A special family of triangles

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Abstract

We study the one-parameter family of triangles that emerges if one side line traces a pencil of lines and the opposite angle is fixed. A description of the traces of triangle centers and the pair of Brocard points in terms of parametrizations and equations is given. The envelopes of the families of circumcircles and nine-point circles are determined. Our approach even allows us to consider and treat the triangle family as a two-parameter family of triangles, *i.e.*, the (interior) angle opposite to the pencil, which is in the beginning fixed, may also change.

Keywords: family of triangles; triangle center; Brocard points; quartic
Subject Classification AMS 2020:

1 Introduction

1.1 Related and prior work

Triangle geometry is rich with elegant results and deep relationships between points, lines, angles, circles and related curves. The study of triangles and their families even nowadays attracts many geometers. Some results in this area, especially for the Euclidean plane can be found in [1, 2, 8, 9, 10, 12] while [3, 4, 5] deal with the situation in the isotropic plane.

1.2 Contributions and aims of the present paper

We intend to approach some results of this particular family of triangles in an analytical way. This is because of two reasons: 1. A synthetic approach is already given in [12]. There is no reason to repeat this and nothing can be added. 2. The synthetic approach is limited, though very elegant, and gives some geometric insight into the problem. We shall omit the discussion of traces and loci of midpoints of changing segments, envelopes of bisectors of angles and segments and all other objects that are not "central" in the sense of [6, 7]. There is only one exception: In Sec. 3.2, we shall have a look at the bicentric pair of Brocard points. They are closely related to some important triangle centers. Further, we determine and discuss the envelopes of special central circles. The envelope of the Euler line appears to be rather unspectacular and is, therefore, not discussed. The envelopes of circles can be determined much easier in a suitable analytic approach than with the synthetic approach.

In Sec. 2, we shall build the analytical environment, *i.e.*, we introduce coordinates in a way that we are able to get through the computations in the present paper. This allows us to parametrize the triangle family under consideration, and further, enables us to give the first results. In particular, we can show that all centers (and points) on the Euler line forming a fixed affine ratio with the centroid and the circumcenter move on hyperbolae. In Sec. 3, the envelopes of the families of circumcircles and and nine-point circles are determined. Then, we move over to the Brocard points and some triangle centers related to them. Finally, in Sec. 4, we shall give the equations of the traces some more triangle centers. As can be expected, some triangle centers run on quartic curves, some on curves of much higher degree. This seems to depend on the algebraic complexity of the construction of the respective centers. Sec. 5 poses open questions and gives hints towards future work.

2 Analytical framework

We assume that the vertex A of the triangle $\Delta = ABC$ coincides with the origin of a Cartesian coordinate system. The line $[A,B]$ shall be the x-axis of the frame and the line $[C,A]$ (which encloses the angle $0<\alpha<\pi$ with the x-axis) is given by the equation $x\sin\alpha - y\cos\alpha = 0$ as indicated in Fig. 2 (left).

The pencil of lines carrying the third side of Δ shall be centered at $P=(\xi,\eta)$ with $\eta\neq 0$ (*i.e.*, $P\notin[A,B]$) and $\xi\sin\alpha-\eta\cos\alpha\neq 0$ (*i.e.*, $P\notin[C,A]$).

Now, we assume that $\phi\in\mathbb{R}$ is a coordinate in the pencil of lines about P and $(\cos\phi,\sin\phi)\in \mathrm{S}^1$ is a unit normal vector of the side line $[B,C]\ni P$. Hence, an equation of $[B,C]$ is given by $x\cos\phi+y\sin\phi=d$, where $d:=\xi\cos\phi+\eta\sin\phi$ is the support function of $[B,C]$.

The remaining vertices of the triangle Δ are then found as the intersection of $[B,C]$ with $[A,B]$ and $[C,A]$. So, the three vertices of Δ are parametrized by

$$A=(0,0),\quad B=\frac{d}{\cos\phi}(1,0)\,,\quad C=\frac{d}{\cos(\alpha-\phi)}(\cos\alpha,\sin\alpha)\,. \tag{1}$$

This describes a one-parameter family $\mathcal{T}$ of triangles. Allowing further α to trace S^1, we have parametrized a two-parameter family of triangles.

