A rarity in geometry: a septic curve

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Abstract

We study the locus $\mathcal{C}$ of all points in the plane whose pedal points on the six sides of a complete quadrangle lie on a conic. In the Euclidean plane, it turns out that $\mathcal{C}$ is an algebraic curve of degree 7 and genus 5 and not of degree 12 as it could be expected. Septic curves occur rather seldom in geometry which motivates a detailed study of this particular curve. We look at its singularities, focal points, and those points on $\mathcal{C}$ whose pedal conics degenerate. Then, we show that the septic curve occurs as the locus curve for a more general question. Further, we describe those cases where $\mathcal{C}$ degenerates or is of degree less than 7 depending on the shape of the initial quadrilateral.

Subject Classification (AMS 2020): 14H45, 14P99, 51F99, 51N15

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1 Introduction

1.1 Septic curves and curves related to a quadrilateral

Algebraic curves of degree two, three, and four (conics, cubics, and quartics) appear frequently in many geometrical problems (see, *e.g.*, [9, 11, 14, 15, 17, 18, 23]). This is caused by the fact that many problems in geometry involve distances between points or angles between lines and a quadratic form is responsible for measuring distances and angles in the Euclidean plane. Curves of odd degrees proved useful in Computer Aided Geometric Design: Cubic, quintic, and even septic curves (in plane and in space) are well suited for solving interpolation tasks with tangent or curvature continuity [6, 7, 13, 19, 21] and are also helpful in spaces of geometric objects, such as lines and spheres [20].

Planar curves of odd degree may be the images of algebraic curves under certain Cremona transformations: Linear components of the image curve will split off if the initial curve passes through base points of the transformation [4, 5, 8] as is the case with many but not all cubic curves and most of the algebraic curves which are related to the geometry of a triangle, see the list on B. Gibert's page [10].

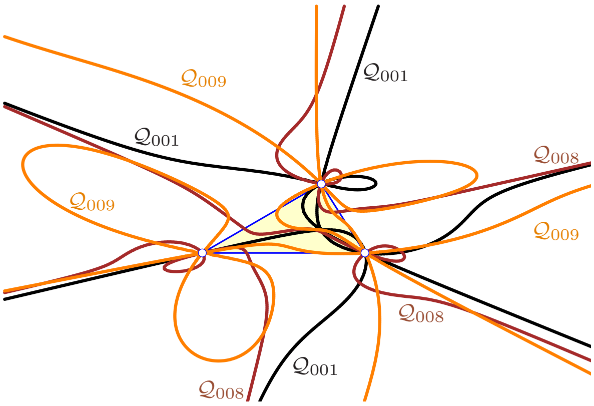

Figure 1: Triangle related septics: The curves $\mathcal{Q}_{001}$, $\mathcal{Q}_{008}$, $\mathcal{Q}_{009}$ are labeled according to Gibert's list [10].

On Gibert's page [10], we find, among many other curves, 12 septic curves related to the geometry of the triangle. Three of these septics are shown in Figure 1. For example, the Darboux septic $\mathcal{Q}_{001}$ is the locus of all 4^{th} pedal points of a point P on the circumconics of a triangle $\Delta = ABC$ such that the circumconic's normals at A, B, C concur in P. This curve was derived and described in [12]. The septic $\mathcal{Q}_{008}$ is the isogonal image of a circular octic which collects the perspectors of pedal and projection triangles of a triangle Δ, while $\mathcal{Q}_{009}$ is related to orthologic triangles.

However, the rational septic also related to a geometric question about triangles found by É. Lemoine (cf. [16]) does not show up in [10]. Compared to the huge amount of special conics, cubics, and quartics related to many geometric questions, these 13 septics are a rather poor aggregation. It seems that K. Fladt [8] may be right when he stated that *"there could hardly be some curves of degree 7 that could be of interest and of geometrical relevance"*, although the space of septic plane curves is 35-dimensional (including even degenerate ones) since the implicit equation of a septic involves 36 coefficients where only the ratio matters.

Cubic curves related to triangles can be characterized by geometric properties [9]. While no vertex of a triangle is distinguished and the ordering of the vertices does not matter, this is not the case with a quadruple of points, say A, B, C, D. There are three different orderings of four points (up to cyclic and reverse rearrangements), and so, they define three different quadrilaterals. Asking for the locus of all points P in the plane of the quadruple with *concyclic* pedal points on four side lines of one particular quadrilateral defined on the point quadruple results in a certain cubic. Since there are three different orderings, the *four points actually define three cubics* one of which passes through the quadrilateral's respective Miquel point (see [3] and cf. Figure 2).

It seems that asking for the locus $\mathcal{C}$ for only one ordering of points may not deliver the complete picture.

In the following, we assume that we are given a planar quadrilateral $\mathcal{Q} = ABCD$ with vertices A, B, C, D, no two of which may coincide and no three shall be collinear. (Later, we shall discuss the case where three of these points are collinear as the only acceptable degenerate case.) Clearly, these four points define six lines $[A, B]$, $[A, C]$, $[A, D]$, $[B, C]$, $[B, D]$, $[C, D]$, *i.e.*, the joins of all six pairs out of the four points. The union of the four points and the six lines is called a *complete quadrangle*.

Now, we raise the following question (cf.

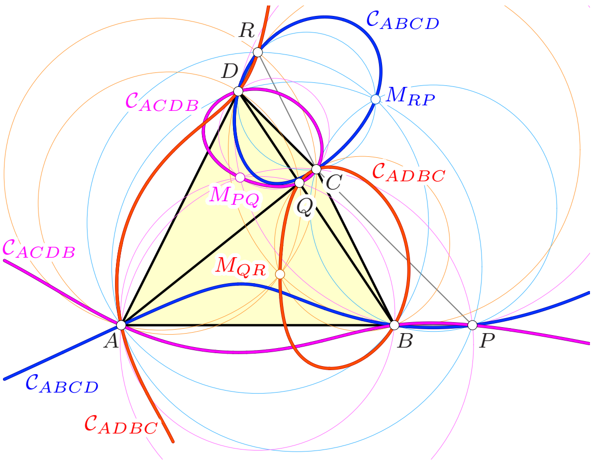

Figure 2: The loci $\mathcal{C}_{ABCD}$, $\mathcal{C}_{ACDB}$, $\mathcal{C}_{ADBC}$ of points with four concyclic pedal points on the sides of the three quadrilaterals on four points A, B, C, D.

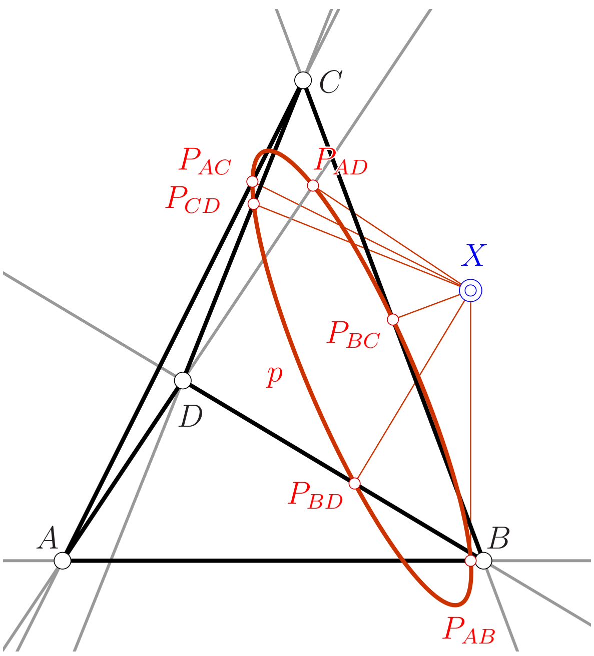

Figure 3: The characteristic property of the points on $\mathcal{C}$: The six pedal points $P_{..}$ of the point X lie on a single conic p.

Figure 3): *What is the locus $\mathcal{C}$ of points X in the quadrilateral's plane such that the pedal points of X on the six lines of the complete quadrilateral are conconic,* i.e., *they are located on a single conic?*

In order to answer this question, the remainder of this section collects necessary notations and provides some basic results. In Section 2, we shall derive the equation of $\mathcal{C}$ for a generic quadrilateral and study $\mathcal{C}$'s algebraic properties. However, the equation of $\mathcal{C}$ is given in the Appendix A in full length because of its complexity (2318 terms). A rather intricate computation will show that beside the diagonal points and three Miquel points there are only 4 further real points on $\mathcal{C}$ that deliver singular pedal conics. Subsequently, Section 3 will show that the curve $\mathcal{C}$ is the locus curve for a more general formulation of the initial problem. Then, Section 4 deals with those quadrilaterals and complete quadrangles where the degree of the curve $\mathcal{C}$ drops. In all these cases, $\mathcal{C}$ becomes a sextic either of genus 1 or 3 and carries no real point off the real (isolated) singularities. We also show that the degree of $\mathcal{C}$ is always larger than 5.

1.2 Prerequisites, notations, and basic results

Although we are mostly dealing with Euclidean geometry, we shall describe points by homogeneous coordinates whenever this is favorable. The Cartesian coordinates (x, y) of a point X can easily made homogeneous by writing $X = 1 : x : y$. On the contrary, from the homogeneous coordinates $x_0 : x_1 : x_2$ of a point, we can change to its Cartesian coordinates by setting $x = x_1x_0^{-1}$ and $y = x_2x_0^{-1}$, provided that $x_0 \neq 0$. In this way, we have performed the projective closure of the Euclidean plane and $x_0 = 0$ is the equation of the ideal line (line at infinity). On this line, we find the absolute points of Euclidean geometry $0 : 1 : \pm\mathrm{i}$ which are henceforth denoted by I and $J = \overline{I}$.

The condition on six points to lie on a single conic can be written in form of a vanishing determinant of a 6×6 matrix whose rows (or columns likewise) are the quadratic Veronese images of the six points in question see [11]. For a point X with homogeneous coordinates $x_0 : x_1 : x_2$, the quadratic Veronese image has the homogeneous coordinates

$$\begin{aligned} & v(x_0, x_1, x_2) = \\ & = x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2. \end{aligned} \tag{1}$$

Each conic c in the plane has a homogeneous equation of the form

$$\sum_{i,j=0}^{2} a_{ij}x_ix_j = 0$$

(with $a_{ik} \in \mathbb{R}$ not simultaneously vanishing). The conic c is regular/singular if, and only if, the symmetric matrix $(a_{ij}) \in \mathbb{R}^{2\times 2}$ is regular/singular. Each point incident with the conic corresponds to a hyperplane in the space $\mathbb{P}^5$ of all Veronese images. Six linearly dependent hyperplanes in $\mathbb{P}^5$ correspond to six conconic points, and hence, the 6×6 matrix of the respective Veronese images is of rank less than 6. A less algebraic and more geometric condition on six points to lie on a conic is given by Pappus's theorem [11]. However, the algebraic formulation of Pappus's theorem is equivalent to (1).

Now, it is natural to conjecture that the locus $\mathcal{C}$ is a curve of degree twelve: The computation/construction of the pedal points of the normals from X to the sides of the complete quadrangle is linear. Algebraically speaking, the coordinates of the six pedal points can be expressed linearly in terms of the coordinates of X.

Therefore, the entries of the 6×6 matrix are quadratic in the coordinates of the pedal points, and thus, quadratic in the coordinates of X. Finally, the determinant of the 6×6 matrix is a polynomial of degree twelve which, set equal to zero, is the equation of an algebraic curve of degree twelve.

Whatever the locus $\mathcal{C}$ may be, the following can be shown without any computation:

Theorem 1.1. *The vertices A, B, C, D and the diagonal points $P = [A, B] \cap [C, D]$, $Q = [A, C] \cap [B, D]$, $R = [A, D] \cap [B, C]$ are located on $\mathcal{C}$.*

Proof. If X coincides with one diagonal point, say P, then the pedal points on $[A, B]$ and $[C, D]$ coincide and equal P. So, there are only five different pedal points naturally having a unique circumconic. The same holds true for the other diagonal points.

