The Geometry of Flags

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Abstract

We present a low dimensional point model for the set of flags in $\mathbb{R}^3$ which is contained in a six-dimensional algebraic variety M_8^6 of degree eight embedded in a nine-dimensional projective space. It turns out that M_8^6 can be parameterized rationally. We study the geometry of the manifold and show the relations between the geometry of flags and kinematics in Euclidean $\mathbb{R}^3$ and Non-Euclidean geometries.

1 Introduction

The study of higher dimensional algebraic (and other) manifolds has a long history. Grassmann, Segre, and Veronese varieties as well as Schubert manifolds have been a field of intensive research [3, 4, 5, 10]. The major benefits of geometries of this type are the following: (1) They allow to treat objects consisting of a collection of elements of projective spaces as points in some (of course higher dimensional) projective space. (2) The transformations of these objects can be represented by collineations (i.e. linear transformations) in the model space. The obvious disadvantage (in view of applications) of these models is their relatively high dimensional model space. Computations can become long winded and costly.

Nowadays point models for some special manifolds appear frequently in application areas: The Klein model of line space as well as other models of line space can be used for approximation and interpolation problems in line space [13, 14]. Even a new model for the set of line elements in Euclidean three-space was developed in [12] for the recognition and reconstruction of spiral surfaces, see [6].

In the following we study the manifold of flags in Euclidean three-space $\mathbb{R}^3$. The investigations are not done for their own sake. We show a tricky way to parameterize this manifold by means of rational functions. This parameterization method also applies to other parameterization problems. Further we discuss the geometry of this six-dimensional manifold in order to get insight and understand it. Applications of this are not well studied until now. Though motion planning by means of subdivision motions on this manifold could be of future interest.

2 Equation of the Manifold M_8^6

2.1 Lines in space

Since we are dealing with Euclidean three-space we use Cartesian coordinates $p = (p_1, p_2, p_3)$ in order to represent points P. A line L in Euclidean three space will be described by normalized Plücker coordinates. Assume that L is parallel to the unit vector $l \neq 0$ (i.e. its Euclidean length $\|l\|$ equals 1) and passes through the point P with coordinate vector p. Then we define the *normalized* Plücker of L coordinates as

$$L = (l, \bar{l}) = (l_1, l_2, l_3; l_4, l_5, l_6), \tag{1}$$

where $\bar{l} = p \times l$ is the *momentum vector* of L. Here and in the following the cross product of vectors in $\mathbb{R}^3$ is denoted by $\times$. The coordinates of L do not depend on the choice of P on L.

With $\langle ., . \rangle$ we denote the standard scalar product of vectors in $\mathbb{R}^3$. It is obvious that

$$M_2^4 : \quad \langle l, \bar{l} \rangle = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0 \tag{2}$$

holds.

If we drop the normalization of l the coordinates l_i can be considered as homogeneous coordinates of points in a projective five-space $\mathbb{P}^5$. M_2^4 is the Klein quadric or Plücker quadric (i.e. the Grassmannian G_1^3). It is a point model for the set of lines in three-space, see [3, 7, 13]. M_2^4 is a regular quadric carrying two three-parameter families of planes corresponding to bundles and fields of lines.

2.2 Line elements in space

The pair (P, L) consisting of a line L and a point P on it will be called *line element*. In order to describe line elements we assign coordinates to them in the following way (cf. [6, 12]): The line L is described by its normalized Plücker coordinates $(l, \bar{l})$. In order to fix the point P on L we add a seventh coordinate $\lambda := \langle p, l \rangle$ to the Plücker coordinates of L and let

$$(P, L) = (l, \bar{l}, \lambda) = (l_1, l_2, l_3; l_4, l_5, l_6; l_7) \tag{3}$$

be the *coordinates of the line element* (P, L).

The point P can be recovered from the line element coordinates by $p = l \times \overline{l} + \lambda l$. Note that this is true because $\|l\| = 1$, which is yet assumed. The line element coordinates of (P, L) thus satisfy (2).

Again we can drop the normalization condition of l. Then we can consider the seven coordinates of line elements as homogeneous coordinates of points in a projective six-space $\mathbb{P}^6$. Since l_7 does not appear in Eq. (2), we see that Eq. (2) (interpreted in $\mathbb{P}^6$) is the Equation of a quadratic cone M_2^5 whose points correspond to line elements in Euclidean three-space. M_2^5 has a point for its vertex and two three-parameter families of three-dimensional generators. The points contained in the generator $l_1 = l_2 = l_3 = 0$ do not correspond to line elements in Euclidean $\mathbb{R}^3$.