In principle, the parameter ϕ is allowed to trace S^1 freely. However, $\phi=\frac{\pi}{2},\frac{3\pi}{2}$ result in an open triangle AB_1C_1 (cf. Fig. 2, right), the vertex B_1 is at infinity and $[B,C]$

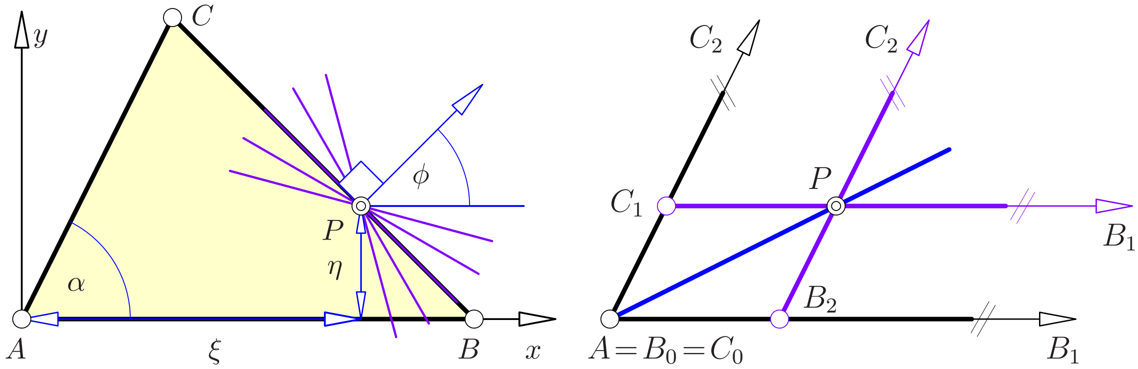

Figure 1: Coordinate frame and geometric meaning of parameters (left), degenerate triangles in the pencil (right).

is either anti-parallel to $[A, B]$ (if $\phi = \frac{\pi}{2}$) or parallel to $[A, B]$ (if $\phi = \frac{3}{2}\pi$). If $\phi = \arctan\frac{\eta}{\xi}, \arctan\frac{\eta}{\xi} + \pi$, the line carrying $[B_0, C_0]$ passes through A and the corresponding two triangles $\Delta_0 = AB_0C_0$ are point-shaped (see also Fig. 2). Finally, if $\phi = \alpha + \frac{\pi}{2}, \alpha + \frac{3\pi}{2}$, we get the second pair of open triangles AB_2C_2 (cf. Fig. 2).

All loci of points (and especially centers) related to the triangles in the family $\mathcal{T}$ are traced twice, since $\Delta(\phi) = AB(\phi)C(\phi) = \Delta(\phi + \pi) = AB(\phi + \pi)C(\phi + \pi)$ agree as congruent triangles with differently oriented side line $[B, C]$.

Now, the analytical representation (1) allows us to formulate:

Theorem 2.1. *The centroid X_2, the circumcenter X_3, and the orthocenter X_4 of Δ run on hyperbolae, while $[B, C]$ traverses the pencil about P.*

Proof. A parametrization of the centroid X_2 in terms of the underlying Cartesian coordinates is obtained as the arithmetic mean of the coordinate vectors (1) of Δ's vertices. This yields

$$X_2(\phi) = \frac{d}{3\cos\phi\cos(\alpha - \phi)} (2\cos\alpha\cos\phi + \sin\alpha\sin\phi, \sin\alpha\cos\phi) \tag{2}$$

which, after implicitization, *i.e.*, after the elimination of the parameter ϕ results in

$$\mathcal{H}_2 : \; 3xy\sin\alpha - 3y^2\cos\alpha + (2\eta\cos\alpha - \xi\sin\alpha)y - x\eta\sin\alpha = 0.$$

It is rather elementary to check that $\mathcal{H}_2$ is a hyperbola (for admissible α, ξ, and η), see [13, 11]. The asymptotes of $\mathcal{H}_2$ are parallel to $[A, B]$ and $[C, A]$.

