EQUIOPTIC POINTS OF A TRIANGLE

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ABSTRACT: The locus of points where two non-concentric circles c_1 and c_2 are seen under equal angles is the equioptic circle e. The equioptic circles of the excircles of a triangle Δ have a common radical axis r. Therefore the excircles of a triangle share up to two real points, i.e., the equioptic points of Δ from which the circles can be seen under equal angles. The line r carries a lot of known triangle centers. Further we find that any triplet of circles tangent to the sides of Δ has up to two real equioptic points. The three radical axes of triplets of circles containing the incircle are concurrent in a new triangle center.

Keywords: triangle, excircle, incircle, equioptic circle, equioptic points, center of similitude, radical axis.

1. INTRODUCTION

Let there be given a triangle Δ with vertices A, B, and C. The incenter shall be denoted by I, the incircle by Γ. The excenters are labeled with I_1, I_2, and I_3. We assume that I_1 is opposite to A, i.e., it is the center of the excircle Γ_1 touching the line $[B,C]$ from the outside of Δ, cf. Fig. 1. Sometimes it is convenient to number vertices as well as sides of Δ: The side (lines) $[B,C]$, $[C,A]$, $[A,B]$ shall be the first, second, third side (line) and A, B, C shall be the first, second, third vertex, respectively. According to [1, 2] we denote the centers of Δ with X_i. For example the incenter I is labeled with X_1.

The set of points where two curves can be seen under equal angles is called *equioptic curve*, see [3]. It is shown that any pair (c_1, c_2) of non-concentric circles has a circle e for its equioptic curve [3]. The circle e is the Thales circle of the segment bounded by the internal and external centers of similitude of either given circle, i.e., the center of e is the midpoint of the two centers of similitude, see Fig. 2. In case of two congruent circles e becomes the bisector of the centers of c_1 and c_2, provided that c_1 and c_2 are not concentric.

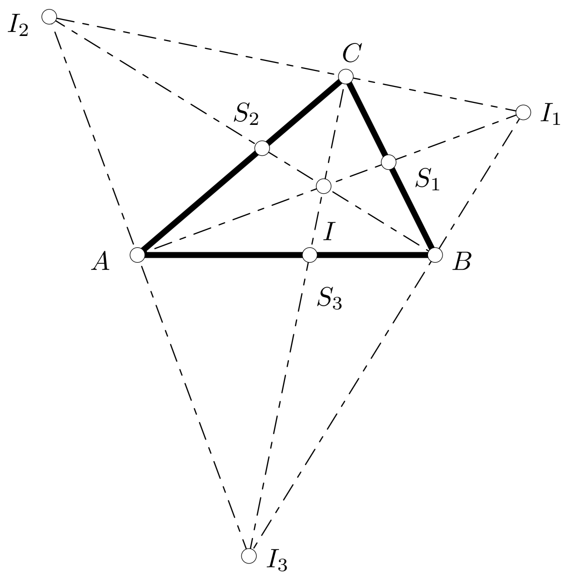

Figure 1: Notations in and around the triangle Δ.

The four circles Γ, Γ_i (with $i \in \{1,2,3\}$) tangent to the sides of a triangle Δ can be arranged in six pairs and thus they define that much equioptic circles. Among them we find four triplets of equioptic circles which have a common radical axis instead of a radical center, i.e., the three circles of such a triplet form a pencil of circles. These shall be the contents of Sec. 2 and Sec. 3.

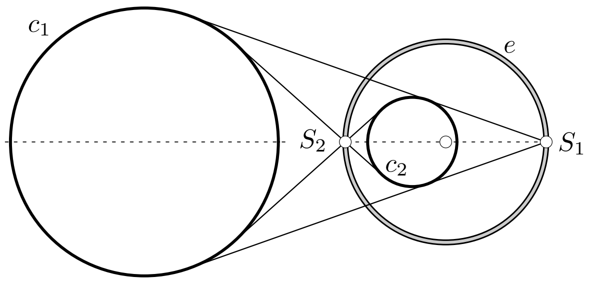

Figure 2: Equioptic circle of c_1 and c_2.