2.3 Flags in space

We follow [10] and define:

Definition 2.1 *A flag $\mathcal{F}$ in Euclidean three-space $\mathbb{R}^3$ is a triplet (P, L, E) where P is a point, L is a line, and E is a plane and $P \in L \subset E$.*

Obviously a flag $\mathcal{F} = (P, L, E)$ consists of a line element (P, L) and a plane carrying it. Assume that the unit vector $\widehat{l}$ is perpendicular to the plane E. In order to assign coordinates to the flag $\mathcal{F}$ we add the vector $\widehat{l}$ to the coordinates of the line element (P, L). So we define

$$(P, L, E) = (l, \overline{l}, \widehat{l}, \lambda) = (l_1, l_2, l_3; l_4, l_5, l_6; l_7, l_8, l_9; l_{10}) \quad (4)$$

as *coordinates of the flag* $\mathcal{F} = (P, L, E)$.

The coordinates of $\mathcal{F}$ satisfy the following conditions:

$$\begin{aligned} \langle l, \overline{l} \rangle &= 0 = l_1 l_4 + l_2 l_5 + l_3 l_6, && (5)\\ \langle l, \widehat{l} \rangle &= 0 = l_1 l_7 + l_2 l_8 + l_3 l_9, && (6)\\ \langle l, l \rangle - \langle \widehat{l}, \widehat{l} \rangle &= 0 = l_1^2 + l_2^2 + l_3^2 - l_7^2 - l_8^2 - l_9^2. && (7) \end{aligned}$$

Any vector $(l_1, \ldots, l_{10})$ satisfying Eqs. (5), (6), and (7) defines (up to orientations) a unique flag $\mathcal{F}$ in $\mathbb{R}^3$. The point P, the line L, and the plane E can be recovered from the coordinates of $\mathcal{F}$ by $p = l \times \overline{l} + \lambda l$, $L = (l, \overline{l})$, and $E : \langle \widehat{l}, x \rangle = det(l, \overline{l}, \widehat{l})$.

3 Some Facts on M_8^6

Now we can drop the norming conditions on l and $\widehat{l}$, respectively. We only assume that they are of equal length, i.e. $\|l\| = \|\widehat{l}\|$. We observe that the coordinates of $\mathcal{F}$ are homogeneous in the following sense: If we scale l and $\widehat{l}$ by a real non-vanishing factor, say c, we find the momentum vector changes to $c\overline{l}$ and λ changes to $c\lambda$. This makes it possible to interpret the coordinates l_i of $\mathcal{F}$ as homogeneous coordinates of points in a projective space $\mathbb{P}^9$ of dimension nine. Eqs. (5), (6), and (7) define an algebraic variety M_8^6 in $\mathbb{P}^9$. We have:

Theorem 3.1 *The algebraic degree and the dimension of M_8^6 equal eight and six, respectively.*

Proof: The dimension d of M_8^6 equals six, which is clear since we need six parameters in order to fix a flag in $\mathbb{R}^3$.

The algebraic degree follows from the Hilbert-Polynomial (for definition and properties see [15])

$$H(t) = \frac{1}{90} t^6 + o(t^6)$$

since $\deg M_8^6 = d! c_6 = 8$, where c_6 is the coefficient of the leading monomial. □

Suprisingly we find the following result:

Theorem 3.2 *The manifold M_8^6 allows a rational parameterization.*

Proof: We construct a rational parameterization Φ of M_8^6. We let P be represented by $p = (u_3, u_4, u_5) \in \mathbb{R}^3$. Further we assume that $l = (2u_1, 2u_2, 1 - u_1^2 - u_2^2)/M$, where $M = 1 + u_1^2 + u_2^2$. The momentum vector is given by $p \times l$ according to its definition.

Since l is a rational isothermal parameterization of the Euclidean unit sphere S^2 we find that the partial derivatives are of equal length and orthogonal to each other, i.e. we have $\|l_{,1}\| = \|l_{,2}\| = 2M^{-1}$ and $\langle l_{,1}, l_{,2} \rangle = 0$ and $\{l_{,1}, l_{,2}, l\}$ is an orthogonal frame.