It is elementary to determine the circumcenter X_3 of all triangles in the family. Along the same elementary and constructive way, we obtain

$$X_3(\phi) = \frac{d}{2\cos\phi}\left(1, \frac{\sin(\alpha - \phi)}{\cos(\alpha - \phi)}\right), \tag{3}$$

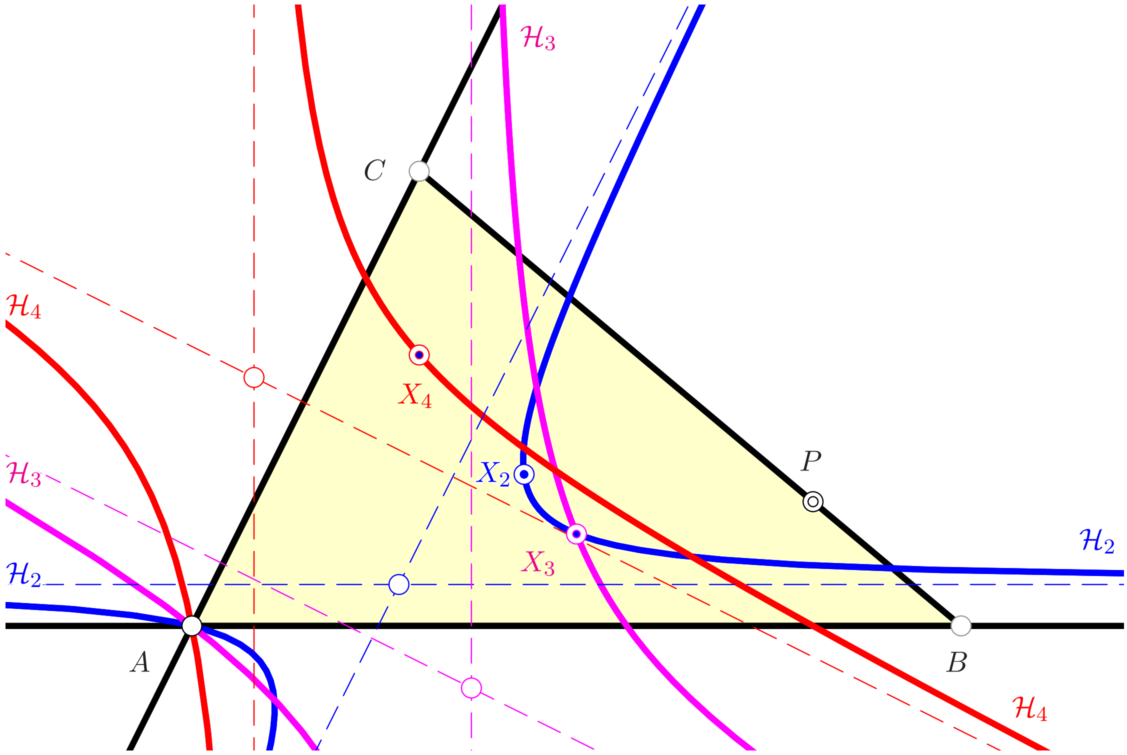

Figure 2: The hyperbolae $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$ as the orbits of the centroid X_2, the circumcenter X_3, and the orthocenter X_4 of the triangles in the pencil of triangles.

and further, by eliminating ϕ, an implicit equation

$$\mathcal{H}_3: \; 2x^2\cos\alpha + 2xy\sin\alpha - (\xi\cos\alpha + \eta\sin\alpha)x - (\xi\sin\alpha - \eta\cos\alpha)y = 0.$$

of a hyperbola (for admissible α, ξ, and η). The conical locus of all circumcenters passes through the ideal points of the lines orthogonal to $[A, B]$ and $[C, A]$, and thus, the asymptotes are orthogonal to lines $[A, B]$ and $[C, A]$. The hyperbola $\mathcal{H}_3$ is centered at

$$\frac{1}{2\sin^2\alpha}\left(\sin\alpha(\xi\sin\alpha - \eta\cos\alpha), \eta(1+\cos^2\alpha) - \xi\cos\alpha\sin\alpha\right). \tag{4}$$