We use homogeneous trilinear coordinates of points and lines, respectively. The homogeneous triplet of real (complex) numbers $(x_0 : x_1 : x_2)$ are said to be the homogeneous trilinear coordinates of a point X if x_i are the oriented distances of X with respect to the sides $[B, C]$, $[C, A]$, and $[A, B]$ up to a common non vanishing factor, see. [1]. When we deal with trilinear coordinates of points expressed in terms of homogeneous polynomials in Δ's side lengths $a = \|BC\|$, $b = \|CA\|$, and $c = \|AB\|$ we need to have a function ζ which produces a homogeneous function out of a homogeneous function by cyclically replacing $a \to b$, $b \to c$, and $c \to a$. However, ζ is more powerful. It also applies to homogeneous polynomials depending on x_i and does $x_i \to x_{i+1}$ (indices $\mod 3$). It also changes $\sin A$ to $\sin B$ and similar for other trigonometric functions.

Another function which frequently appears will be denoted by σ. It acts on (homogeneous coordinate) triplets in the following way $\sigma(x_0 : x_1 : x_2) = (x_2 : x_0 : x_1)$. In this paper mappings will be written as superscripts, e.g., $\sigma \circ \zeta(X) = X^{\sigma\zeta} = X^{\zeta\sigma}$ if applied to points. Note that $\zeta \circ \sigma = \sigma \circ \zeta$, provided that x_i are homogeneous functions in a, b, c.

2 EQUIOPTIC CIRCLES OF THE EXCIRCLES

In order to construct the equioptic circles of a pair of excircles we determine the respective centers of similitude. First we observe that the internal centers of similitude of Γ_i and Γ_j is the k-th vertex of Δ, where $(i, j, k) \in \mathbb{I}^3 := \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Second we have to find the external centers of similitude. For any pair (Γ_i, Γ_j) the k-th side of Δ is an exterior common tangent of both Γ_i and Γ_j, respectively, and thus $[I_i, I_j]$ and the k-th side of Δ meet in the external center of similitude S_{ij} of Γ_i and Γ_j. Now we are able to show a first result:

Corollary 1.
The external centers of similitude S_{ij} of the excircles Γ_i and Γ_j of a triangle Δ are collinear. The line carrying these points is the polar of X_1 with respect to Δ and the polar line with respect to the excentral triangle of Δ at the same time.

Proof. We construct the polar line of the incenter X_1 with respect to Δ. For that purpose we project I from C to the line $[A, B]$. This gives $S_3 := [A, B] \cap [I, C]$. Then we determine a fourth point C' on $[A, B]$ such that (A, B, S_3, C') is a harmonic quadrupel. The four lines $[C, A]$, $[C, B]$, $[C, I]$, and $[I_1, I_2]$ obviously form a harmonic quadrupel and thus any line (which is not passing through C) meets these four lines in four points of a harmonnic quadrupel. So we have $C' = [I_1, I_2] \cap [A, B]$ and obvioiusly $C' = S_{12}$. Cyclically shifting labels of points gives S_{23}, S_{31} which are collinear with S_{12} and lie on the polar of X_1. On the other hand we have the harmonic quadruples (I_1, I_2, C, S_{12}) (cyclic) which shows that $[S_{12}, S_{23}]$ is the polar of X_1 with respect to the excentral triangle, see Fig. 3. □

The centers T_{ij} of the equioptic circles e_{ij} are the midpoints of the line segments bounded by S_{ij} and the k-th vertex of Δ with $(i, j, k) \in$

$\mathbb{I}^3$. Their trilinears are

$$S_{12} = (-1 : 1 : 0), \quad S_{23} = S_{12}^{\sigma}, \; S_{31} = S_{23}^{\sigma}.$$

The centers T_{ij} are thus

$$T_{12} = (c : -c : a - b), \quad T_{23} = T_{12}^{\sigma\zeta}, \; T_{31} = T_{23}^{\sigma\zeta}, \tag{1}$$

and we can easily prove:

Corollary 2.
The centers T_{ij} of the equioptic circles e_{ij} of any pair (Γ_i, Γ_j) of excircles of Δ are collinear.

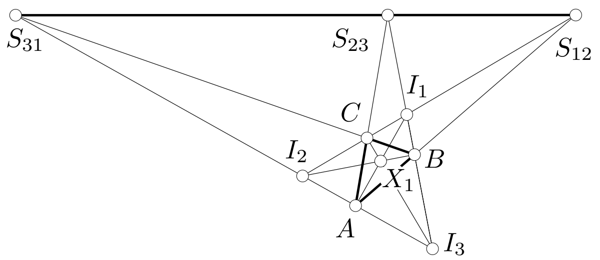

Figure 3: Centers of similtude and harmonic quadruples.