If X equals a vertex of $\mathcal{Q}$, say A, then even three pedal points fall in one point, *i.e.*, the pedal points of A on $[A, B]$, $[A, C]$, and $[A, D]$ (the three side lines through A). Therefore, the four vertices of $\mathcal{Q}$ are located on $\mathcal{C}$ and are singular points on $\mathcal{C}$. □

We shall also verify that A, B, C, and D are double points on $\mathcal{C}$ by computation in Thm. 2.2.

Remark:

The pedal conic of a vertex of $\mathcal{Q}$, say A, is not uniquely determined. It passes through the three pedal points on $[B, C]$, $[C, D]$, $[D, B]$, and A. These four points will, in general, serve as the base points of a pencil of pedal conics (cf. [11]). $\diamondsuit$

2 The equation of $\mathcal{C}$

2.1 The generic quadrilateral

In order to give an equation of $\mathcal{C}$, we attach a Cartesian coordinate system to the given quadrilateral. It means no loss of generality, if we assume that the vertices of the quadrilateral are given by the homogenized Cartesian coordinates

$$\begin{aligned} A &= 1:0:0, \quad B = 1:a:0, \\ C &= 1:b:c, \quad D = 1:d:e. \end{aligned}$$

We could simplify the coordinates of these four points a little bit more by setting $a = 1$. Regarding the question we are trying to answer, this is admissible, since it would only scale the quadrilateral and the problem of conconic pedal points is invariant under equiform transformations in general. However, we do not set $a = 1$ in order to keep the coefficients of $\mathcal{C}$ homogeneous (polynomials in a, b, c, d, e).

Later, some quadratic functions in terms of a, b, c, d, e shall occur frequently and in order to simplify many expressions, we label the *squares* of the six Euclidean lengths between the given points by

$$\begin{aligned} l_1 &:= \overline{AB} = a^2, \\ l_2 &:= \overline{AC} = b^2 + c^2, \\ l_3 &:= \overline{AD} = d^2 + e^2, \\ l_4 &:= \overline{BC} = (b-a)^2 + c^2, \\ l_5 &:= \overline{BD} = (d-a)^2 + e^2, \\ l_6 &:= \overline{CD} = (d-b)^2 + (e-c)^2. \end{aligned} \tag{2}$$

For the same reason, we denote the areas of the four subtriangles of $\mathcal{Q}$ by

$$\begin{aligned} F_D &:= \text{area}(ABC) = \tfrac{1}{2}ac, \\ F_C &:= \text{area}(ABD) = \tfrac{1}{2}ae, \\ F_B &:= \text{area}(ACD) = \tfrac{1}{2}(be - cd), \\ F_A &:= \text{area}(BCD) = \tfrac{1}{2}(ac - ae + be - cd), \end{aligned} \tag{3}$$

where, for example, F_A is the area of the triangle BCD (*i.e.*, the area is labeled by the point that does not contribute).

Now, let $X = (x, y)$ (or likewise $1 : x : y$) be a point in the plane of $\mathcal{Q}$. It is elementary to compute the six pedal points from X to the sides of the complete quadrilateral. Then, we replace the Cartesian coordinates of X by homogeneous coordinates according to $x \to x_1 x_0^{-1}$ and $y \to x_2 x_0^{-1}$. For example, the pedal point P_{AC} on the side line $[A, C]$ has the homogeneous coordinates

$$P_{AC} = l_2 x_0 : b(bx_1 + cx_2) : c(bx_1 + cx_2).$$

Subsequently, we apply the Veronese mapping (1) and compute the determinant of the 6×6 matrix

$$\begin{aligned} V := (&v(P_{AB}), v(P_{AC}), v(P_{AD}), \\ &v(P_{BC}), v(P_{BD}), v(P_{CD})) . \end{aligned} \tag{4}$$

This results in a homogeneous polynomial of degree 12 in the variable homogeneous coordinates $x_0 : x_1 : x_2$ of X. Surprisingly, $\det V$ factors and we have

$$\det V = -2^8 l_1^{-1} F_A^2 F_B^2 F_C^2 F_D^2 \cdot x_0^5 \cdot P_7, \quad (5)$$

where $P_7 = \sum_{k=0}^{7} q_k x_0^k$ is a degree 7 form in $x_0 : x_1 : x_2$ with

$$\begin{aligned}
&q_7 = q_6 = 0,\\
&q_5 = 2^4 l_1 l_2 F_A F_B F_C F_D (x_1^2 + x_2^2),\\
&q_4 = \dots,\ q_3 = \dots,\\
&q_2 = (\dots)(x_1^2 + x_2^2),\ q_1 = (\dots)(x_1^2 + x_2^2)^2,\\
&q_0 = 4(al_1)^{-1}(4(F_C - F_D)(l_1 F_B (F_B - F_C)\cdot\\
&\cdot(F_B + F_D) + l_2 F_C^2 (F_C - F_B) - l_3 F_D^2\cdot\\
&\cdot(F_B + F_D))x_1 + (l_1^2 F_B^2 (F_B - F_C - F_D) -\\
&- l_2^2 F_C^2 F_D - 2 l_3^2 F_D^3 + l_1 l_2 F_C ((4F_B - 5F_C)\cdot\\
&\cdot(F_B - F_C) + (F_B - F_C) F_D) +\\
&+ l_1 l_3 F_D (4F_B^2 - 4F_B F_C - F_C^2 +\\
&+ 3F_D (F_B - F_C + F_D)) + l_3 l_4 F_C^2 F_D +\\
&+ l_2 l_3 F_C F_D (F_C + 2F_D) - l_2 l_4 F_C^2 (2F_C - F_D) -\\
&- 16 F_B F_C F_D ((F_B - F_C)\cdot\\
&\cdot(F_C + F_D) + F_D^2))x_2)(x_1^2 + x_2^2)^3.
\end{aligned} \quad (6)$$

The polynomial P_7 is given in full length in the Appendix A in term of inhomogeneous (Cartesian) coordinates.

Now, we have:

Theorem 2.1. *The locus $\mathcal{C}$ of points X in the Euclidean plane with conconic pedal points on the six lines of a complete quadrangle is, in general, a tricyclic algebraic curve of degree 7 with the equation $P_7 = 0$ having one real point at infinity.*

We have added the phrase *in general* since we shall soon see that for some special configurations of the four points A, B, C, D the degree will drop.

Proof. By virtue of (5), we can see that the (in general) non-degenerate factor of $\det V$ is a polynomial P_7 of degree 7. Obviously, the factor x_0^5 splits off from $\det V$, and thus, the line at infinity is a component with multiplicity 5. However, this component does not matter, since one cannot draw normals from ideal points to proper lines. Therefore, the affine part of $\mathcal{C}$ is only of degree 7. (An example is shown in Figure 4.)

In the projective closure and the complex extension of the Euclidean plane, the term q_0 of degree 7 (given in (6)) consists of a linear factor corresponding to the one and only real point at infinity and the term $(x_1^2+x_2^2)^3 = (x_1+\mathrm{i}x_2)^3(x_1-\mathrm{i}x_2)^3$ whose solutions are the absolute points (circle points) of Euclidean geometry each with multiplicity 3. □

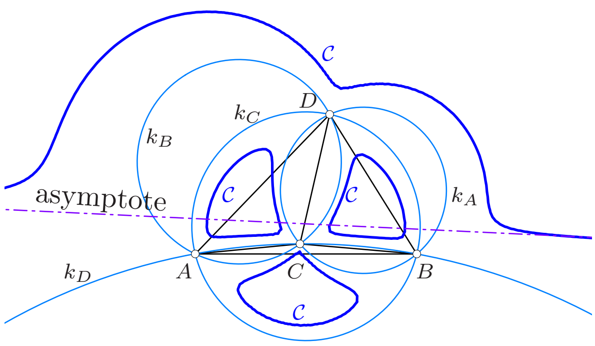

Figure 4: The septic locus $\mathcal{C}$ of points whose six pedal points on the sides of a complete quadrilateral $\mathcal{Q} = ABCD$ lie on a conic.

Later, we shall have a look at all types of quadrilaterals including those with symmetry. In some cases the degree of the curve $\mathcal{C}$ will drop. For some special quadrilaterals, the curve $\mathcal{C}$ will consist of a finite number of isolated real points and complex branches without any real point.

Remark:
The equations of the cubics showing up in [3] as the loci of points with four concyclic pedal points on the four sides of a quadrilateral are also the irreducible parts of polynomials of degree 8. The concyclicity of the four pedal points is equivalent to the vanishing of the determinant of the 4×4 matrix whose rows (columns) are Veronese images

$$(p_1^2+p_2^2, p_0p_1, p_0p_2, p_0^2)$$

(cf. [11, p. 241]) of the four homogenized pedal points. Surprisingly, from this degree 8 polynomial the factor x_0^5 (the ideal line) also splits off with multiplicity 5. $\diamondsuit$

We can state and prove:

Theorem 2.2. *The vertices of the quadrilateral $\mathcal{Q} = ABCD$ are isolated double points on the septic $\mathcal{C}$. The four vertices are focal points of $\mathcal{C}$. The curve $\mathcal{C}$ is of class 22 and genus 5.*

Proof. From (6), we see that q_7 and q_6 are equal to zero, and therefore, A is a double point on $\mathcal{C}$. The coefficient $q_5 \neq 0$ (cf. (6)) tells us that the point A is a double point on $\mathcal{C}$. The linear factors of q_5 are the equations of $\mathcal{C}$'s tangents at the double point. Since

$$x_1^2 + x_2^2 = (x_1 + \mathrm{i}x_2)(x_1 - \mathrm{i}x_2) = 0,$$

we see that the tangents at A are isotropic lines and A is an isolated double point.

We recall von Staudt's definition of focal points on algebraic curves: *A point F is a* focal point *of an algebraic curve if the curve's tangents at F are isotropic lines* (cf. [1, 5]). According to this, A is a focal point since the tangents of the curve at A are isotropic lines.

The other vertices B, C, D are of the like kind. This can be shown by applying translations to $\mathcal{Q}$ and to the septic curve $\mathcal{C}$ such that each vertex of $\mathcal{Q}$ coincides with the origin of the coordinate system (three different translations). This does not change the algebraic and geometric properties of $\mathcal{C}$ and the linear factors of q_0 are the equations of the tangents at the origin. In all three cases, q_0 will turn out to be a scalar multiple of $x_1^2 + x_2^2$ (since this quadratic form is invariant under Euclidean transformations). Consequently, all four vertices of $\mathcal{C}$ are isolated double points and focal points of $\mathcal{C}$.

There are no further singularities on $\mathcal{C}$ (different from A, B, C, D, I, J). This can be shown either with a CAS (like Maple) or by considering the following: At a singular point of $\mathcal{C}$ at least three pedal points have to coincide which is not possible for any other point (different from the already known singularities).

With the Plücker formulae for planar algebraic curves (cf. [2, 4, 5, 8, 14]), we find the genus g and the class m of $\mathcal{C}$:

$$g = \tfrac{1}{2}(7-1)\cdot(6-1) - 1\cdot 4 - 3\cdot 2 = 5,$$
$$m = 7\cdot(7-1) - 2\cdot 4 - 6\cdot 2 = 22$$

since there are 4 ordinary double points and 2 ordinary triple points on $\mathcal{C}$. $\square$

Figure 5 shows that the curve $\mathcal{C}$ can have up to six real separated components as is to be expected for a curve of genus 5. These six components occur if one vertex lies close to one side.