In order to complete the proof, we determine the orthocenter X_4 of Δ and find

$$X_4(\phi) = \frac{d\cos\alpha}{\cos\phi\cos(\alpha-\phi)}\left(\cos\phi, \sin\phi\right),$$

a parametrization annihilating the equation

$$\mathcal{H}_4: \; x^2\cos\alpha + xy\sin\alpha - \eta y\cos\alpha - \xi x\cos\alpha = 0.$$

$\mathcal{H}_4$ describes a hyperbola centered at

$$\left(\eta\frac{\cos\alpha}{\sin\alpha}, \xi\frac{\cos\alpha}{\sin\alpha} - \eta\frac{\cos^2\alpha}{\sin^2\alpha}\right), \tag{5}$$

and passing through the ideal points of the normals to the fixed side lines of the triangles in the family. □

We shall also note that the centers (4) of the hyperbolae housing the circumcenters of the triangles in the family trace the parabola

$$8x^2 - 6\xi x - 2y\eta + \xi^2 + \eta^2 = 0$$

if the triangles' interior angle α traces S^1. The centers (5) of the hyperbolae generated by the orthocenters run on the parabola

$$2x^2 - \xi x + \eta y = 0.$$

The parabolae as loci of centers of the hyperbolic orbits fit well with a much larger concept. So far, we have considered the orbits of three centers which are collinear, *i.e.*, they lie on the Euler line. In this respect, we can show the following result:

Theorem 2.2. *Any but two points on the Euler line moves on hyperbolae while the moving triangle side traverses its pencil. The exceptional points move on the interior and exterior angle bisector through A and deliver the only degenerate conical loci of points on the Euler line. The centers of the hyperbolae move on a parabola if the angle α at A traverses S^1.*

Proof. The parametrizations (2) and (3) of the orbits of X_2 and X_3 can be used to parametrize the range of points on the Euler line $\mathcal{L}_{2,3}$ and we have

$$\mathcal{L}_{2,3}(w) = X_2(1-w) + X_3 w, \quad w \in \mathbb{R}. \tag{6}$$

We eliminate the parameter ϕ in the pencil of lines about P. For the sake of simplicity, we replace the trigonometric functions of α by their rational equivalents:

$$\cos\alpha = \frac{1-a^2}{1+a^2}, \quad \sin\alpha = \frac{2a}{1+a^2}.$$

Then, we find the quadratic equation of the orbits of the points on $\mathcal{L}_{2,3}(w)$:

$$\begin{gathered} 6\Big((w(3a^4-2a^2+3)+8a^2)x+(4aw(1-a^2)+4a(a^2-1))y\Big)\Big(3(a^2-1)wx-2a(w+2)y\Big) \\ +(w(3a^2-1)+4)(w(a^2-3)-4a^2)\big(3w(a^2-1)\xi-2a(w+2)\eta\big)x \\ -\big(2a(w+2)\xi - (w(a^2-1)-4(a^2-1))\eta\big)y = 0 \end{gathered}$$

This is the equation of a hyperbola, since it shares the ideal points with the lines

$$\begin{gathered} (w(3a^4-2a^2+3)+8a^2)x + (4wa(1-a^2)+4a(a^2-1))y = 0, \\ 3w(a^2-1)x - 2a(w+2)y = 0. \end{gathered}$$

The hyperbolae degenerate if

$$w \in \left\{ \frac{4}{1-3a^2}, \frac{4a^2}{a^2-3} \right\}$$

and become either the repeated line $ay + x = 0$ or the repeated line $ax - y = 0$. Since $a = \tan\frac{\alpha}{2}$, these repeated lines are the interior and exterior angle bisector of Δ at A.