Proof. The coordinate vectors of T_{ij} given in (1) are linearly dependent. □

Remark 1.
The line t connecting any T_{ij} with any T_{jk} (with $(i,j,k) \in \mathbb{I}^3$) has trilinear coordinates $[\lambda_0 : \lambda_1 : \lambda_2]$, where $\lambda_0 = a(-a+b+c)$ and $\lambda_i = \zeta^i(\lambda_0)$ with $i \in \{0,1,2\}$. Obviously t is the polar of X_{55} with respect to Δ. The center X_{55} is the center of homothety of the tangential triangle, the intangent triangle, and the extangent triangle, see [1]. Further it is the internal center of similitude of the incircle and the circumcircle of the base triangle.

We use the formula for the distance of two points given by their *actual trilinear coordinates* given in [1, p. 31] and compute the radii ρ_{ij} of the equioptic circles c_{ij} and find

$$\begin{aligned} \rho_{12} &= \operatorname{dist}(C, T_{12}) = \operatorname{dist}(S_{12}, T_{12}) \\ &= \frac{ab}{a-b} \sin \frac{C}{2}, \\ \rho_{23} &= \zeta(\rho_{12}), \;\; \rho_{31} = \zeta(\rho_{23}). \end{aligned} \tag{2}$$

By means of the distance formula from [1, p. 31] or equivalently by means of the more complicated equation for a circle given by center and radius from [1, p. 223] we write down the equations of the equioptic circles

$$\begin{aligned} e_{12} : \;\; & -\mathrm{c_A}\, x_0^2 + \mathrm{c_B}\, x_1^2 + \\ & +(1-\mathrm{c_C})x_2(x_1 - x_0) \\ & +(\mathrm{c_B} - \mathrm{c_A})x_0 x_1 = 0, \\ e_{23} &= \zeta(e_{12}), \;\; e_{31} = \zeta(e_{23}), \end{aligned} \tag{3}$$

where $\mathrm{c_A}$, $\mathrm{c_B}$, and $\mathrm{c_C}$ are shorthand for $\cos A$, $\cos B$, and $\cos C$, respectively. Now it is easily verified that the following holds:

Theorem 1.
1. The three equioptic circles of the excircles of generic triangle Δ have a common radical axis r and thus they have up to two common real points, i.e., the equioptic points of the excircles from which the excircles can be seen under equal angles.
2. The radical axis r contains X_4 (ortho center), X_9 (Mittenpunkt), X_{10} (Spieker center), and further X_i with

$$\begin{aligned} i \in \{&19, 40, 71, 169, 242, 281, 516, 573, \\ &966, 1276, 1277, 1512, 1542, 1544, \\ &1753, 1766, 1826, 1839, 1842, 1855, \\ &1861, 1869, 1890, 2183, 2270, 2333, \\ &2345, 2354, 2550, 2551, 3496, 3501\}. \end{aligned}$$

Proof. 1. Let $P_1 := \mu e_{12} + \nu e_{23}$, $P_2 := \mu e_{23} + \nu e_{31}$, and $P_3 := \mu\, e_{31} + \nu e_{12}$ be the equations of the conic sections in the pencils panned by any pair of equioptic circles e_{ij}. We compute the singular conic sections in the pencils and find that for all pencils the real singular conic sections consist of the ideal line $\omega : \;\; ax_0 + bx_1 + cx_2$ and the line

$$\sum_{cyclic} (b-c)\, \mathrm{c_A}\, x_0 = 0, \tag{4}$$

which is the radical axis of any pair of equioptic circles. 2. In [1, pp. 64 ff.] we find $X_4 = [\operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C]$ and $X_9 = [b{+}c{-}a, c{+}a{-}b, a{+}b{-}c]$ and obviously these coordinate vectors annihilate Eq. (4). By inserting the trilinears of the other points mentioned in the theorem we proof the incidence. The trilinears of points X_i with $i \leq 360$ can be found in [1] whereas the trilinears for $i > 360$ can be found in [2]. □

In Fig. 6 the equioptic circles as well as the equioptic points of an acute triangle are depicted. Fig. 4 shows some of the centers mentioned in Th. 1 located on r.

