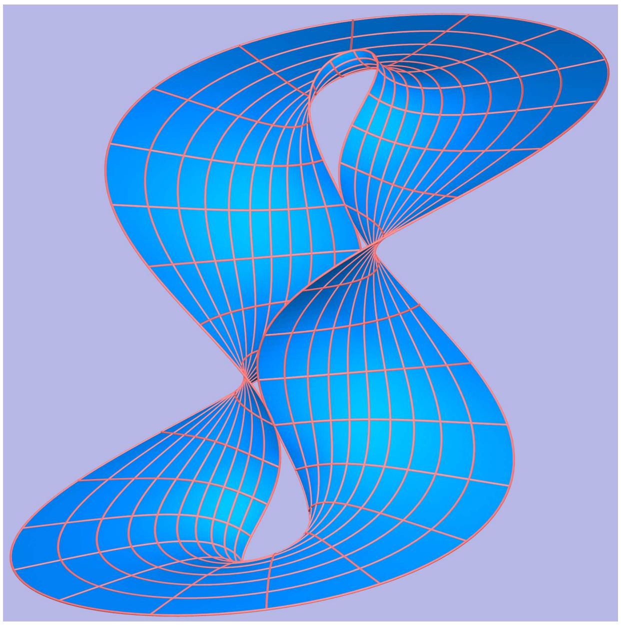

Figure 11: The minimal surface $\mathscr{M}(\frac{\pi}{2})$.

Figure 11 shows an axonometric view of $\mathscr{M}(\frac{\pi}{2})$. However, $\mathscr{M}(\frac{\pi}{2})$ is not of the topological type of a Möbius strip. The parametrizations of the minimal surfaces in the present associate family are linear combinations of $\mathfrak{Re}\Phi$ and $\mathfrak{Im}\Phi$ as we know from (13).

With Lem. 5.1 and Thm. 2.2 we can formulate

Theorem 5.1. *The minimal surfaces associated to $\mathscr{M}$ are algebraic minimal surfaces admitting rational parametrizations. The algebraic degree of the minimal surfaces in the associate family equals 22, except for $\mathscr{M}(0) = \mathscr{M}$.*

Proof. A rational parametrization of the surfaces in the associate family is obtained by substituting the rational expressions for the trigonometric and the hyperbolic functions as given in the proof of Thm. 2.2. The algebraic degree can be found by implicitizing the parametrization $f^\star(\tau)$ after the rational substitution like done in the proof of Thm. 2.2. It is also advantageous to substitute

$$\cos\tau = \frac{1-T^2}{1+T^2}, \quad \sin\tau = \frac{2T}{1+T^2}.$$

For the implicitization it is recommended to use a CAS. □

Figure 12 shows a sequence of minimal surfaces in the associate family.

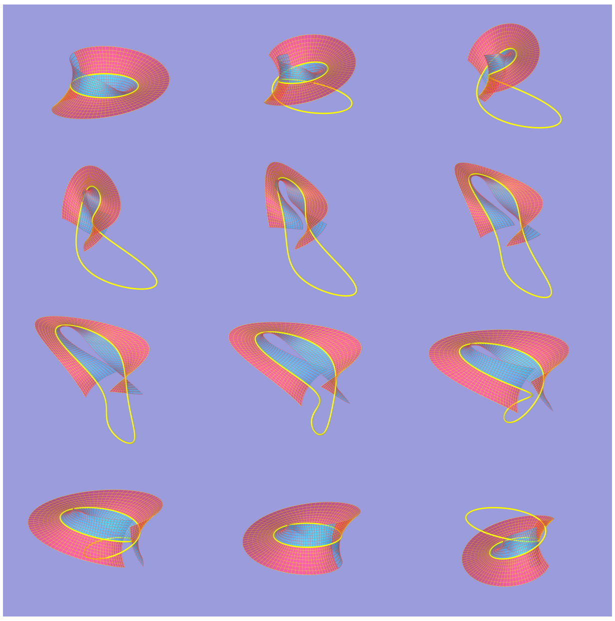

Figure 12: Slightly more than a half turn in the family of associated minimal surfaces: $\tau = \frac{k\pi}{10}$ with $k \in \{0, \ldots, 11\}$.

Finally, we can give the generating functions A and B in the Weierstraß-representation (11) of the minimal surfaces in the associate family. The computation does not differ from that shown in Sec. 4 and we arrive at

$$A(\tau) = \mathrm{e}^{i\tau} A, \;\; B(\tau) = B.$$

Rational equivalents can be found by substituting $\mathrm{e}^{-iw} = t$ and simultaneously replacing $\cos\tau$ and $\sin\tau$ by their rational equivalents.