Remark:
The well-known Plücker formulae (cf. [2, 4, 5, 8, 14, 23]) for the genus and class of a planar algebraic curve have to be adapted if the degree d is larger than or equal to 4

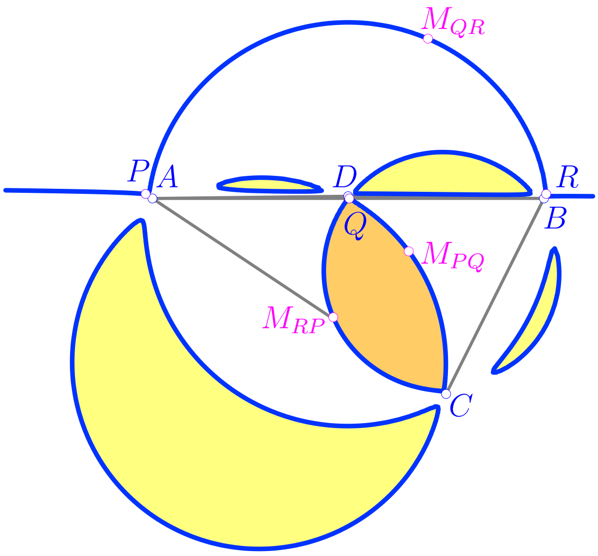

Figure 5: If one vertex (here D) comes close to one side line (here $[A, B]$), then the curve $\mathcal{C}$ consists of 6 separated real components.

since curves of sufficiently high degree may have singularities of multiplicity larger than 2.

In the present case with $d = 7$ and ordinary triple points, the formulae for the class m, the number w of inflection points, and the genus g read

$$\begin{aligned} m &= d(d-1) - 2d - 3s - 6t, \\ w &= 3d(d-2) - 6d - 8s - 18t, \\ g &= \tfrac{1}{2}(d-1)(d-2) - \sum \delta_i. \end{aligned}$$

Herein, d, s, t, δ_i are the numbers of (ordinary) double points, cusps (of the first kind), (ordinary) triple points, and the δ-invariants of all singularities. The δ-invariant can be computed with Maple's function `singularities` provided by the `algcurves` package.

It is rather technical to show that each (ordinary) triple point has to be weighted with the factors 6 and 18 in the class and inflection point formula.

This allows us to conjecture that

$$w = 3 \cdot 7 \cdot (7-2) - 6 \cdot 4 - 18 \cdot 2 = 45.$$

is an *upper bound* for the number of *real* inflection points on $\mathcal{C}$. $\Diamond$

2.2 Miquel points determine singular pedal conics

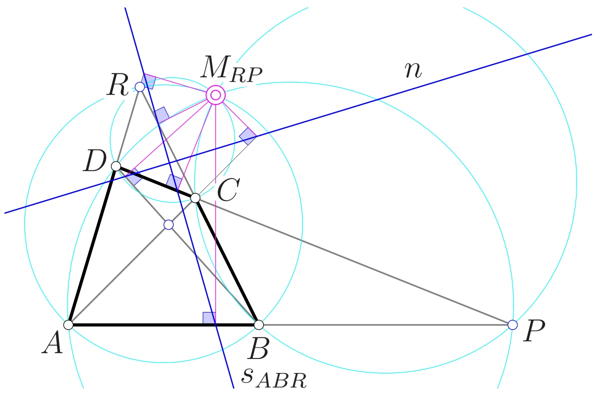

Figure 6: The Miquel point M_{RP} lies on the septic $\mathcal{C}$, for its six pedals with respect to the lines of a complete quadrilateral form a degenerate conic $m = s_{ABR} \cup n$.

Each quadrilateral $\mathcal{Q} = ABCD$ defines three Miquel points each of which is common to four circles on two pairs of opposite vertices and the respective diagonal points of $\mathcal{Q}$ (cf. [22]). We shall denote the Miquel points by M_{PQ}, M_{QR}, M_{RP} pointing to the diagonal points involved. It is well-known that the Miquel points are located on the following circles (cf. [22]):

$$\begin{aligned} M_{PQ} &\in \quad k_{ACP},\ k_{BDP},\ k_{ABQ},\ k_{CDQ}, \\ M_{QR} &\in \quad k_{ADQ},\ k_{BCQ},\ k_{ACR},\ k_{BDR}, \\ M_{RP} &\in \quad k_{ABR},\ k_{CDR},\ k_{ADP},\ k_{BCP}, \end{aligned}$$

where k_{XYZ} denotes the circle on the three (pairwise different) points X, Y, and Z. We are able to show that these points play an outstanding role:

Theorem 2.3. *The three Miquel points M_{PQ}, M_{QR}, M_{RP} are located on the septic $\mathcal{C}$. The three pedal conics defined by the six pedal points of each Miquel point are degenerate and split into pairs of lines.*

Proof. It is sufficient to show the validity of the above theorem for one particular Miquel point, say M_{RP}. For the remaining two the proof uses the same arguments for different subtriangles.

The Miquel point M_{RP} is the common point of the circumcircles k_{ABR}, k_{CDR}, k_{ADP}, k_{BCP} of the respective subtriangles.

Since $M_{RP} \in k_{ABR}$, the three pedal points of M_{RP}'s normals to $[A, B]$, $[B, R]$, $[R, A]$ are collinear: They lie on the Simson line of the triangle ABR. The triangles ABR and CDR share two side lines: $[A, R] = [D, R]$ and $[B, R] = [C, R]$. Thus, two by two pedal points coincide: $P_{M_{RP},[A,R]} = P_{M_{RP},[D,R]}$ and $P_{M_{RP},[B,R]} = P_{M_{RP},[C,R]}$. So, the two triangles ABR and CDR share the Simson line $s_{ABR} = s_{CDR}$ on which also the pedal points $P_{M_{RP},[A,B]}$ and $P_{M_{RP},[C,D]}$ have to lie. This makes in total four collinear pedal points.

The remaining two pedal points $P_{M_{RP},[A,C]}$ and $P_{M_{RP},[B,D]}$ span a second line n. The union of s_{ABR} and n is the singular conic m. Since m is a (singular) conic, M_{RP} has to lie on $\mathcal{C}$ by the very definition. □

Figure 7 shows the three Miquel points of the complete quadrangle $\mathcal{Q}$ together with the three singular pedal conics. Each point and line displayed in Figure 7 can be constructed only with a ruler (linearly): Each Miquel point is a common point of two circles sharing an already known point. The singular pedal conics of the Miquel points are Simson lines which require only linear constructions.

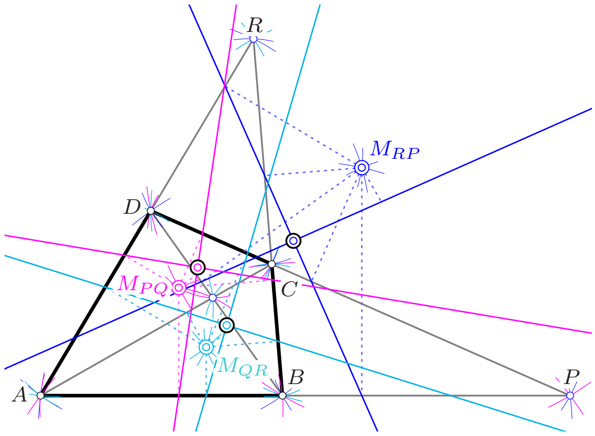

Figure 7: The three Miquel points and their singular pedal conics.

It is noteworthy that the triangle built by the centers of the singular conics is perspective to the diagonal triangle PQR of $\mathcal{Q}$:

$$PQR \,\overline{\wedge}\, C_{QR}C_{RP}C_{PQ}$$

(with C_{QR} denoting the center of the singular pedal conic of M_{QR}. Further, the triangle formed by the three Miquel points is also perspective to the diagonal triangle, *i.e.*,

$$PQR \,\overline{\wedge}\, M_{QR}M_{RP}M_{PQ}.$$

Remark:
Theorem 2.3 can also be verified by means of computation. For that purpose, only the coordinates

$$\begin{aligned} M_{RP} = {} & 2(l_1 - l_2 + l_3 + l_4 - l_5 + l_6) : \\ & : a(l_1 - l_2 + 2l_3 - l_5 + l_6) : \\ & : 4a(F_C - F_B), \end{aligned}$$

$$
\begin{aligned}
M_{PQ} = {} & 2(l_1 + l_2 - l_3 - l_4 + l_5 + l_6) : \\
& : a(l_1 + 2l_2 - l_3 - l_4 + l_6) : \\
& : 4a(F_B + F_D),
\end{aligned}
$$

$$
\begin{aligned}
M_{QR} = {} & 4a(l_1 - l_2 - l_3 - l_4 - l_5 + l_6) : \\
& : l_1(l_1 - l_2 - l_3 - l_4 - l_5) + \\
& \quad + (l_4 - 3l_2)l_3 + \\
& \quad + (l_2 + l_4)l_5 - 16F_C F_D : \\
& : 8(l_1(F_C - F_B) - F_D l_3 - l_4 F_C),
\end{aligned}
$$

of the three Miquel points (with the abbreviations given in (2) and (3)) have to be inserted into (5). ◊

We are able to show that the Miquel points are not the only points whose six pedal points lie on a singular conic:

Theorem 2.4. *In the Euclidean plane of a generic quadrilateral $\mathcal{Q}$ there exist, in general, 4 real points (different from the Miquel point, the diagonal points, and the vertices of $\mathcal{Q}$) whose pedal conics are singular.*

Proof. Unfortunately, this proof requires some computation. We assume that $W = 1 : \xi : \eta$ is a point on $\mathcal{C}$, and thus, its coordinates annihilate P_7 from (5) and (6). By the very definition of $\mathcal{C}$, the six pedal points of W lie on a conic. We can use (4) to determine the equation of the conic c_{CD} on the pedals P_{AB}, P_{AC}, P_{AD}, P_{BC}, P_{BD} of W (note that P_{CD} is missing). The determinant of the coefficient matrix M_{CD} has to vanish in order to make c_{CD} singular. Surprisingly, $\det M_{CD}$ splits into quadratic factors:

$$
\begin{aligned}
& \det M_{CD} = \iota_A \cdot \iota_B \cdot k_C \cdot k_D \cdot \\
& \cdot k_{ABR} \cdot k_{ABQ} \cdot k_{BCQ} \cdot k_{ADQ} \cdot k_{ACR} \cdot k_{BDR}.
\end{aligned}
$$

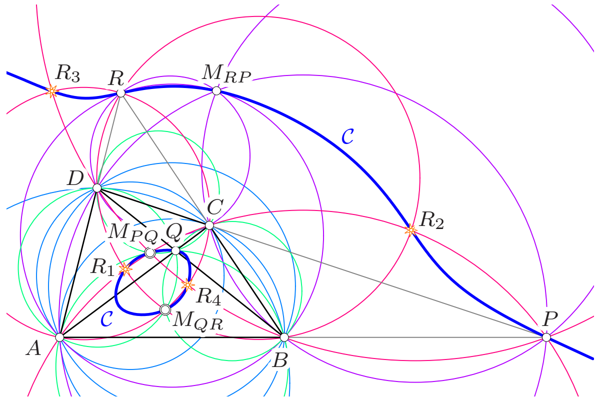

Figure 8: The cycle $\mathcal{L}$ consists of 16 circles and 8 isotropic lines. It intersects $\mathcal{C}$ in possible candidates of points with degenerate pedal conics.

The factors in the latter product are the equations of some circles and pairs of isotropic lines. For example, $\iota_A = \xi^2 + \eta^2$ is the equation of the pair of isotropic lines through A, k_A is the (equation of the) circumcircle k_A of BCD, and k_{ABR} is the (equation of the) circumcircle of ABR (with P, Q, and R still being $\mathcal{Q}$'s diagonal points as defined in Thm. 1.1).