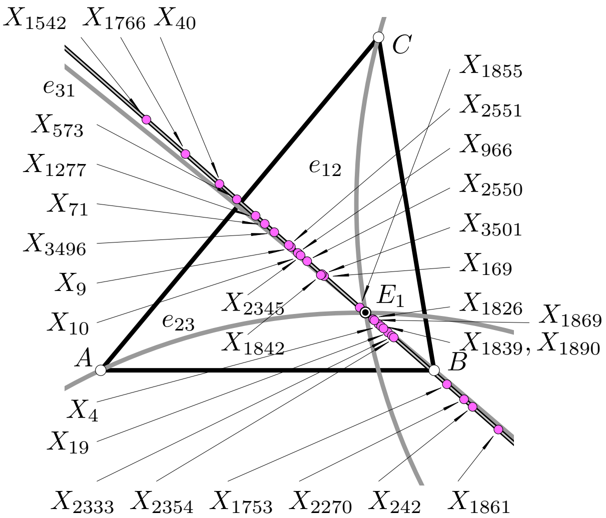

Figure 4: Some of the triangle centers on the radical axis, cf. Th. 1.

The circles in a pencil of circles can share two real points, one real point with multiplicity two, or no real points. Thus a triangle has either two equioptic points, or a single equioptic point.

Remark 2.
In case of an equilateral triangle Δ there is only one equioptic point E that coincides with the center of Δ. The three equioptic circle become straight lines: e_{ij} is the k-th interior angle bisector and the k-th altitude of Δ. Fig. 5 illustrates this case. From E any excircle Γ_j can be seen under $\arccos\left(-\frac{1}{8}\right) \approx 97.180756°$.

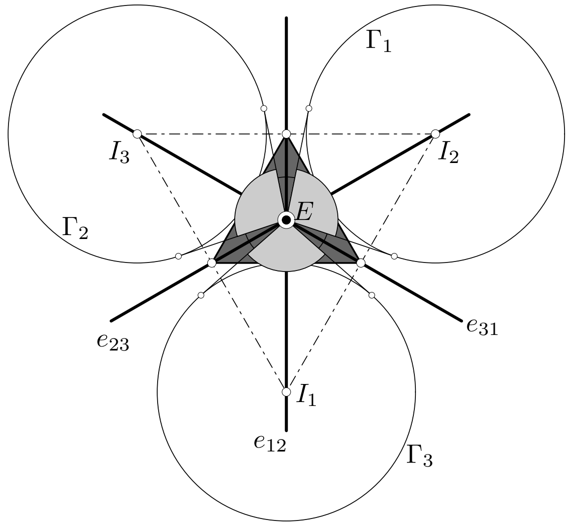

Figure 5: The only equioptic point of an equilateral triangle.

The case of a single equioptic point is illustrated in Fig. 8 at hand of an isosceles triangle. It is easily shown that in this case one has to choose $\angle ACB = 2\arcsin(\sqrt{3}-1) \approx 94.11719432°$ in order to have a unique equioptic point. Thus the triangle is obtuse. The unique (real) equioptic angle now equals $2\arcsin\left(\frac{3-\sqrt{3}}{2}\right) \approx 78.68794716°$.

A generic triangle Δ has a unique equioptic point if and only if

$$6a^2b^2c^2 + 4\sum_{cyclic} a^3b(b^2 - \widehat{a}c) =$$
$$= \sum_{cyclic} (a^6 + 2\widehat{a}a^5 - a^4(\widehat{a}^2 + 4bc)).$$

Here and in the following we use the abbreviations $\widehat{a} = b + c$, $\widehat{b} = c + a$, and $\widehat{c} = a + b$. We skip the proof since it is merely based on simple and straight foreward computations.