6. REMARKS AND GENERALIZATIONS

We started with the unit normal vector field ν given in (2) assuming that $\nu(0)$ is pointing in the direction of the x-axis. Changing ν from (2) to

$$\nu(t,\phi) = \left(\mathrm{c}_{\frac{t}{2}+\phi}\,\mathrm{c}_t, \mathrm{c}_{\frac{t}{2}+\phi}\,\mathrm{s}_t, \mathrm{s}_{\frac{t}{2}+\phi}\right) \qquad (14)$$

with $\phi \in]0,\pi[$ does not produce substantially new minimal surfaces. Though these surfaces are still of the topological type of a Möbius strip, admit rational parametrizations, and are algebraic, they are only rotated copies of $\mathscr{M}$ from (5) or (6). The axis of the rotation transforming the minimal surface with unit normal vector field (2) into the minimal surface with unit normal vector field (14) is the z-axis. The angle of rotation equals ϕ.

A wide class of minimal surfaces containing rational (and thus algebraic) minimal surfaces of Möbius type can be generated with the choice

$$\nu(t) = (\cos qt \cos t \cos qt \sin t, \sin qt) \qquad (15)$$

with $q \in \mathbb{R} \setminus \{-1,1\}$ instead of ν from (2). These minimal surfaces can be called *rotoidal minimal surfaces*, since they, somehow, behave like surfaces generated by two uniform and proportional rotations about skew axes, cf. [1, 2].

Like in Sec. 2, we can compute a parametrization by inserting γ from (1) and the unit normal vector field ν from (15) into (3) and obtain

$$f = \begin{pmatrix} \mathrm{c}_u \mathrm{C}_v \\ \mathrm{s}_u \mathrm{C}_v \\ \frac{1}{q}\mathrm{c}_{qu}\mathrm{S}_{qv} \end{pmatrix} + \\ + \frac{\mathrm{S}_{(q+1)v}}{2(q+1)} \begin{pmatrix} -\mathrm{s}_{(q+1)u} \\ \mathrm{c}_{(q+1)u} \\ 0 \end{pmatrix} - \frac{\mathrm{S}_{(q-1)v}}{2(q-1)} \begin{pmatrix} \mathrm{s}_{(q-1)u} \\ \mathrm{c}_{(q-1)u} \\ 0 \end{pmatrix}. \qquad (16)$$

The parametrization (16) delivers the parametrization (5) with $q = \frac{1}{2}$ after changing $u \to 2u$ and $v \to 2v$.

The rotoidal minimal surfaces are closed, rational (and algebraic) if $q \in \mathbb{Q} \setminus \{-1,1\}$. Rotoidal minimal surfaces of Möbius type can especially be found by choosing $q = \frac{1}{2}r$ with an odd integer r. Figure 13 shows the rotoidal Möbius type minimal surface with $q = \frac{5}{2}$.

The rotoidal minimal surfaces that correspond to $q = \pm 1$ are not algebraic. This is caused by the fact that the antiderivatives of $\sin^2 t$ or $\cos^2 t$ are not purely trigonometric functions.

The degree of the algebraic rotoidal minimal surfaces grows rapidly: If $q = 2$, the corresponding rotoidal minimal surface is of degree 26.

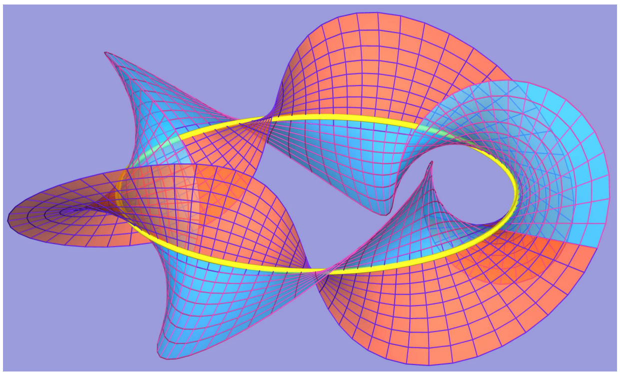

Figure 13: A closed and rational rotoidal minimal Möbius strip with $q = \frac{5}{2}$.

In analogy to Sec. 4, we compute the generating meromorphic functions A and B for the Weierstraß-representation of the rotoidal minimal surfaces. First, we insert (1) and (15) into (3). From the resulting real parametrization, we create the vector function $(F^1, F^2, F^3) = \frac{\partial}{\partial u} f - i \frac{\partial}{\partial v} f$. Finally, we find $A = F^1 - iF^2$ and $B = F^3/A$ (provided that $A \neq 0$) and arrive at

$$A = -i\mathrm{e}^{-iz} + \tfrac{1}{2}\mathrm{e}^{i(q-1)z} - \tfrac{1}{2}\mathrm{e}^{-i(q+1)z}, \quad F^3 = -\tfrac{1}{2}i\left(\mathrm{e}^{iqz} + \mathrm{e}^{-iqz}\right)$$

which can also be written in terms of algebraic functions if $q \in \mathbb{Q} \setminus \{-1, 1\}$ by substituting $t = \mathrm{e}^{-iz}$. If $q = \frac{n}{m}$ with $\gcd(m,n) = 1$, we obtain rational functions with $t \to u^m$.