So far, it seems that the pedal point P_{CD} does not play a role. In order not to miss a single pedal point, we compute the least common multiple L of all determinants $\det M_{kl}$ (with $k \neq l$ and $(k, l) \in$

$\{A, B, C, D\}$) and find

$$L = \underbrace{\iota_A \cdot \iota_B \cdot \iota_C \cdot \iota_D}_{\substack{\text{isotropic lines}\\ \text{through vertices}}} \cdot \underbrace{k_A \cdot k_B \cdot k_C \cdot k_D}_{\substack{\text{circumcircles}\\ \text{of subtriangles}}} \cdot$$

$$\cdot \underbrace{k_{ABR} \cdot k_{CDR} \cdot k_{ADP} \cdot k_{BCP}}_{\substack{\text{circles through}\\ \text{the Miquel point } M_{RP}}} \cdot$$

$$\cdot \underbrace{k_{ACP} \cdot k_{BDP} \cdot k_{ABQ} \cdot k_{CDQ}}_{\substack{\text{circles through}\\ \text{the Miquel point } M_{PQ}}} \cdot$$

$$\cdot \underbrace{k_{ADQ} \cdot k_{BCQ} \cdot k_{ACR} \cdot k_{BDR}}_{\substack{\text{circles through}\\ \text{the Miquel point } M_{QR}}} .$$

The points on $\mathcal{C}$ with degenerate conics through their pedal points are found as the intersection of the curve $\mathcal{C}: \ P_7 = 0$ and the cycle $\mathcal{L}: \ L = 0$ of degree 40. The cycle $\mathcal{L}$ consists of 16 circles and the 8 isotropic lines passing through the four vertices of $\mathcal{Q}$, cf. Figure 8. According to Bézout's theorem, we have to expect up to 280 common points of $\mathcal{C}$ and $\mathcal{L}$. As we shall see, many of them are not real and a huge amount of them coincides with already known points.

In order to get rid of solutions that we already now and, further, in order to simplify the computation we have to discuss the intersection of the components of $\mathcal{L}$ with $\mathcal{C}$.

The four pairs of isotropic lines can be cut out immediately: The pair described by $\iota_A = 0$ intersects $\mathcal{C}$ in 14 points 6 of which coincide with A (since A is an ordinary double point on ι_A and $\mathcal{C}$ and both (isotropic) components of ι_A are tangents to $\mathcal{C}$ at A). Three intersection points each are located at I and J (since they are ordinary triple points on $\mathcal{C}$ (cf. Thm. 2.1) and regular points on ι_A). The two remaining points cannot be real since ι_A does not contain any real point different from A. The same arguments hold for the other pairs. Therefore, we can cut out the cycle of degree 8 given by the equation $\iota_A \cdot \iota_B \cdot \iota_C \cdot \iota_D = 0$.

The circumcircles can also be canceled: For example, the circle k_A (passing through B, C, D) intersects $\mathcal{C}$ at B, C, D with multiplicity 2 at each point (since they are double points on $\mathcal{C}$, cf. Thm. 1.1 and Thm. 2.2). At both absolute points I and J, the intersection multiplicity of k_A and $\mathcal{C}$ equals 3. Further, k_A and $\mathcal{C}$ have a pair of complex conjugate proper points in common. These two points are never real since the discriminant Δ_A of the respective quadratic equations is a full square with a minus ahead:

$$\begin{aligned}\Delta_A = -4l_1^{-1}(l_1F_B + l_3F_D - l_2F_C)^2 \cdot \\ \cdot (l_1(l_3 - l_2 - l_4 + l_5) + \\ + ad(l_2 - l_3 + l_6) - 4F_BFC)^2.\end{aligned}$$

Hence, k_A does not lead to new real points on $\mathcal{C}$ with singular pedal conics, as is the case with k_B, k_C, k_D for the same reasons. Therefore, the cycle $k_A \cdot k_B \cdot k_C \cdot k_D = 0$ of degree eight being the union of the circumcircles of the four subtriangles can also be cut out.

Finally, we have to study the last three quadruples of circles passing through their respective Miquel point: At first, we shall have a look at the four circles passing through one particular Miquel point. For example the circles k_{ABR}, k_{CDR}, k_{ADP}, k_{BCP} share only the points A, B, C, D, R, P, M_{RP}, I, and J with $\mathcal{C}$ (with multiplicities 4, 4, 4, 4, 2, 2, 4, 16, 16). Which is similarily true for the other quadruples of circles passing through the Miquel points M_{PQ} and M_{QR} and does not deliver new points.

A	B	C	D	P	Q	R	M_{PQ}	M_{QR}	M_{RP}	I	J
24	24	24	24	4	4	4	4	4	4	60	60

R_1	R_2	R_3	R_4	compl. pts.	$\sum$
2	2	2	2	32	280

Table 1: The common points of $\mathcal{L}$ and $\mathcal{C}$ algebraically counted.

Surprisingly, the following combinations of circles yield real points on $\mathcal{C}$

$$k_{ACP} \cap k_{BDR} = \{R_1, R_2\},$$
$$k_{ACR} \cap k_{BDP} = \{R_3, R_4\}$$

while all other combinations of circles lead to intersections which are either already known or not on $\mathcal{C}$, or, if on $\mathcal{C}$, two points which can never be real. □

Table 1 lists the intersection points of $\mathcal{L}$ and $\mathcal{C}$ with their respective multiplicities, and thus, it summarizes the proof of Thm. 2.4.

Remark:
The cycle $\mathcal{L}$ is of degree 40 and it is the union of 16 circles and 8 isotropic lines. It has four 11-fold points at A, B, C, D; six 4-fold points at P, Q, R, M_{PQ}, M_{QR}, M_{RP}; and the absolute points I, J are 20-fold points. Further it has 128 ordinary double points (among them $R_1, \ldots, R_4$). ◊

2.3 Degenerate quadrilaterals

Quadrilaterals may degenerate in many ways. Until now, we have assumed that none of the four vertices falls into a line spanned by two others, *i.e.*, $\mathcal{Q} = ABCD$ is a proper quadrilateral. If we exclude cases where two or more vertices coincide, the only possible degenerate quadrilaterals are those where one vertex, say C, lies on the side line $[A, B]$. In any other case, we can relabel the points. In this rather special case, we can state:

Theorem 2.5. *Assume that all vertices of $\mathcal{Q}$ are pairwise different, but, for example, $C \in [A, B]$. Then, the septic curve $\mathcal{C}$ becomes the septic cycle consisting of the line $[A, B]$ and the circumcircles of the three non-degenerate subtriangles ABD, ACD, and BCD.*

The line $[A, B]$ serves as the degenerate circumcircle of the improper triangle ABC.

Proof. If C lies on $[A, B]$, then $C = 1 : b : 0$, *i.e.*, $c = 0$. Inserting this into P_7, yields

$$P_7 = (a-b)^2 b^2 \cdot x_2 \cdot (e(x_1^2+x_2^2) - bex_0x_1 + \\ +(bd-d^2-e^2)x_0x_2) \cdot \\ \cdot (e(x_1^2+x_2^2) - aex_0x_1 + (ad-d^2-e^2)x_0x_2) \cdot \\ \cdot (e(x_1^2+x_2^2) - (a+b)ex_0x_1 + \\ +((a+d)(d-b)-e^2)x_0x_2 + abex_0^2).$$

The linear factor is the equation of $[A, B]$, the quadratic factors are the equations of the circumcircles k_C, k_B, k_A of ABD, ACD, BCD. □

The points on the septic cycle described in Theorem 2.5 define only degenerate conics: Let X be some point on the circumcircle of $\Delta_C = ABD$. The pedal points P_{AB}, P_{AD}, P_{BD} of X on the sides of Δ_C are collinear and lie on the Simson line s_{ABD}. Since $C \in [A, B]$, $[A, B] = [A, C] = [B, C]$, and thus, $P_{AB} = P_{AC} = P_{BC}$. Therefore, the conic on the six pedals is the union of two lines, the Simson line s_{ABD} and the line $[P_{CD}, P_{AB}]$.

Here, we have only four different pedal points, and four points always lie on at least one conic, indeed, they form the basis of a pencil of conics.

3 A more general point of view

We have drawn the normals from some point X to the lines of a complete quadrilateral and determined the pedal points. However, these six pedal points are very special points on the six normals through P.

Let again P_{kl} denote the pedal point of X on the line $[k,l]$ (with $k \neq l$ and $(k,l) \in \{A,B,C,D\}$) and let further denote P_{kl}^{ω} the ideal point of the normal of $[k,l]$ through X. Then, we shall determine the points P_{kl}^{δ} on the normal such that the crossratio of P_{kl}, P_{kl}^{ω}, X, and P_{kl}^{δ} equals $\delta \in \mathbb{R} \setminus \{0\}$.

Now, we can ask for the set $\mathcal{C}^{\delta}$ of all points X such that the six points P_{kl}^{δ} lie on a single conic. We can show the astonishing result:

Theorem 3.1. *Let $\mathcal{Q} = ABCD$ be a quadrilateral in the projectively extended Euclidean plane. Then, define six perspective collineations κ_{kl}^{δ} whose axes are the six lines $[k,l]$ ($k \neq l$, $k,l \in \{A,B,C,D\}$) of the complete quadrangle determined by $\mathcal{Q}$, their centers P_{kl}^{δ} being the ideal points of the normals of $[k,l]$, and $\delta \in \mathbb{R} \setminus \{0\}$ be their (common) characteristic crossratio.*

Then, the set $\mathcal{C}^{\delta}$ of all points X whose images P_{kl}^{δ} under the six perspective collineations κ_{kl}^{δ} lie on a single conic form the septic curve $\mathcal{C}$ described in Theorem 2.1 independent of the choice of $\delta \neq 0$.

Proof. With the Cartesian coordinates of X and P_{kl} and the characteristic cross ratio $\delta \in \mathbb{R}$, the points P_{kl}^{δ} can be written as a linear combination of X and and the respective pedal point P_{kl}

$$P_{kl}^{\delta} = (1-\delta)X + \delta\, P_{kl}$$

(where $\delta \neq 0$, $(k,l) \in \{A,B,C,D\}$, and $k \neq l$) since P_{kl}^{ω} is a point at infinity. Again, the determinant of the matrix (4) factors and equals

$$\det V = -2^8 l_1^{-1} F_A^2 F_B^2 F_C^2 F_D^2 \cdot \boldsymbol{\delta}^8 \cdot x_0^5 \cdot P_7$$

with the same polynomial P_7 of degree 7 as we know from (5) and (6) which is independent of δ. Hence $P_7 = 0$ is the equation of $\mathcal{C}^{\delta} = \mathcal{C}$. □

Theorem 3.1 contains a very special case: If $\delta = -1$, then the collinear images of X are the reflections of X in the six side lines of the complete quadrilateral. Obviously, these points are conconic if X lies on the septic $\mathcal{C}$. Figure 9 shows the septic together with some point $X \in \mathcal{C}$ and the conics on the six pedal points P_{kl} and the six reflections R_{kl}. It is clear that the conics corresponding to two different characteristic cross ratios $\delta_1, \delta_2 \neq 0$ are related by a central similarity with center X and similarity factor $\delta_1\delta_2^{-1}$ (or its reciprocal).