3 EQUIOPTIC CIRCLES OF THE INCIRCLE AND AN EXCIRCLE

We recall that the equioptic circles of a pair (Γ, Γ_i) is the Thales circle of the line segment bounded by the internal and external center of similitude of the incircle Γ and the i-th excircle

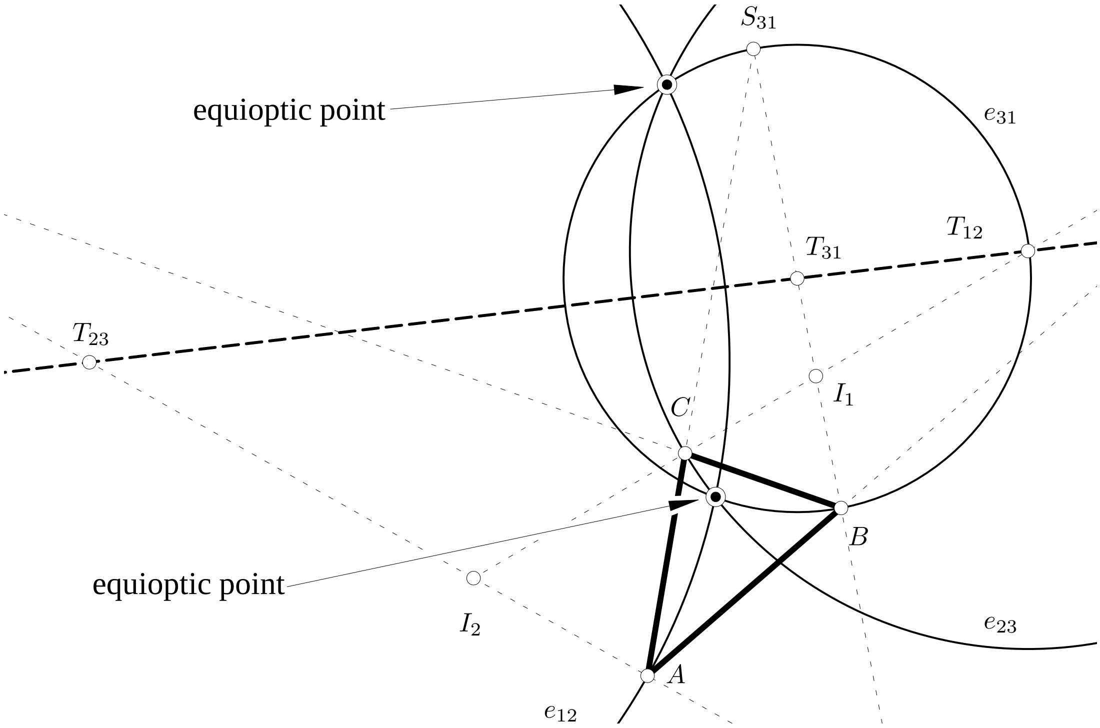

Figure 6: The equioptic circles and equioptic points of a triangle and the radical axis through some triangle centers.

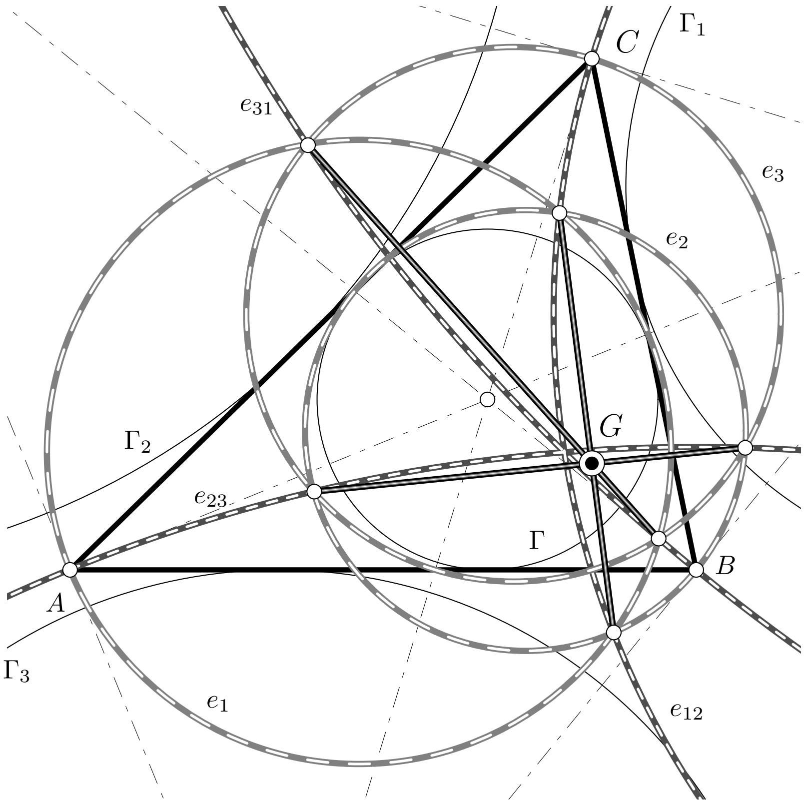

Figure 7: The six equioptic circles of the incircle and the excircle, the three concurrent radical axes, and the center G from Th. 3.