We can summarize our results in

Theorem 6.1. *The rotoidal minimal surfaces through the scroll* (γ, ν) *with* γ *from* (1) *and* ν *given in* (15) *can be parametrized by* (16). *The rotoidal minimal surfaces* (16) *are closed, admit rational parametrizations, and are algebraic if, and only if,* $q \in \mathbb{Q} \setminus \{-1, 1\}$.

The top views of the u-curves (curves with $v = \text{const.}$*) are cycloidal curves whose order is at most three. (The algebraic order of these curves is at most 6.) The v-curves (curves with* $u = \text{const.}$*) are rational cubic space curves, and thus, all linear images of the v-curves are rational cubic plane curves, provided that the center of the projection is not located on such a curve.*

Proof. Since the parametrization (16) is already found, the existence of rational parametrizations with the techniques from the proof of Thm. 2.2, provided that $q \in \mathbb{Q} \setminus \{-1, 1\}$. If $q = \frac{n}{m}$ with $\gcd(m,n) = 1$, we have to change the parameters according to

$$u' = u\frac{n}{m}, \quad v' = v\frac{n}{m}$$

before we replace the trigonometric and hyperbolic functions with the equivalent rational expressions.

Once a rational parametrization is found, an implicit equation can be computed for given $q \in \mathbb{Q} \setminus \{-1, 1\}$.

The study of special curves on the rotoidal minimal surfaces is simple and works in the same way as demonstrated in the proof of Thm. 3.2 and Thm. 3.3. For example: Building the complex number $w = x + iy$ with x and y being the first and second coordinate function of (16), we find the top views of the u-curves as

$$w = \mathrm{C}_v \mathrm{e}^{iu} + \frac{i\,\mathrm{S}_{(1+q)v}}{2(1+q)} \mathrm{e}^{(1+q)iu} - \frac{i\,\mathrm{S}_{(1-q)v}}{2(1-q)} \mathrm{e}^{(1-q)iu}.$$

With [13, Eq. (6), p. 280], we find

$$a_1 = \mathrm{C}_v, \quad a_2 = \frac{\mathrm{S}_{(1+q)v}}{2(1+q)}, \quad a_3 = \frac{\mathrm{S}_{(1-q)v}}{2(1-q)}$$

and

$$\omega_1 : \omega_2 : \omega_3 = 1 : 1+q : 1-q$$

which determine the radii of the polhodes. □

The ratio $\omega_1 : \omega_2 : \omega_3$ of angular velocities for $q = \frac{1}{2}$ agrees with that given in the proof of Thm. 3.2 only up to permutations. However, cycloids of order three allow $3! = 6$ different generations.

REFERENCES

[1] G. GLAESER: *Über die aus zwei gleichförmigen Drehungen um rechtwinkelig kreuzende Achsen zusammengesetzte Raumbewegung*. PhD thesis, Vienna University of Technology, 1980.

[2] F. HOHENBERG: *Über die Zusammensetzung zweier gleichförmiger Schraubungen.* Monatsh. Math. **54** (1950), 221–234.

[3] J. KRAMES: *Die Regelfläche dritter Ordnung, deren Striktionslinie eine Ellipse ist.* Sb. Akad. Wiss. Wien **127** (1918), 1–22.

[4] J.D. LAWRENCE: *A Catalog of Special Plane Curves.* Dover Publications, New York, 1972.

[5] S. LIE: *Gesammelte Abhandlungen.* Friedrich Engel, Poul Heegaard, eds., B.G. Teubner, Leipzig, 1922–1960.

[6] J.C.C. NITSCHE: *Vorlesungen über Minimalflächen.* Springer-Verlag, 1975.

[7] M. PFURNER, J. SCHADLBAUER, H.-P. SCHRÖCKER: *A recyclable Möbius strip.* Proceedings of ICGG 2014, The 16th International Conference on Geometry and Graphics, August 4–8, 2014, Innsbruck, Austria, article no. 001.

[8] H. POTTMANN: *Zur Geometrie höherer Planetenumschwungbewegungen.* Mh. Math. **97** (1984), 141–156.

[9] H.W. RICHMOND: *On the simplest algebraic minimal curves, and the derived real minimal surfaces.* Transact. Cambridge Phil. Soc. **19** (1904), 69–82.

[10] E. STUDY: *Über einige imaginäre Minimalflächen.* Ber. königl. sächs. Ges. Wiss., math.-phys. Kl. **LXIII** (1911), 14–26.

[11] H. WIELEITNER: *Spezielle Ebene Kurven.* G.J. Göschen'sche Verlagshandlung, Leipzig, 1908.

[12] W. WUNDERLICH: *Ebene Kinematik.* Bibliograph. Inst. Mannheim, 1970.

[13] W. WUNDERLICH: *Höhere Radlinien.* Österr. Ingen. Archiv **1** (1947), 277–296.

[14] W. WUNDERLICH: *Über ein abwickelbares Möbiusband.* Monatsh. Math. **66** (1962), 276–289.

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