4 Exceptional quadrilaterals, degree reduction

4.1 Special configurations

In the case of the locus curve described in [3], the cubic may degenerate, *i.e.*, it splits

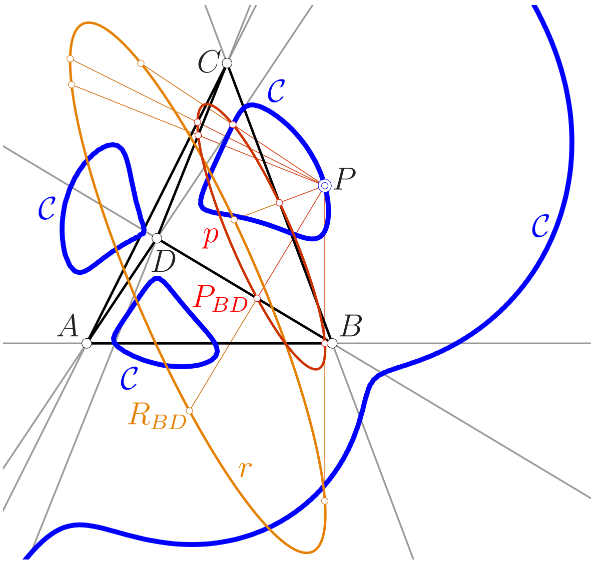

Figure 9: The conics p and r collect the pedal points and reflections of $P \in \mathcal{C}$. Here, the conic r is the image of p under the central similarity with center P and similarity factor 2.

into lower degree parts, depending on the shape of the quadrilateral. From Thm. 2.5, we know that $\mathcal{C}$ becomes the union of three circles and a straight line if three points out of $\{A, B, C, D\}$ are collinear (while still being pairwise different). This seems to be the only case (as is indicated by a detailed study of the curve $\mathcal{C}$ for all possible types of quadrilaterals – up to Euclidean transformations).

Now, we shall ask under what circumstances the degree of $\mathcal{C}$ is less than 7. We have the following:

Theorem 4.1. *Let $\mathcal{Q} = ABCD$ be a proper quadrilateral such that, for example, the point D is the orthocenter of ABC. The curve $\mathcal{C}$ associated with the complete quadrangle on $\mathcal{Q}$ is of degree 6 and genus 1, has 9 (isolated) double points and no further singularities. It is of class 12 and has no real branch.*

Proof. The contents of this theorem can be verified by setting

$$A = 1 : 0 : 0, \; B = 1 : a : 0, \; C = 1 : b : c,$$

and since D has to be the orthocenter of ABC, we have

$$D = c : bc : b(a - b).$$

With (4), we find the (homogeneous) equation of $\mathcal{C}$ as

$$\begin{gathered}\mathcal{C} : c^2(x_1^2 + x_2^2)^3 - \\ -2c((a+b)cx_1^3 + 3bcx_1x_2^2 + (ab-b^2+c^2)x_2^3)\cdot \\ \cdot(x_1^2 + x_2^2 + abx_0^2)x_0 + (c^2(a^2 + 4ab + b^2)x_1^4 + \\ +6(a + b)bc^2x_1^2x_2^2 + +4bc(ab - b^2 + c^2)x_1x_2^3 + \\ +(a^2b^2 - 2ab^3 + 4abc^2 + b^4 - b^2c^2 + c^4)x_2^4)x_0^2 + \\ +a^2b^2c^2(x_1^2 + x_2^2)x_0^4 = 0\end{gathered} \tag{7}$$

which is obviously of degree 6 and allows us to locate the singularities (isolated double points) at the three diagonal points of $\mathcal{Q}$. (According to Thm. 2.2, the vertices of $\mathcal{Q}$ are singular points on $\mathcal{C}$ in any case.) Although the leading term in (7) is $(x_1^2 + y_2^2)^3$, the absolute points I and J are only double points. (This can be shown at hand or the ranks of the tensors of the partial derivatives of order 3 of (7) with respect to the three variables x_i or using the `singularities` command in Maple's `algcurves` package.) Besides A, B, C, D, P, Q, R, I, J there are no further singularities.

With the Plücker formulae (cf. [2, 4, 5, 8, 14]), we find

$$\begin{aligned} g &= \tfrac{1}{2}(6-1)\cdot(6-2) - 9\cdot 1 = 1, \\ m &= 6\cdot(6-1) - 2\cdot 9 = 12 \end{aligned}$$

for the genus and the class of $\mathcal{C}$. □

Symmetries of the initial quadrilateral may not necessarily cause a reduction of the degree of $\mathcal{C}$. However, if two diagonal points of $\mathcal{Q}$ move to the line at infinity, then their join splits off from $\mathcal{C}$. This yields to the following result:

Theorem 4.2. *Let $\mathcal{Q} = ABCD$ be a parallelogram. The curve $\mathcal{C}$ associated with the complete quadrangle on $\mathcal{Q}$ is of degree 6 and genus 3, has 7 (isolated) double points, is of class 16 and has no real branch.*

Proof. We proceed in a similar way as in the proof of Thm. 4.1 with

$$A = 1:0:0, \quad B = 1:a:0,$$
$$C = 1:a+u:c, \quad D = 1:u:c.$$

It is not necessary to write down the rather lengthy equation of $\mathcal{C}$. (The reader may convince her-/himself by using a CAS that it is of degree 6.)

Now, the singularities are still the vertices of $\mathcal{Q}$ (according to Thm. 2.2), the absolute points I, J are double points, and the diagonal point $Q = [A,B] \cap [C,D]$ is the seventh (isolated) double point. Since there are no further singularities, the genus equals 3 and the class equals 16. □

We shall make explicit the fact that Thm. 4.2 contains the cases of *rhombi*, *rectangles*, and *squares*.

For *trapezoids*, in general, (no matter if they are symmetric, cyclic, tangential, or bicentric, equipped with right angles, or three equally long sides (as long as they are none of the above) the degree of $\mathcal{C}$ equals 7.

Kites (different from rhombi), *cyclic*, *tangential*, and *bicentric* quadrilaterals (as long as they do not fall into one of the above mentioned classes of quadrilaterals) always defined a *septic* $\mathcal{C}$ as the locus of points with six conconic pedal points on the complete quadrangle's sides.

4.2 Degree less than 6?

Finally, we want to show that the degree of $\mathcal{C}$ cannot be less than 6: Prior to Thm. 4.2, we have pointed out that a parallelogram has two diagonal points on the line ω at infinity, and thus, ω splits off from $\mathcal{C}$ once and $\deg \mathcal{C} = 6$. In a *classical projective plane*, the diagonal points of a quadrilateral are never collinear. Therefore, the ideal line will never splits off with multiplicity 3.

However, by virtue of (6), we see that the greatest common divisor of coefficients q_i of P_7 for $i \in \{0,1,2,5,6,7\}$ equals $x_1^2+x_2^2 = \Omega$. The degree of P_7 would reduce about 2 if $\gcd(q_3, q_4) = \Omega$. In this case the resultant

$$r_3 := \operatorname{res}(q_3, \Omega, x_i), \quad r_4 := \operatorname{res}(q_4, \Omega, x_i)$$

for any variable x_i ($i \in \{0,1,2\}$) have to be equal to zero. We build the resultants with respect to x_1 (and would find the same results if we would eliminate x_2):

$$r_3 = x_2^8 \cdot l_2^2\, l_3^2\, l_4\, l_5\, l_6 \cdot (l_1^2 l_4 - l_1 l_2 l_5 - 2 l_1 l_3 l_4 + \\ + l_1 l_3 l_5 + l_2^2 l_5 - l_2 l_3 l_5 + l_3^2 l_4),$$
$$r_4 = x_2^6 \cdot l_1\, l_2^2\, l_3^2\, l_4\, l_5\, l_6 \cdot (2aF_B - e l_2 + c l_3)^2 .$$

By assumption, $l_i \neq 0$ for all $i \in \{1, \ldots, 6\}$, hence $r_4 = 0$ yields

$$a = \frac{e\, l_2 - c\, l_3}{2F_B},$$

and after inserting into r_3, we find

$$r_4 = x_2^8 \cdot l_2^4\, l_3^4\, l_6^6\, F_C^4\, F_D^4\, F_B^{-8}.$$

None of the (squares of the) lengths l_i and none of the areas of the subtriangles are allowed to vanish, otherwise $\mathcal{Q}$ would degenerate. Therefore, neither r_3 nor r_4 can vanish, and thus, Ω is a common divisor of q_3 and q_4. Since there are no other (non-constant) factors of q_5, Ω cannot split off from P_7 and $\deg \mathcal{C}$ cannot be equal to 5.

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A Equation of $\mathcal{C}$

For the sake of completeness, we add the equation of $\mathcal{C}$ in terms of inhomogeneous coordinates.