The centers of the hyperbolae depending on the angle α are given by

$$\frac{1}{12a^2}\left(a(w(3a^2\eta+2a\xi-3\eta)+4a\xi), w(3a(a^2-1)\xi+(3a^4-4a^2+3)\eta)+4a^2\eta\right),$$

which clearly shows that for a fixed angle α (a family of triangles with a common angle at A), the centers of the orbits of points $\mathcal{L}_{2,3}(w)$ trace a straight line. However, we aim at the description of the locus of the orbits of the centers for varying α. For that purpose, we eliminate α (or a) and obtain the parabolae

$$72x^2 - 6\xi(w+8)x - 18w\eta y + 3w^2\eta^2 - w^2\xi^2 + 6w\eta^2 + 2w\xi^2 + 8\xi^2 = 0$$

that touch the ideal line in the ideal point of y-axis. □

3 Some envelopes

3.1 The one-parameter family of circumcircles

We have already found an analytic representation (cf. (3)) of the circumcenters of the triangles in the family $\mathcal{T}$. An equation of the one-parameter family of circumcircles of the triangles in $\mathcal{T}$ can be obtained, since the radius function equals $R = \overline{X_3A}$. The circumcircles have the equations

$$\mathcal{U}: \quad \cos\phi\cos(\alpha-\phi)(x^2+y^2) - d\cos(\alpha-\phi)x - d\sin(\alpha-\phi)x = 0, \tag{7}$$

where $d = d(\phi)$ is the support function of the line $[B, C]$ which still depends on ϕ (a fact that should be taken into account when it comes to the computation of the envelope).

The envelope of the circles (7) is now found by first differentiating $\mathcal{U}$ with respect to ϕ and the subsequent elimination of ϕ from both $\mathcal{U}$ and $\partial\mathcal{U}/\partial\phi$. The elimination is simplified by replacing $\cos\phi$ and $\sin\phi$ by their rational equivalents. Besides some constant factors, the resultant of $\mathcal{U}$ and $\partial\mathcal{U}/\partial\phi$ contains the factors

$$(x^2+y^2)^2, \quad ((x-\xi)^2+(y-\eta)^2)^2,$$

which can be canceled, for they describe two pairs of isotropic lines (of Euclidean Geometry, cf. [13, p. 253]). Any isotropic line splits off with multiplicity two from the envelope.

The essential part of the resultant yields the equation

$$\begin{gathered}\mathcal{E}_{\mathcal{U}}: \quad \sin^2\alpha(x^2+y^2)^2 \\ -2(x^2+y^2)\left(\sin\alpha(\xi\sin\alpha-\eta\cos\alpha)x+(\eta(1+\cos^2\alpha)-\xi\sin\alpha\cos\alpha)y\right) \\ +((\eta\cos\alpha-\xi\sin\alpha)x+(\xi\cos\alpha+\eta\sin\alpha)y)^2 = 0.\end{gathered} \tag{8}$$

We can summarize the results in:

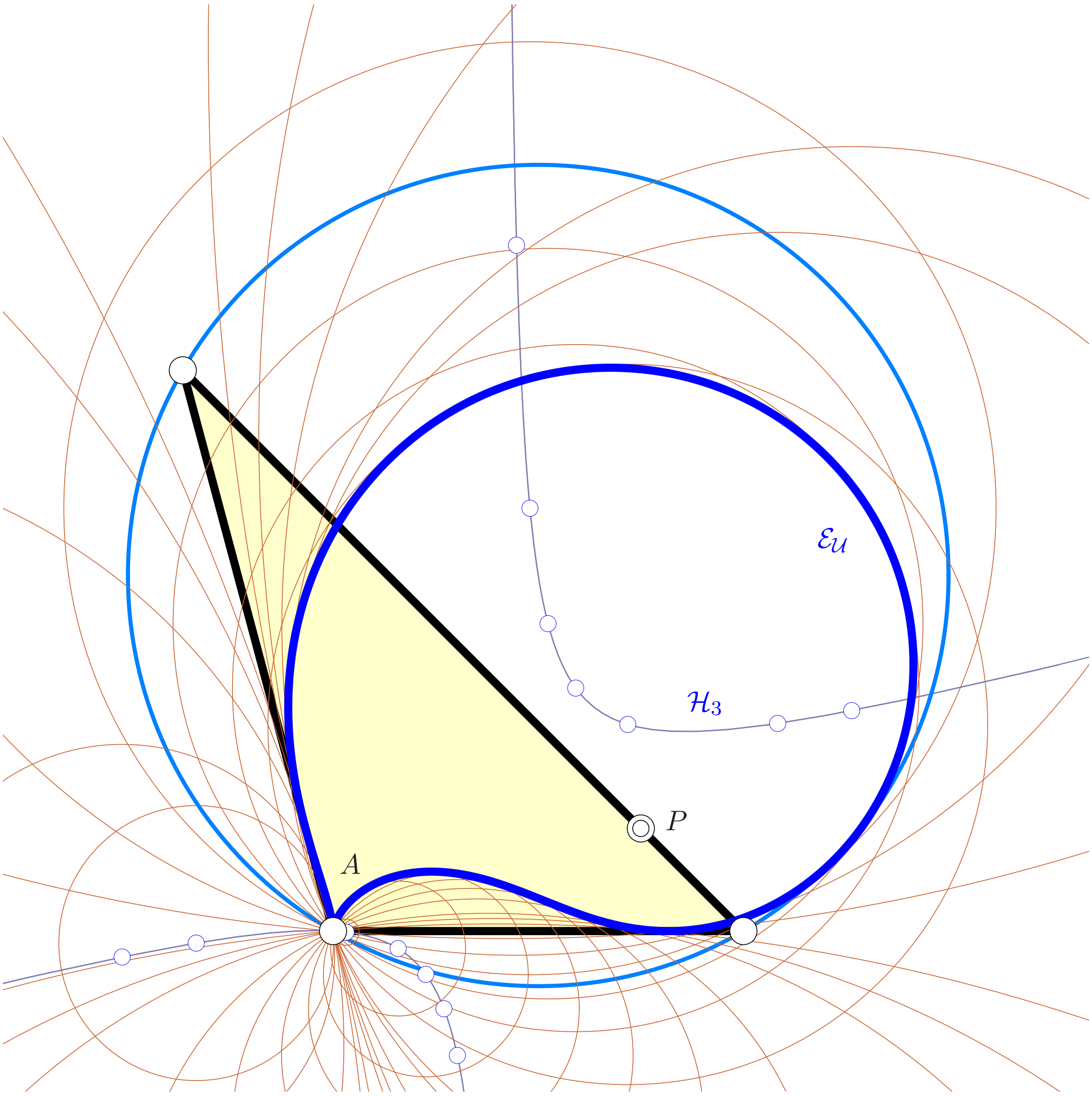

Figure 3: The envelope $\mathcal{E}_\mathcal{U}$ of the one-parameter family of circumcircles always has a cusp at A.

Theorem 3.1. *The envelope of the circumcircles of all triangles in the one-parameter triangle family $\mathcal{T}$ is a rational and bicircular quartic curve $\mathcal{E}_\mathcal{U}$ with ordinary double points at the absolute points of Euclidean geometry and a cusp of the second kind at the point A. The tangent to the super-linear branch at A is given by the equation*

$$(\eta \cos\alpha - \xi \sin\alpha)x + (\xi \cos\alpha + \eta \sin\alpha)y = 0$$

and encloses the angle $|\alpha - \psi|$ with the line $[A, P]$, where $\psi = \angle PAB$.

Fig. 3 shows the quartic curve $\mathcal{E}_\mathcal{U}$ for a specific choice of α.

The computation of the nine-point circles as the circumcircles of the medial triangles of the totality of triangles in $\mathcal{T}$ is nearby. Their equations are

$$\begin{gathered}\mathcal{N}: \ 2\cos\phi\cos(\alpha-\phi)(x^2+y^2)\\ -d(2\cos\phi\cos\alpha+\cos(\alpha-\phi))x-d(\sin(\alpha+\phi))y+d^2\cos\alpha=0\end{gathered}$$

and the envelope $\mathcal{E}_{\mathcal{N}}$ is computed in the same way as the envelope of the circumcircles. This results in

$$\begin{gathered}\mathcal{E}_{\mathcal{N}}: \ 4\sin^2\alpha(x^2+y^2)^2\\ -4(x^2+y^2)\big(\sin\alpha(\eta\cos\alpha+\xi\sin\alpha)x+(\xi\cos\alpha\sin\alpha+\eta(1-3\cos^2\alpha))y\big)\\ +(\eta\cos\alpha+\xi\sin\alpha)^2x^2-2(3\eta\cos\alpha-\xi\sin\alpha)(\xi\cos\alpha-\eta\sin\alpha)xy\\ +(\xi^2\cos^2\alpha+6\xi\eta\cos\alpha\sin\alpha+(1-9\cos^2\alpha)\eta^2)y^2=0\end{gathered}$$

where the equations of the two pairs of repeated isotropic lines about $(\frac{1}{2}\xi,\frac{1}{2}\eta)$ and $(\cos\alpha(\xi\cos\alpha+\eta\sin\alpha),\cos\alpha(\xi\sin\alpha-\eta\cos\alpha))$ are cut out.