Eqs. (5), (6), and (7) tell us that $\widehat{l}$ is orthogonal to l and thus we can write $\widehat{l} = l_{,1}/\|l_{,1}\| c_3 + l_{,2}/\|l_{,2}\| s_3$. We substitute $c_3 = (1 - u_3^2)/N$ and $s_3 = 2u_3/N$ with $N = 1 + u_3^2$ which guarantees that $\|l\| = \|\widehat{l}\|$ and find

$$\begin{aligned} MN\Phi(u_1, u_2, u_3, u_4, u_5, u_6) = [&2u_1 N, 2u_2 N, NS;\\ &N(u_5 S - 2u_2 u_6), N(2u_1 u_6 - u_4 S),\\ &2N(u_2 u_4 - u_1 u_5);\\ &(1 - u_3^2)(M - 2u_1^2) - 4u_1 u_2 u_3,\\ &2u_3(M - 2u_2^2) - 2(1 - u_3^2) u_1 u_2,\\ &-2u_1(1 - u_3^2) - 4u_2 u_3;\\ &N(2u_1 u_4 + 2u_2 u_5 + u_6 S)], \end{aligned} \quad (8)$$

where $S = 1 - u_1^2 - u_2^2$. □

We note that $(u_1, \ldots, u_6)$ is an affine parameter and (8) does not reach the entire surface M_8^6.

4 Geometric Properties of M_8^6

In Eqs. (5), (6), and (7) the coordinate l_{10} does not show up. Thus we have:

Theorem 4.1 *The manifold M_8^6 is a cone with the thenth base point of the standard projective frame for its vertex.*

From Eqs. (5), (6), and (7) we conclude that M_8^6 is the intersection of three quadratic cones Δ_i. At least from the viewpoint of projective geometry (over the real number field) these three cones do not differ: One can find collineations of $\mathbb{P}^9$ (i.e. linear mappings in $\mathbb{R}^{10}$) transforming one cone Δ_i into another cone Δ_j.

The cone Δ_1 has the Klein quadric M_2^4 contained in the subspace $B_1 : \; x_7 = x_8 = x_9 = x_{10} = 0$ for a director quadric. Its vertex is the three-dimensional subspace $V_1 : \; x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$. Since M_2^4 carries two three-parameter families of planes Δ_1 has two three-parameter families of six-dimensional projective subspaces for its generators. The cones Δ_2 and Δ_3 have similar properties. The vertex space of Δ_1 is entirely contained in Δ_2 and vice versa.

We use the following definition:

Definition 4.1 *The set of flags sharing two components is called pencil of flags.*

We can distiguish between three types of pencils of flags: (1) flags with fixed line and plane component, (2) flags with fixed point and plane component, and (3) flags with fixed point and line component (see Fig. 1).

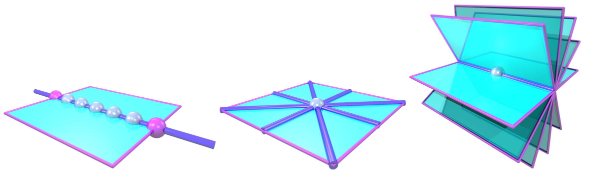

Figure 1: Flags sharing two components.

The three types of pencils of flags correspond to certain subspaces in M_8^6:

Theorem 4.2 *Pencils of flags correspond to lines in M_8^6.*

Proof: Parameterizing these pencils and computing their flag coordinates leads to linear parameterizations of the one-dimensional subspaces in M_8^6 corresponding to the pencils. □

There are other simple manifolds of flags:

Definition 4.2 *The set of flags sharing one component is called bundle of flags.*

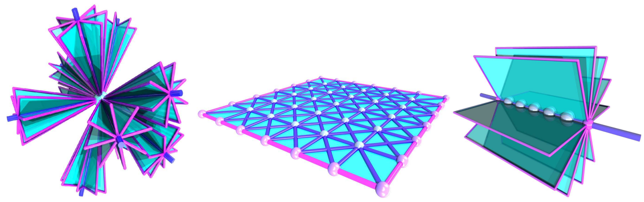

Figure 2: Flags sharing only one component.