Γ_i. We observe that the i-th vertex of Δ is the external center of similitude of the above given pair of circles. The internal center is the meet of a common internal tangent, i.e, Δ's i-th side and the line $[I, I_i]$ connecting the respective centers.

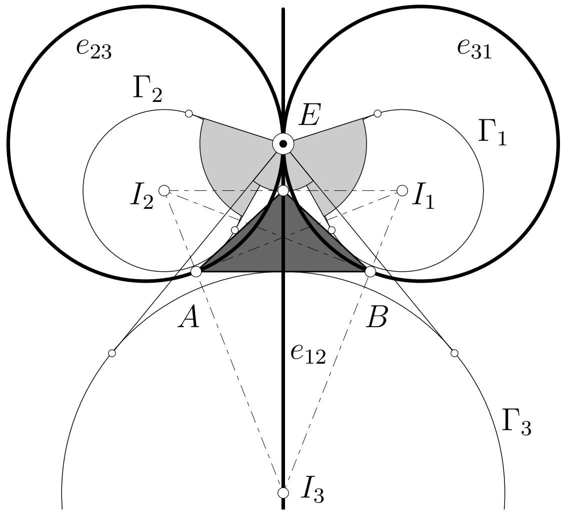

Figure 8: An iscosceles triangle with a unique equioptic point.

Consequently the internal centers of similitude are the points S_i ($i \in \{1, 2, 3\}$). We have

$$S_1 = (0 : 1 : 1), S_2 = S_1^\sigma, \quad S_3 = S_2^\sigma.$$

As a consequence of Cor. 1 we have:

Corollary 3.
The two internal centers S_i, S_j of similitude of Γ and Γ_i, Γ_j are collinear with the external center S_{ij} of similitude of Γ_i and Γ_j for $(i, j) \in \{(1, 2), (2, 3), (3, 1)\}$.

Proof. The collinearity is easily checked by showing the linear dependency of the respective coordinate vectors. □

The centers T_i of equioptic circles e_i of Γ and Γ_i are the midpoints of Δ's i-th vertex and S_i. Thus we have

$$T_1 = (b + c : a : a), \quad T_{i+1} = T_i^{\sigma\zeta}. \tag{5}$$

Now we observe the following:

Corollary 4.
The two centers T_i and T_j of equioptic circles e_i, e_j of Γ and the i-th and j-th excircle are collinear with the center T_{ij} of the equioptic circle e_{ij} of Γ_i and Γ_j.

Proof. We simply show dependencies of vectors given in Eqs. (1) and (5). □

Again we use the formulae given in [1, p. 223] in order to compute the equations of the equioptic circles e_i of the incircle Γ and the i-th excircle Γ_i. Note that these homogeneous equations can always be written in the form $e_i : x^{\mathrm{T}} \cdot A_i \cdot x = 0$ with a regular and symmetric 3×3-matrix A_i. The matrix A_1 reads

$$\begin{bmatrix} 2a^2s(a-s) & 2abs(a-s) & 2acs(a-s) \\ 2abs(a-s) & (\star\star) & (\star) \\ 2acs(a-s) & (\star) & (\star\star\star) \end{bmatrix}, \tag{6}$$
$$A_2 = \zeta(A_1), \quad A_3 = \zeta(A_2),$$

where $(\star) = 2b^2c^2 + 3b^3c + 3bc^3 - 3a^2bc + 2b^4 + 2c^4 - 2a^2b^2 - 2a^2c^2$, $(\star\star) = b(\widehat{a}^2(4c - b) + a^2b - 8b^2c)$, and $(\star\star\star) = c(2bc^2 + 7b^2c - c^3 + a^2c + 4b^3)$. Here $s = (a + b + c)/2$ is the halfperimeter of Δ.