Summarizing, we can state:

Theorem 3.2. *The envelope of the nine-point circles of the triangles in the family $\mathcal{T}$ is a rational and bicircular quartic $\mathcal{E}_{\mathcal{N}}$ with an ordinary node at A.*

If P is chosen on $[B,C]$, then $\eta\cos\alpha-\xi\sin\alpha=0$ and the quartic $\mathcal{E}_{\mathcal{N}}$ becomes a repeated circle centered at $\frac{\xi}{2}(1,-\cot(2\alpha))$ and with radius $\frac{1}{2}\xi\mathrm{cosec}(2\alpha)$.

Fig. 3.1 shows the envelope $\mathcal{E}_{\mathcal{N}}$ of the one-parameter family of nine-point circle for a specific choice of α.

3.2 Bicentric pairs

We shall have look at a special pair of bicentric points instead of browsing through a huge collection. The special pair shall be the pair of Brocard points.

The first Brocard point B_1 is common to the three circles b_A, b_B, b_C, where b_C touches $[B,C]$ at B and passes through A (the other circles are obtained by cyclically replacing the ingredients). In order to determine the second Brocard point, we look for the meet of the circles c_A, c_B, and c_C, where c_C touches $[A,B]$ at A and passes through B and the other circles are constructed by cyclical shifts of points and tangents. We omit the lengthy representations of B_1 and B_2 and give the locus of the 1$^{\text{st}}$ Brocard points of the triangles in the pencil of triangles by the equation

$$\begin{gathered}\mathcal{B}_1: \ \sin^4\alpha(x^2+y^2)^2\\ +\sin^2\alpha(x^2+y^2)\big(\sin\alpha(\eta\cos\alpha-\xi\sin\alpha)x+((\cos^2\alpha-2)\eta-\cos\alpha\sin\alpha\xi)y\big)\\ +\big(\eta\sin\alpha(\xi\sin\alpha-\eta\cos\alpha)x+(\xi^2\sin^2\alpha-\xi\eta\sin\alpha\cos\alpha+\eta^2)y\big)^2=0\end{gathered}$$

The locus of the 2$^{\text{nd}}$ Brocard points of the triangles in the one-parameter family can be described by

$$\begin{gathered}\mathcal{B}_2: \ \sin^4\alpha(x^2+y^2)^2-\sin^2\alpha(x^2+y^2)\big(2\xi\sin^2\alpha x-(2\xi\sin\alpha\cos\alpha-\eta)y\big)\\ -\sin^4\alpha(\xi^2+\eta^2)x^2-\sin\alpha(2\sin^2\alpha\cos\alpha\xi^2-\sin\alpha\xi\eta+\cos\alpha(2\cos^2\alpha-3)\eta^2)xy\\ +\cos\alpha(\sin^2\alpha\cos\alpha\xi^2-\sin\alpha\xi\eta+\cos\alpha(1+\sin^2\alpha)\eta^2)y^2=0\end{gathered}$$