$$\mathcal{C}: \ (\mathbf{x^2+y^2})^{\mathbf{3}}\cdot$$
$$\cdot\,(4c(c^2F_Bl_3-2c^2F_Cl_3+c^2F_Dl_3-ceF_Bl_3+ceF_Bl_5+ceF_Cl_3-e^2F_Bl_5+$$
$$+e^2F_Bl_6-4F_B^3+4F_B^2F_C-4F_B^2F_D)\mathbf{x}+(c^3l_2l_3+c^3l_3l_4-2c^3l_3l_5-2c^3l_3l_6+$$
$$+16c^2eF_B^2-16c^2eF_BF_C-2c^2el_2l_3+c^2el_3^2+c^2el_3l_5-2c^2el_3l_6-16ce^2F_B^2+$$
$$+16ce^2F_BF_C-32aF_B^3+16aF_B^2F_C-24aF_B^2F_D+24aF_BF_CF_D-4cF_B^2l_2+$$
$$+4cF_B^2l_3-4cF_B^2l_4+12cF_B^2l_5+8cF_B^2l_6+8cF_BF_Cl_2-12cF_BF_Cl_3-12cF_BF_Cl_5+$$
$$+24cF_BF_Dl_3-12cF_C^2l_2-4cF_CF_Dl_3+16cF_D^2l_3+8eF_B^2l_2)\mathbf{y})+$$
$$+2(\mathbf{x^2+y^2})^{\mathbf{2}}\cdot$$
$$\cdot((c^4l_2l_3-c^4l_3^2-c^4l_3l_6+16c^3eF_B^2-16c^3eF_BF_C-c^3el_2l_3+c^3el_3^2-2c^3el_3l_6-$$
$$-16c^2e^2F_B^2+16c^2e^2F_BF_C-4c^2F_B^2l_2-4c^2F_B^2l_3+8c^2F_B^2l_5+4c^2F_B^2l_6+$$
$$+4c^2F_BF_Cl_2+20c^2F_BF_Cl_3-12c^2F_BF_Cl_5-8c^2F_BF_Dl_3-4c^2F_C^2l_2-12c^2F_C^2l_3+$$
$$+24c^2F_CF_Dl_3-8c^2F_D^2l_3+8ceF_B^2l_2+12ceF_B^2l_3-8ceF_B^2l_6-12ceF_BF_Cl_3+$$
$$+8ceF_BF_Cl_5-32F_B^3F_D+16F_BF_CF_D^2)\mathbf{x^2}+(2c^3dl_3l_6-8c^3eF_Bl_5+8c^2e^2F_Bl_5+$$
$$-8c^2e^2F_Bl_6-96c^2F_B^2F_C+64c^2F_BF_C^2-32c^2F_BF_CF_D+10c^2F_Bl_2l_3+$$
$$+4c^2F_Bl_2l_5-2c^2F_Bl_3^2-10c^2F_Bl_3l_4+6c^2F_Bl_3l_5+4c^2F_Bl_3l_6-8c^2F_Cl_2l_3+2c^2F_Cl_3^2+$$
$$+12c^2F_Cl_3l_4-6c^2F_Cl_3l_5+2c^2F_Dl_2l_3+4c^2F_Dl_3^2-6c^2F_Dl_3l_4+32ceF_B^3$$
$$-32ceF_BF_C^2-2ceF_Bl_2^2-6ceF_Bl_2l_3+2ceF_Bl_2l_6-16F_B^3l_1-56F_B^3l_2+$$
$$+40F_B^3l_4+8F_B^2F_Cl_1+96F_B^2F_Cl_2-16F_B^2F_Dl_1-40F_B^2F_Dl_2-48F_B^2F_Dl_3+$$
$$+24F_B^2F_Dl_4-8F_BF_C^2l_2+32F_BF_CF_Dl_2+64F_BF_CF_Dl_3-16F_BF_D^2l_2-$$
$$-24F_BF_D^2l_3-24F_C^3l_2+24F_C^2F_Dl_2)\mathbf{xy}+(-c^4l_2l_3+c^4l_3^2+c^4l_3l_6-16c^3eF_B^2+$$
$$+16c^3eF_BF_C+c^3el_2l_3-c^3el_3^2+2c^3el_3l_6+16c^2e^2F_B^2-16c^2e^2F_BF_C+4c^2F_B^2l_2+$$
$$+12c^2F_B^2l_3-16c^2F_B^2l_5-4c^2F_B^2l_6-12c^2F_BF_Cl_2-12c^2F_BF_Cl_3+12c^2F_BF_Cl_5+$$
$$+16c^2F_BF_Dl_3+12c^2F_C^2l_2+12c^2F_C^2l_3-24c^2F_CF_Dl_3-2c^2l_2l_3^2-c^2l_2l_3l_4+2c^2l_3^3+$$
$$+c^2l_2l_3l_5+c^2l_2l_3l_6+c^2l_3^2l_4-c^2l_3^2l_5-c^2l_3^2l_6-8ceF_B^2l_2+4ceF_B^2l_3-4ceF_BF_Cl_3+$$
$$+2cel_2^2l_3-2cel_2l_3^2-cel_2l_3l_6+64F_B^3F_D-32F_B^2F_CF_D+64F_B^2F_D^2-4F_B^2l_1l_2+$$
$$+4F_B^2l_1l_3-12F_B^2l_2l_3+4F_B^2l_2l_4+4F_B^2l_2l_5-4F_B^2l_2l_6-48F_BF_CF_D^2+4F_BF_Cl_2^2+$$
$$+20F_BF_Cl_2l_3+4F_BF_Cl_2l_5+8F_BF_Dl_1l_3-36F_BF_Dl_2l_3+12F_BF_Dl_3^2-4F_BF_Dl_3l_4-$$
$$-12F_C^2l_2l_3+4F_C^2l_2l_4-4F_C^2l_2l_5-4F_CF_Dl_1l_3+20F_CF_Dl_2l_3+4F_CF_Dl_3l_4+4F_D^2l_1l_3-$$
$$-12F_D^2l_2l_3+4F_D^2l_3^2-4F_D^2l_3l_4)\mathbf{y^2})+$$
$$+(\mathbf{x^2+y^2})\cdot$$
$$\cdot((128c^3F_BF_C^2-128c^3F_B^2F_C+12c^3F_Bl_2l_3-12c^3F_Bl_3^2-8c^3F_Bl_3l_4+12c^3F_Cl_3^2+$$
$$+24c^3F_Bl_3l_5-4c^3F_Bl_3l_6+12c^3F_Cl_2l_3-24c^3F_Dl_3^2+64c^2eF_B^3+64c^2eF_B^2F_C-$$
$$-128c^2eF_BF_C^2-12c^2eF_Bl_2l_3-4c^2eF_Bl_3l_5+4c^2eF_Bl_3l_6-64aF_B^3F_D-32aF_B^2F_D^2+$$
$$+96aF_B^2F_CF_D+32aF_BF_CF_D^2-48cF_B^3l_2-96cF_B^3l_3+32cF_B^3l_4+48cF_B^3l_5+16cF_B^3l_6+$$
$$+80cF_B^2F_Cl_2+48cF_B^2F_Cl_3-48cF_B^2F_Cl_5-192cF_B^2F_Dl_3+48cF_BF_C^2l_3-16cF_BF_C^2l_5-$$
$$-32cF_BF_CF_Dl_2+64cF_BF_CF_Dl_3-112cF_BF_D^2l_3-48cF_C^3l_2+32cF_C^2F_Dl_2+16cF_D^3l_3+$$
$$+48cF_C^2F_Dl_3-48cF_CF_D^2l_3)\mathbf{x^3}+(112c^3F_B^2l_3-96c^3F_B^2l_5-288c^3F_BF_Cl_3-2c^3l_2^2l_3+$$

$$+64c^3F_BF_Cl_5+96c^3F_BF_Dl_3+7c^3l_2l_3^2+7c^3l_2l_3l_5+2c^3l_2l_3l_6-5c^3l_3^3-14c^3l_3^2l_4+$$
$$+7c^3l_3^2l_5-48c^2eF_B^2l_2-80c^2eF_B^2l_3+48c^2eF_B^2l_5+16c^2eF_B^2l_6+80c^2eF_BF_Cl_3-$$
$$-64c^2eF_BF_Cl_5+4c^2el_2^2l_3-4c^2el_2l_3^2+2c^2el_2l_3l_6+16aF_B^3l_1+120aF_B^3l_2-$$
$$-88aF_B^3l_4-8aF_B^2F_Cl_1-272aF_B^2F_Cl_2-8aF_B^2F_Dl_1-24aF_B^2F_Dl_2-120aF_B^2F_Dl_3+$$
$$+104aF_BF_C^2l_2+112aF_BF_CF_Dl_2+32aF_BF_CF_Dl_3-48aF_BF_D^2l_3+24aF_C^3l_2-$$
$$-96aF_C^2F_Dl_2-72aF_C^2F_Dl_3+120aF_CF_D^2l_3+24aF_D^3l_3-64cF_B^4-192cF_B^3F_C+$$
$$+384cF_B^3F_D+192cF_B^2F_C^2+192cF_B^2F_CF_D+384cF_B^2F_D^2+8cF_B^2l_2^2-68cF_B^2l_2l_3+$$
$$+12cF_B^2l_2l_5-8cF_B^2l_2l_6+56cF_B^2l_3l_4+64cF_BF_C^3-320cF_BF_C^2F_D-320cF_BF_CF_D^2+$$
$$-24cF_C^2l_2l_3-140cF_BF_Dl_2l_3-72cF_BF_Dl_3^2+44cF_BF_Dl_3l_4-4cF_C^2l_2^2+36cF_C^2l_3l_4-$$
$$+168cF_BF_Cl_2l_3+36cF_C^2l_2l_4+40cF_CF_Dl_2l_3+32cF_CF_Dl_3^2-72cF_CF_Dl_3l_4-$$
$$28cF_D^2l_2l_3-16cF_D^2l_3^2+8eF_B^2l_2^2)\mathbf{x^2y}+(384c^3F_B^2F_C-128c^3F_BF_C^2-36c^3F_Bl_2l_3+$$
$$+52c^3F_Bl_3^2+24c^3F_Bl_3l_4-40c^3F_Bl_3l_5-4c^3F_Bl_3l_6-20c^3F_Cl_2l_3+4c^2dl_2l_3^2-$$
$$-20c^3F_Cl_3^2+40c^3F_Dl_3^2-2c^2dl_2^2l_3-2c^2dl_3^3-2c^2dl_2l_3l_6+2c^2dl_3^2l_6+64c^2eF_B^3-$$
$$-192c^2eF_B^2F_C+128c^2eF_BF_C^2-8c^2eF_Bl_3^2+16c^2eF_Bl_2^2-4c^2eF_Bl_2l_3+32aF_B^2l_2^2-$$
$$-16c^2eF_Bl_2l_6-4c^2eF_Bl_3l_5+4c^2eF_Bl_3l_6+192aF_B^3F_D-32aF_B^2F_CF_D+8aF_B^2l_2l_6-$$
$$-32aF_B^2F_D^2+40aF_B^2l_1l_2-24aF_B^2l_1l_3-8aF_B^2l_2l_3-32aF_B^2l_2l_4+8aF_BF_Dl_2^2-$$
$$-32aF_B^2l_2l_5+32aF_BF_CF_D^2-16aF_BF_Cl_1l_2-72aF_BF_Cl_2^2-40aF_BF_Cl_2l_3+$$
$$+8aF_BF_Cl_2l_5+32aF_BF_Dl_2l_3+16aF_BF_Dl_3^2+24aF_C^2l_2^2-8aF_C^2l_2l_4+8aF_C^2l_2l_5-$$
$$-8aF_BF_Dl_3l_4+48aF_C^2l_2l_3+8aF_CF_Dl_1l_3-104aF_CF_Dl_2l_3-32aF_CF_Dl_3^2-$$
$$-8aF_CF_Dl_3l_4-8aF_D^2l_1l_3+16aF_D^2l_2l_3+48aF_D^2l_3^2-16cF_B^3l_5+16cF_B^3l_6+8aF_D^2l_3l_4+$$
$$+144cF_B^3l_2-32cF_B^3l_3-96cF_B^3l_4-240cF_B^2F_Cl_2-16cF_B^2F_Cl_3+16cF_B^2F_Cl_5+$$
$$+192cF_B^2F_Dl_3-192cF_BF_C^2l_2+48cF_BF_C^2l_3-16cF_BF_C^2l_5-4cF_Bl_2^2l_3-4cF_Bl_2^2l_5+$$
$$+224cF_BF_CF_Dl_2-64cF_BF_CF_Dl_3+80cF_BF_D^2l_3-32cF_Bl_2l_3^2-4cF_Bl_2l_3l_6-4cF_Bl_3^2l_4+$$
$$+4cF_Bl_2l_3l_4+32cF_Bl_2l_3l_5+144cF_C^3l_2-160cF_C^2F_Dl_2-144cF_C^2F_Dl_3+144cF_CF_D^2l_3+$$
$$+44cF_Cl_2^2l_3+12cF_Cl_2l_3^2-16cF_Cl_2l_3l_4+16cF_Cl_3^2l_4+4cF_Dl_3^3-16cF_Dl_3^2l_4-4eF_Bl_2^3+$$
$$+16cF_D^3l_3-12cF_Dl_2^2l_3-48cF_Dl_2l_3^2+16cF_Dl_2l_3l_4+12eF_Bl_2^2l_3+4eF_Bl_2^2l_6)\mathbf{xy^2}+$$
$$(32c^3F_B^2l_5-16c^3F_B^2l_3+32c^3F_BF_Cl_3-32c^3F_BF_Dl_3+2c^3l_2^2l_3-c^3l_2l_3^2-c^3l_2l_3l_5+$$
$$+16c^2eF_B^2l_2-2c^3l_2l_3l_6-c^3l_3^3+2c^3l_3^2l_4-c^3l_3^2l_5-16c^2eF_B^2l_3-16c^2eF_B^2l_5-16aF_B^3l_1-$$
$$+16c^2eF_B^2l_6+16c^2eF_BF_Cl_3-4c^2el_2^2l_3+4c^2el_2l_3^2-2c^2el_2l_3l_6-40aF_B^3l_2+$$
$$+40aF_B^3l_4-8aF_B^2F_Cl_1+112aF_B^2F_Cl_2-8aF_B^2F_Dl_1+8aF_B^2F_Dl_2+8aF_B^2F_Dl_3+$$
$$-24aF_BF_C^2l_2+16aF_BF_CF_Dl_2-64aF_BF_CF_Dl_3+16aF_BF_D^2l_3-2aF_Bl_1^2l_2+$$
$$+2aF_Bl_1^2l_3+2aF_Bl_1l_2^2-12aF_Bl_1l_2l_3+2aF_Bl_1l_2l_5-2aF_Bl_1l_3^2-2aF_Bl_1l_3l_4-$$
$$-2aF_Bl_2^3+12aF_Bl_2^2l_3-2aF_Bl_2^2l_5+2aF_Bl_2^2l_6-10aF_Bl_2l_3^2-88cF_BF_Cl_2l_3+$$
$$+8aF_Bl_2l_3l_4-8aF_Bl_2l_3l_5-8aF_Bl_2l_3l_6+2aF_Bl_3^2l_4-8aF_C^3l_2+32aF_C^2F_Dl_2+$$
$$+24aF_C^2F_Dl_3-40aF_CF_D^2l_3+2aF_Cl_1l_2^2+4aF_Cl_1l_2l_3-2aF_Cl_2^2l_3-2aF_Cl_2^2l_5+$$
$$+8aF_Cl_2l_3^2-12aF_Cl_2l_3l_4+8aF_Cl_2l_3l_5-8aF_D^3l_3-8aF_Dl_1l_2l_3+2aF_Dl_1l_3^2-$$
$$-4aF_Dl_2l_3^2+8aF_Dl_2l_3l_4-2aF_Dl_3^3-2aF_Dl_3^2l_4-64cF_B^4+64cF_B^3F_C-128cF_B^3F_D-$$
$$-64cF_B^2F_C^2-320cF_B^2F_CF_D-128cF_B^2F_D^2-8cF_B^2l_2^2+12cF_B^2l_2l_3-4cF_B^2l_2l_5+$$
$$+8cF_B^2l_2l_6-8cF_B^2l_3l_4+64cF_BF_C^3+192cF_BF_C^2F_D-64cF_BF_CF_D^2+16cF_BF_Cl_2^2-$$