Theorem 2.
The equioptic circles e_i, e_j, and e_{ij} defined by the incircle Γ and the excircles Γ_i, Γ_j have a common radical axis r_k (with $(i, j, k) \in \mathbb{I}^3$) and thus Γ, Γ_i, and Γ_j have up to two real equioptic points.

Proof. With Eq. (3) and (6) we compute the radical axis r_k of Γ, Γ_i, and Γ_j (where $(i, j, k) \in \mathbb{I}^3$) as the singular conic sections in the pencil of conics spanned by either two circles, cf. the proof of Th. 1. The radical axis r_3 is given by

$$\begin{aligned} r_3 = [&-ba^5 - (\widehat{a}^2 + 2bc)a^4 + (\widehat{a}^2b + c(2\widehat{a}^2 - bc))a^3 \\ &+\widehat{a}^2(\widehat{a}^2 + 6c^2)a^2 + c^2\widehat{a}^2(b + 4c) : \\ &: ab^5 + 2(\widehat{b}^2 + 2ac)b^4 - (a\widehat{b}^2 + c(2\widehat{b}^2 - ac))b^3 \\ &-\widehat{b}^2(\widehat{b}^2 + 6c^2)b^2 - c^2\widehat{b}^2(a + 4c)\widehat{b} : \\ &: (b - a)c^5 + 4(a - b)\widehat{c}c^4 + (a - b)(7\widehat{c}^3 - 3ab)c^3 \\ &+4(a - b)\widehat{c}(\widehat{c}^2 + ab)c^2 \\ &+7ab(a - b)\widehat{c}^2c + 4a^2b^2(a - b)\widehat{c})]. \end{aligned} \tag{7}$$

Finally we have $r_1 = r_3^{\sigma\zeta}$ and $r_2 = r_1^{\sigma\zeta}$. □

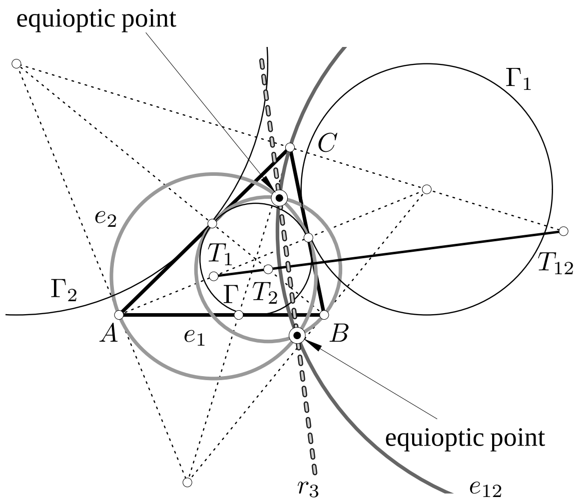

Figure 9: Equioptic circles and points of Γ, Γ_1, Γ_2.

A certain triplet of equioptic circles is shown in Fig. 9. Finally we have:

Theorem 3.
The three radical axes r_k (cf. Th. 2) are concurrent in a new triangle center.

Proof. The homogeneous coordinate vectors of the lines r_i given in (7) are linearly dependent. This proves the concurrency.

We compute the meet $G = (g_0 : g_1 : g_2)$ of any pair (r_i, r_j) of radical axes and find

$$\begin{aligned} g_0 = {} & bc\widehat{a}^5(b-c)^2 + 2bc\widehat{a}^2(2\widehat{a}^4 - 10\widehat{a}^2bc + 5b^2c^2)a \\ & +\widehat{a}^3(\widehat{a}^4 - 8\widehat{a}^2bc + 4b^2c^2)a^2 \\ & -2(\widehat{a}^6 + 3bc\widehat{a}^4 - 10b^2c^2\widehat{a}^2 + b^3c^3)a \\ & -\widehat{a}(8\widehat{a}^4 - 23bc\widehat{a}^2 + 4b^2c^2)a^4 \\ & -2(\widehat{a}^4 - 8bc\widehat{a}^2 + 5b^2c^2)a^5 \\ & +\widehat{a}(7\widehat{a}^2 - 4bc)a^6 + 2(2\widehat{a}^2 - bc)a^7. \end{aligned} \quad (8)$$

Since $g_1 = \zeta(g_0)$ and $g_2 = \zeta(g_1)$ we find that G is a center of Δ which is not mentioned in [2] and shown in Fig. 7. $\square$

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