We find three different types of bundles of flags (see Fig. 2): (1) flags with a common point, (2) flags with a common plane, and (3) flags with a common line. Unfortunately only one type of these corresponds to projective subspaces in M_8^6. We have:

Theorem 4.3 *The bundels of flags with a common line component correspond to planes in M_8^6.*

Proof: The proof is as simple as in the case of pencils. □

5 Euclidean Kinematics and Non-Euclidean Geometries

In this section we finish the discussion of bundels of flags and corresponding subspaces. For that we look closer to the definition of flag coordinates. We recall that the vectors l and $\widehat{l}$ were unit vectors in the beginning. Consequently the line L (and thus the line element) and the plane E are oriented. Therefore a flag can be oriented in four different ways depending on the respective choices of the orientations of l and $\widehat{l}$.

As is clear from the definition the triplet $\{l, \widehat{l}, l \times \widehat{l}\}$ is a Cartesian frame attached to the flag $\mathcal{F} = (l, \bar{l}, \widehat{l}, \lambda)$. Assume an orientation is fixed and a certain proto flag $\mathcal{F}_0$ (system of reference, referred to as the fixed system) is chosen. Then there exits a unique Euclidean motion transforming $\mathcal{F}_0$ into $\mathcal{F}$. Thus we have:

Theorem 5.1 *There is a bijective correspondence between the set of oriented flags in Euclidean three-space and the set of Euclidean motions.*

Using the calculus of dual quaternions (see [7, 8]) one finds that Euclidean motions can be mapped to points on a certain quadric $S_2^6 \subset \mathbb{P}^7$, the so called Study quadric, see [2, 7, 16]. By identifying Euclidean motions and oriented flags we find:

Theorem 5.2 *There is a one-to-one correspondence between the set of oriented flags in Euclidean three-space and the points of the* Study *quadric S_2^6.*

The Study quadric is a hyper surface in $\mathbb{P}^7$ and it is not possible to find a lower dimensional point model for the set of (oriented) flags in Euclidean three-space.

Note that the geometric object $\mathcal{F} = (P, L, E)$ carries four different orientations. These correspond to four Euclidean motions transforming the proto flag into $\mathcal{F}$. Thus the manifold of Euclidean motions is covered four times by the set of flags (without orientations) in Euclidean three space.

The bundles of flags with fixed plane or point component are closely related to Non-Euclidean geometries:

Theorem 5.3
(1) The set of oriented flags through a fixed point form an elliptic three-space.
(2) The set of flags with a fixed plane component form a quasi elliptic three-space.

Proof: (1) Fix one oriented flag in the bundle, it then serves as proto flag $\mathcal{F}_0$. The Euclidean motions transforming $\mathcal{F}_0$ into any other flag $\mathcal{F}$ in the bundle are rotations about the common point. It is well known that the rotations about one fixed point form an elliptic three space, see [1, 4, 9].
(2) Fix an oriented line element (P_0, L_0) in the plane E as proto element. Now there exists a unique planar Euclidean motion transforming the proto element in a certain oriented line element (P, L). So any oriented line element in E can be identyfied with a certain planar Euclidean motion, which can be mapped to exactly one point of a quasi-elliptic three-space (via the Blaschke-Grünwald mapping), see [4, 7, 13]. □

Note that the elliptic three-space as well as the quasi-elliptic three-space appearing in the above theorem are covered more than once.

6 Final Remarks

The manifold of oriented flags in Euclidean $\mathbb{R}^3$ which can also be seen as the Euclidean motion group can be used for motion planning. Several algorithms for that are developed. Most of them use orthogonal projection onto the group of motions. Subdivision schemes are not restricted to polygons/polyhedra in Euclidean spaces they can also be applied in any Riemannian manifold such as the Euclidean motion group, see [17]. A manifold with explicitly known rational parameterization of relatively low degree may support subdivion algorithms.

The construction of the rational parameterization of M_8^6 given in (8) does not use an algorithm and is mainly discovered by close inspection. A similar approach works for another algebraic manifold M_5^5 which serves as a point model for the set of line elements in projective space $\mathbb{P}^3$ [11].

It would be of interest to find techniques or algorithms for the construction of low degree parameterizations of such manifolds. Maybe the study of geometric properties and a deeper insight in projective generations of such manifolds can help to find appropriate algorithms.

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