$$+36cF_BF_Dl_2l_3+8cF_BF_Dl_3^2-20cF_BF_Dl_3l_4-20cF_C^2l_2^2+24cF_C^2l_2l_3-12cF_C^2l_2l_4-$$
$$-12cF_C^2l_3l_4+8cF_CF_Dl_2l_3-32cF_CF_Dl_3^2+24cF_CF_Dl_3l_4+4cF_D^2l_2l_3-cl_2^3l_3+$$
$$+16cF_D^2l_3^2+7cl_2^2l_3^2+cl_2^2l_3l_4+3cl_2^2l_3l_5-6cl_2l_3^3-8cl_2l_3^2l_4+3cl_2l_3^2l_5+4cl_2l_3^2l_6+$$
$$+cl_3^3l_4-8eF_B^2l_2^2-2el_2^3l_3+2el_2^2l_3^2+4el_2^2l_3l_6)\mathbf{y^3})+$$

$$+(128c^2F_B^2F_C^2-256c^2F_B^3F_C+128c^2F_B^2F_CF_D+16c^2F_B^2l_2l_3+8c^2F_B^2l_3^2-$$
$$-16c^2F_B^2l_3l_4-8c^2F_B^2l_3l_5+128c^2F_BF_C^3-128c^2F_BF_C^2F_D+8c^2F_BF_Cl_2l_3-$$
$$-16c^2F_BF_Cl_3^2+16c^2F_BF_Cl_3l_5-8c^2F_BF_Dl_2l_3-8c^2F_BF_Dl_3^2+8c^2F_BF_Dl_3l_4+$$
$$+48c^2F_C^2l_2l_3-24c^2F_CF_Dl_2l_3-24c^2F_CF_Dl_3^2+8ceF_B^2l_2l_3-64F_B^4l_1-64F_B^4l_2-$$
$$+64F_B^4l_4+64F_B^3F_Cl_1+128F_B^3F_Cl_2-32F_B^3F_Dl_1+32F_B^3F_Dl_2+96F_B^3F_Dl_3+$$
$$-32F_B^3F_Dl_4+32F_B^2F_CF_Dl_1-128F_B^2F_CF_Dl_2-96F_B^2F_CF_Dl_3+96F_B^2F_D^2l_3+$$
$$-64F_BF_C^3l_2+64F_BF_C^2F_Dl_2-32F_BF_C^2F_Dl_3+32F_BF_CF_D^2l_2-128F_BF_CF_D^2l_3-$$
$$+32F_BF_D^3l_3+32F_C^3F_Dl_2-32F_C^2F_D^2l_2-32F_C^2F_D^2l_3+32F_CF_D^3l_3)\mathbf{x^4}+$$

$$+(64c^2F_B^3l_3-64c^2F_B^3l_5+128c^2F_B^2F_Cl_2+64c^2F_B^2F_Cl_3+64c^2F_B^2F_Dl_3-8c^2F_Dl_3^3-$$
$$-192c^2F_BF_C^2l_2-128c^2F_BF_C^2l_3+64c^2F_BF_C^2l_5+512c^2F_BF_CF_Dl_3+8c^2F_Bl_2l_3l_4-$$
$$-8c^2F_Bl_2^2l_3+8c^2F_Bl_2l_3^2-8c^2F_Bl_2l_3l_5+64c^2F_C^3l_2-64c^2F_CF_D^2l_3+8c^2F_Cl_2^2l_3+$$
$$+24c^2F_Cl_2l_3^2-16c^2F_Cl_2l_3l_4-16c^2F_Cl_3^2l_4-24c^2F_Dl_2l_3^2+32c^2F_Dl_3^2l_4-32F_B^3l_1l_2+$$
$$+256F_B^4F_D+256F_B^3F_D^2-32F_B^3l_1^2+32F_B^3l_2^2-32F_B^3l_2l_4-256F_B^2F_C^2F_D+32F_B^2F_Cl_1l_2+$$
$$-96F_B^2F_Cl_2^2+160F_B^2F_Dl_1l_3+160F_B^2F_Dl_2l_3-128F_B^2F_Dl_3l_4-768F_BF_C^2F_D^2+$$
$$+128F_BF_C^2l_1l_2+96F_BF_C^2l_2^2-32F_BF_CF_Dl_1l_2-32F_BF_CF_Dl_1l_3-64F_C^3l_1l_2-$$
$$-384F_BF_CF_Dl_2l_3+64F_BF_D^2l_1l_3+64F_BF_D^2l_2l_3-32F_BF_D^2l_3^2+32F_C^2F_Dl_1l_2+$$
$$+32F_C^2F_Dl_1l_3-64F_C^3l_2^2+64F_C^3l_2l_4+64F_C^2F_Dl_2l_3-32F_C^2F_Dl_2l_4-32F_C^2F_Dl_3l_4-$$
$$+96F_CF_D^2l_2l_3+32F_CF_D^2l_3^2-128F_D^3l_3^2)\mathbf{x^3y}+$$

$$+(256c^2F_B^2F_C^2-256c^2F_B^3F_C-1024c^2F_B^2F_CF_D-48c^2F_B^2l_2l_3+64c^2F_B^2l_2l_5-$$
$$-16c^2F_B^2l_3l_4+80c^2F_BF_Cl_2l_3-16c^2F_BF_Cl_3^2+16c^2F_BF_Cl_3l_5-16c^2F_BF_Dl_3^2-$$
$$-64c^2F_BF_Dl_3l_4-96c^2F_C^2l_2l_3+48c^2F_CF_Dl_2l_3+48c^2F_CF_Dl_3^2+2c^2l_2^2l_3^2-2c^2l_2l_3^3-$$
$$-4c^2l_2^2l_3l_5+4c^2l_2l_3^2l_4+4c^2l_2l_3^2l_5-4c^2l_3^3l_4+16ceF_B^2l_2l_3-2cel_2^3l_3+2cel_2^2l_3^2+$$
$$+2cel_2^2l_3l_6-64F_B^4l_1-64F_B^4l_2+64F_B^4l_4+64F_B^3F_Cl_1+128F_B^3F_Cl_2+16F_B^2l_1^2l_2-$$
$$-320F_B^3F_Dl_1-512F_B^3F_Dl_2+64F_B^3F_Dl_3+256F_B^3F_Dl_4+64F_B^2F_CF_Dl_1-24F_D^2l_1l_2l_3+$$
$$+768F_B^2F_CF_Dl_2-64F_B^2F_CF_Dl_3-256F_B^2F_D^2l_2+64F_B^2F_D^2l_3-16F_C^2l_1l_2^2-32F_C^2l_1l_2l_3+$$
$$+16F_B^2l_1^2l_3+16F_B^2l_1l_2^2+24F_B^2l_2^2l_3-16F_B^2l_2^2l_5-16F_B^2l_2l_3l_4+32F_D^2l_2l_3^2+8F_D^2l_3^3-$$
$$-64F_BF_C^3l_2-64F_BF_C^2F_Dl_2-64F_BF_C^2F_Dl_3+64F_BF_CF_D^2l_2+16F_BF_Cl_1^2l_2-$$
$$-48F_BF_Cl_1l_2^2+8F_BF_Cl_1l_2l_3-16F_BF_Cl_1l_2l_5-72F_BF_Cl_2^2l_3+16F_BF_Cl_2^2l_5+$$
$$+320F_BF_D^3l_3-16F_BF_Dl_1^2l_3+8F_BF_Dl_1l_2l_3+16F_BF_Dl_1l_3l_4+64F_BF_Dl_2^2l_3+$$
$$+8F_BF_Dl_2l_3^2-16F_BF_Dl_3^2l_4-192F_C^3F_Dl_2+192F_C^2F_D^2l_2+192F_C^2F_D^2l_3+$$
$$+16F_C^2l_2^3+56F_C^2l_2^2l_3-24F_C^2l_2^2l_4+16F_C^2l_2^2l_5+8F_C^2l_2l_3l_4-192F_CF_D^3l_3+$$
$$+88F_CF_Dl_1l_2l_3+24F_CF_Dl_1l_3^2-80F_CF_Dl_2^2l_3-48F_CF_Dl_2l_3^2-24F_CF_Dl_3^2l_4$$
$$+8F_CF_Dl_2l_3l_4-40F_D^2l_1l_3^2+16F_D^2l_2^2l_3+16F_D^2l_3^2l_4)\mathbf{x^2y^2}+$$

$$
\begin{gathered}
(64c^2F_B^3l_3-64c^2F_B^3l_5-128c^2F_B^2F_Cl_2+64c^2F_B^2F_Cl_3+64c^2F_B^2F_Dl_3+64c^2F_BF_C^2l_2\\
-128c^2F_BF_C^2l_3+64c^2F_BF_C^2l_5+8c^2F_Bl_2^2l_3-8c^2F_Bl_2l_3^2-8c^2F_Bl_2l_3l_4+8c^2F_Bl_2l_3l_5+\\
+64c^2F_C^3l_2-64c^2F_CF_D^2l_3+8c^2F_Cl_2^2l_3-8c^2F_Cl_2l_3^2-8c^2F_Dl_2l_3^2+8c^2F_Dl_3^3+2cdl_2^3l_3-\\
-4cdl_2^2l_3^2-2cdl_2^2l_3l_6+2cdl_2l_3^3-2cdl_2l_3^2l_6-8ceF_Bl_2^2l_3+8ceF_Bl_2l_3^2+256F_B^4F_D-64F_D^3l_3^2+\\
+256F_B^3F_D^2+32F_B^3l_1^2-32F_B^3l_1l_2-32F_B^3l_2^2+32F_B^3l_2l_4-256F_B^2F_C^2F_D-128F_BF_C^2l_1l_2+\\
+32F_B^2F_Cl_1l_2+96F_B^2F_Cl_2^2+32F_B^2F_Dl_1l_3-32F_B^2F_Dl_2l_3+256F_BF_C^2F_D^2-32F_BF_C^2l_2^2+\\
+32F_BF_CF_Dl_1l_2+32F_BF_CF_Dl_1l_3-64F_BF_D^2l_1l_3+32F_BF_D^2l_3^2-8F_Bl_1^2l_2^2+16F_Bl_1^2l_2l_3+\\
+8F_Bl_1^2l_3^2-16F_Bl_1l_2^2l_3+8F_Bl_1l_2^2l_5-8F_Bl_1l_2l_3^2+4F_Bl_1l_2l_3l_4-4F_Bl_1l_2l_3l_5-4F_Dl_1l_3^2l_4-\\
-8F_Bl_1l_3^2l_4+4F_Bl_2^3l_3-4F_Bl_2^2l_3^2-4F_Bl_2^2l_3l_5-4F_Bl_2^2l_3l_6+4F_Bl_2l_3^2l_4-12F_Dl_2l_3^3+\\
+64F_C^3l_1l_2-64F_C^3l_2l_4-32F_C^2F_Dl_1l_2-32F_C^2F_Dl_1l_3+64F_C^2F_Dl_2^2-64F_C^2F_Dl_2l_3+\\
+32F_C^2F_Dl_2l_4+32F_C^2F_Dl_3l_4-32F_CF_D^2l_2l_3+96F_CF_D^2l_3^2+4F_Cl_1^2l_2^2-12F_Cl_1^2l_2l_3+\\
+4F_Cl_1l_2^3+8F_Cl_1l_2^2l_3-4F_Cl_1l_2^2l_5-8F_Cl_1l_2l_3^2+16F_Cl_1l_2l_3l_4-4F_Cl_1l_2l_3l_5-\\
-4F_Cl_2^3l_5+16F_Dl_2l_3^2l_4+12F_Cl_2^2l_3^2-12F_Cl_2^2l_3l_4-4F_Dl_3^3l_4+16F_Cl_2^2l_3l_5-12F_Cl_2l_3^2l_4+\\
+4F_Dl_1^2l_2l_3+4F_Dl_1^2l_3^2-20F_Dl_1l_2^2l_3+12F_Dl_1l_2l_3^2-4F_Dl_1l_2l_3l_4+4F_Dl_1l_3^3)\mathbf{xy^3}+
\end{gathered}
$$

$$
\begin{gathered}
+(128c^2F_B^2F_C^2-128c^2F_B^2F_CF_D-8c^2F_B^2l_3^2+8c^2F_B^2l_3l_5-128c^2F_BF_C^3+128c^2F_BF_C^2F_D+\\
+8c^2F_BF_Cl_2l_3+8c^2F_BF_Dl_2l_3-8c^2F_BF_Dl_3^2-8c^2F_BF_Dl_3l_4-16c^2F_C^2l_2l_3+8c^2F_CF_Dl_2l_3+\\
+3l_1l_2^2l_3^2+8c^2F_CF_Dl_3^2-2c^2l_2^2l_3^2+2c^2l_2l_3^3+8ceF_B^2l_2l_3+2cel_2^3l_3-2cel_2^2l_3^2-2cel_2^2l_3l_6-\\
-32F_B^3F_Dl_1-32F_B^3F_Dl_2-32F_B^3F_Dl_3+32F_B^3F_Dl_4+32F_B^2F_CF_Dl_1+128F_B^2F_CF_Dl_2+\\
+32F_B^2F_CF_Dl_3-32F_B^2F_D^2l_3+16F_B^2l_1l_2l_3+8F_B^2l_2^2l_3-128F_BF_C^2F_Dl_2-32F_BF_C^2F_Dl_3+\\
+32F_BF_CF_D^2l_2+128F_BF_CF_D^2l_3-24F_BF_Cl_1l_2l_3-8F_BF_Cl_2^2l_3+32F_BF_D^3l_3+8F_BF_Dl_1l_2l_3+\\
+16F_BF_Dl_1l_3^2+8F_BF_Dl_2l_3^2+32F_C^3F_Dl_2-32F_C^2F_D^2l_2-32F_C^2F_D^2l_3+16F_C^2l_1l_2l_3-l_2^2l_3^3-\\
-8F_C^2l_2^2l_3+3l_2^2l_3^2l_4+8F_C^2l_2^2l_4-8F_C^2l_2l_3l_4+32F_CF_D^3l_3-8F_CF_Dl_1l_2l_3-8F_CF_Dl_1l_3^2-\\
-16F_CF_Dl_2l_3^2+3l_2^2l_3^2l_5-8F_CF_Dl_2l_3l_4+8F_CF_Dl_3^2l_4+8F_D^2l_1l_2l_3+8F_D^2l_1l_3^2-16F_D^2l_2l_3^2+\\
+8F_D^2l_3^3+l_1^3l_2l_3-2l_1^2l_2^2l_3-2l_1^2l_2l_3^2-l_1^2l_2l_3l_4-l_1^2l_2l_3l_5+l_1l_2^3l_3-l_1l_2^2l_3l_4-l_2l_3^3l_4-l_2l_3^2l_4l_5+\\
+2l_1l_2^2l_3l_5+l_1l_2l_3^3+2l_1l_2l_3^2l_4-l_1l_2l_3^2l_5+l_1l_2l_3l_4l_5-l_2^3l_3^2-l_2^3l_3l_5-l_2^2l_3l_4l_5-2l_2^2l_3^2l_6)\mathbf{y^4}+
\end{gathered}
$$

$$
\begin{gathered}
+(32aF_B^4l_2-32aF_B^4l_4-64aF_B^3F_Cl_2-32aF_B^3F_Dl_3+32aF_B^2F_C^2l_2+32aF_B^2F_CF_Dl_2+\\
+32aF_B^2F_CF_Dl_3-32aF_BF_C^2F_Dl_2+32aF_BF_CF_D^2l_3+256cF_B^3F_CF_D-256cF_B^2F_C^2F_D-\\
-32cF_B^2F_Cl_2l_3-16cF_B^2F_Dl_2l_3+16cF_B^2F_Dl_3l_4+16cF_BF_C^2l_2l_3-32cF_BF_CF_Dl_2l_3+\\
+16cF_BF_D^2l_3^2+16cF_C^3l_2l_3-32cF_C^2F_Dl_2l_3+16cF_CF_D^2l_2l_3)\mathbf{x^3}+
\end{gathered}
$$

$$
\begin{gathered}
+(8aF_B^3l_1l_2-16aF_B^3l_2^2+16aF_B^3l_2l_4-8aF_B^2F_Cl_1l_2+48aF_B^2F_Cl_2^2-8aF_B^2F_Dl_1l_3-\\
-56aF_BF_C^2l_2^2+24aF_C^3l_2^2-48aF_BF_CF_Dl_2l_3-8aF_BF_D^2l_2l_3+32aF_BF_D^2l_3^2+24aF_D^3l_3^2+\\
+24aF_C^2F_Dl_2l_3-48aF_CF_D^2l_2l_3-24aF_CF_D^2l_3^2-64cF_B^2F_C^2l_2-128cF_B^2F_CF_Dl_2-\\
-64cF_B^2F_CF_Dl_3+64cF_BF_C^3l_2+128cF_BF_C^2F_Dl_2+64cF_BF_C^2F_Dl_3-64cF_BF_CF_D^2l_3+\\
+8cF_D^2l_3^3+8cF_BF_Dl_2^2l_3-8cF_BF_Dl_2l_3l_4+16cF_C^2l_2^2l_3-24cF_C^2l_2l_3l_4-4cF_CF_Dl_2^2l_3-\\
-20cF_CF_Dl_2l_3^2+12cF_CF_Dl_2l_3l_4+12cF_CF_Dl_3^2l_4)\mathbf{x^2y}+
\end{gathered}
$$

$$
\begin{gathered}
+(32aF_B^4l_2-32aF_B^4l_4-64aF_B^3F_Cl_2-32aF_B^3F_Dl_3+32aF_B^2F_C^2l_2+32aF_B^2F_CF_Dl_2+\\
+32aF_B^2F_CF_Dl_3-8aF_B^2l_1l_2^2+8aF_B^2l_1l_2l_3-8aF_B^2l_2^2l_3+8aF_B^2l_2^2l_5+8aF_B^2l_2l_3l_4-\\
-32aF_BF_C^2F_Dl_2+32aF_BF_CF_D^2l_3+16aF_BF_Cl_1l_2^2-8aF_BF_Cl_1l_2l_3+24aF_BF_Cl_2^2l_3-\\
-16aF_BF_Cl_2^2l_5-8aF_BF_Dl_1l_2l_3-16aF_BF_Dl_1l_3^2-8aF_BF_Dl_2^2l_3-8aF_BF_Dl_2l_3^2+\\
+16aF_BF_Dl_3^2l_4-8aF_C^2l_1l_2^2+8aF_C^2l_1l_2l_3-24aF_C^2l_2^2l_3+8aF_C^2l_2^2l_5-8aF_C^2l_2l_3l_4+\\
+8aF_CF_Dl_1l_2l_3+24aF_CF_Dl_2^2l_3+32aF_CF_Dl_2l_3^2-8aF_CF_Dl_2l_3l_4-8aF_D^2l_1l_3^2-\\
-24aF_D^2l_2l_3^2-8aF_D^2l_3^3+8aF_D^2l_3^2l_4+256cF_B^3F_CF_D-256cF_B^2F_C^2F_D-32cF_B^2F_Cl_2l_3-\\
-16cF_B^2F_Dl_2l_3+16cF_B^2F_Dl_3l_4+64cF_BF_C^2l_2^2+16cF_BF_C^2l_2l_3-96cF_BF_CF_Dl_2l_3+\\
+16cF_BF_D^2l_3^2+4cF_Bl_2^2l_3^2-4cF_Bl_2^2l_3l_5-64cF_C^3l_2^2+16cF_C^3l_2l_3+96cF_C^2F_Dl_2l_3+\\
+16cF_CF_D^2l_2l_3-64cF_CF_D^2l_3^2-4cF_Cl_2^3l_3-4cF_Cl_2^2l_3^2+4cF_Cl_2^2l_3l_4-4cF_Cl_2l_3^2l_4+\\
+8cF_Dl_2^2l_3^2-4cF_Dl_2l_3^2l_4+4cF_Dl_3^3l_4)\mathbf{xy^2}+
\end{gathered}
$$

$$
\begin{gathered}
+(8aF_B^3l_1l_2+16aF_B^3l_2^2-16aF_B^3l_2l_4-8aF_B^2F_Cl_1l_2-48aF_B^2F_Cl_2^2-8aF_B^2F_Dl_1l_3+\\
+16aF_BF_CF_Dl_2l_3-8aF_BF_D^2l_2l_3-32aF_BF_D^2l_3^2+4aF_Bl_1l_2l_3^2-2aF_Bl_1l_2l_3l_4+\\
+2aF_Bl_1l_2l_3l_5-2aF_Bl_2^3l_3+2aF_Bl_2^2l_3^2-2aF_Bl_2^2l_3l_5+2aF_Bl_2^2l_3l_6+2aF_Bl_2l_3^2l_4-\\
-8aF_C^3l_2^2-8aF_C^2F_Dl_2l_3+16aF_CF_D^2l_2l_3+8aF_CF_D^2l_3^2-2aF_Cl_1l_2l_3^2+2aF_Cl_2^3l_3-\\
-4aF_Cl_2^2l_3^2+2aF_Cl_2^2l_3l_4-2aF_Cl_2^2l_3l_5+2aF_Cl_2l_3^2l_4-8aF_D^3l_3^2+2aF_Dl_1l_2l_3^2+\\
+2aF_Dl_2l_3^3-2aF_Dl_2l_3^2l_4-64cF_B^2F_C^2l_2+128cF_B^2F_CF_Dl_2-64cF_B^2F_CF_Dl_3-\\
+64cF_BF_C^2F_Dl_3-64cF_BF_CF_D^2l_3+16cF_BF_Cl_2^2l_3-8cF_BF_Dl_2^2l_3-8cF_D^2l_3^3+\\
+8cF_C^2l_2l_3l_4-4cF_CF_Dl_2^2l_3+12cF_CF_Dl_2l_3^2-4cF_CF_Dl_2l_3l_4-4cF_CF_Dl_3^2l_4+\\
-128cF_BF_C^2F_Dl_2+8cF_BF_Dl_2l_3l_4+64cF_BF_C^3l_2+40aF_BF_C^2l_2^2+\\
+cl_2^3l_3^2+cl_2^2l_3^3+cl_2^2l_3^2l_4-2cl_2^2l_3^2l_5+cl_2l_3^3l_4)\mathbf{y^3}+\\
+32l_2l_3F_BF_CF_D(F_B-F_C+F_D)(\mathbf{x^2}+\mathbf{y^2})=0
\end{gathered}
$$