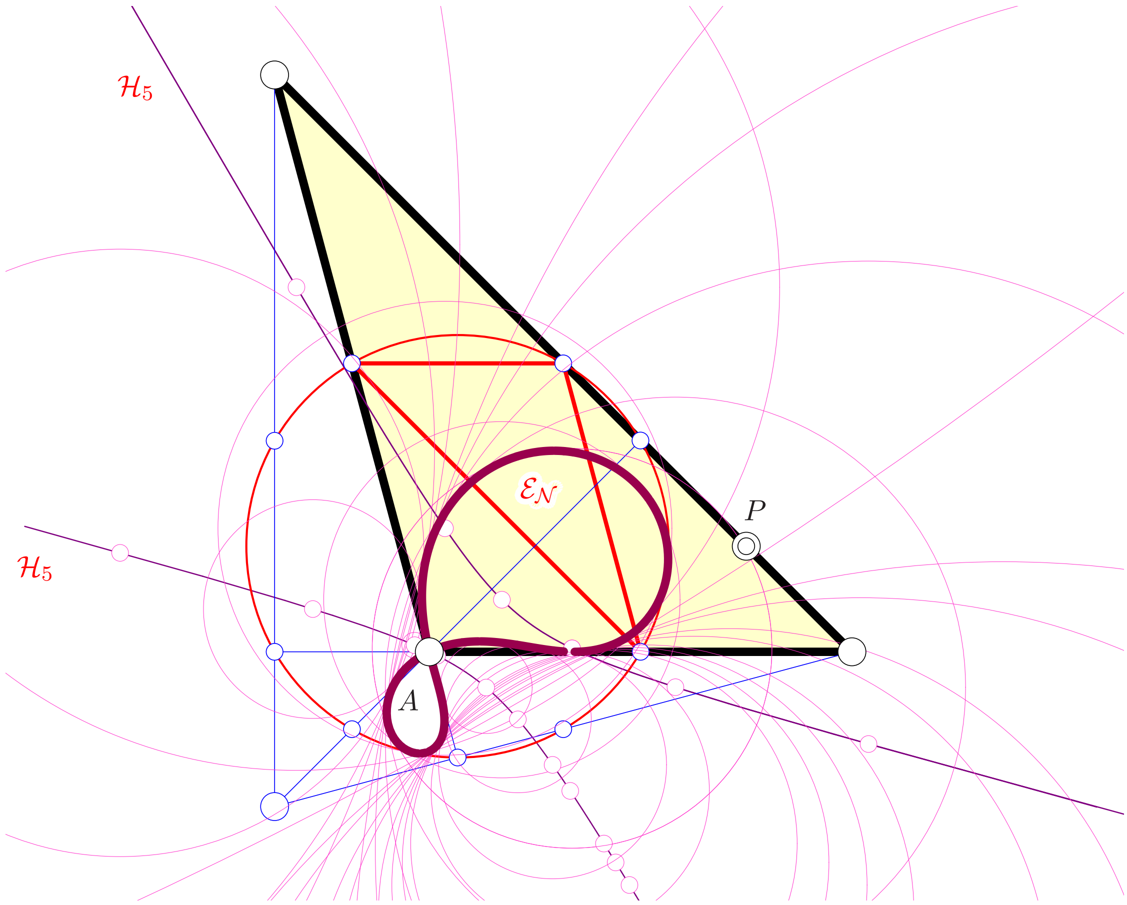

Figure 4: The envelope $\mathcal{E}_\mathcal{N}$ of the one-parameter family of nine-point circles has an ordinary double point at A. The locus of the nine-point centers X_5 of the triangle family is a hyperbola $\mathcal{H}_5$ according to Thm. 2.2.

We can summarize in:

Theorem 3.3. *The* 1^{st} *and* 2^{nd} *Brocard points of the triangles in the family* $\mathcal{T}$ *trace rational and bicircular quartic curves* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *with ordinary double points at* A*.* $\mathcal{B}_1$ *one touches the line* $[A, B]$*,* $\mathcal{B}_2$ *touches the line* $[C, A]$ *at* A*.*

Fig. 3.2 shows the two bicircular quartics occurring as the loci of the two Brocard points of the triangles in the one-parameter family $\mathcal{T}$ of triangles inscribed in the angle at A.

From [6, 7] we know, that the midpoint of the segment B_1B_2 is the center X_{39}, called the *Brocard midpoint*. Further, the Brocard circle b is now well-defined as the circumcircle of Δ's circumcenter X_3 and the two Brocard points. The center of b is the triangle center X_{182} usually referred to as the *Midpoint of the Brocard Diameter* (cf. [6]). Finally, the Symmedian point X_6 is the reflection of X_3 in X_{182}. Fig. 3.2 shows the traces of the triangle centers X_i with $i \in \{3, 6, 39, 182\}$.

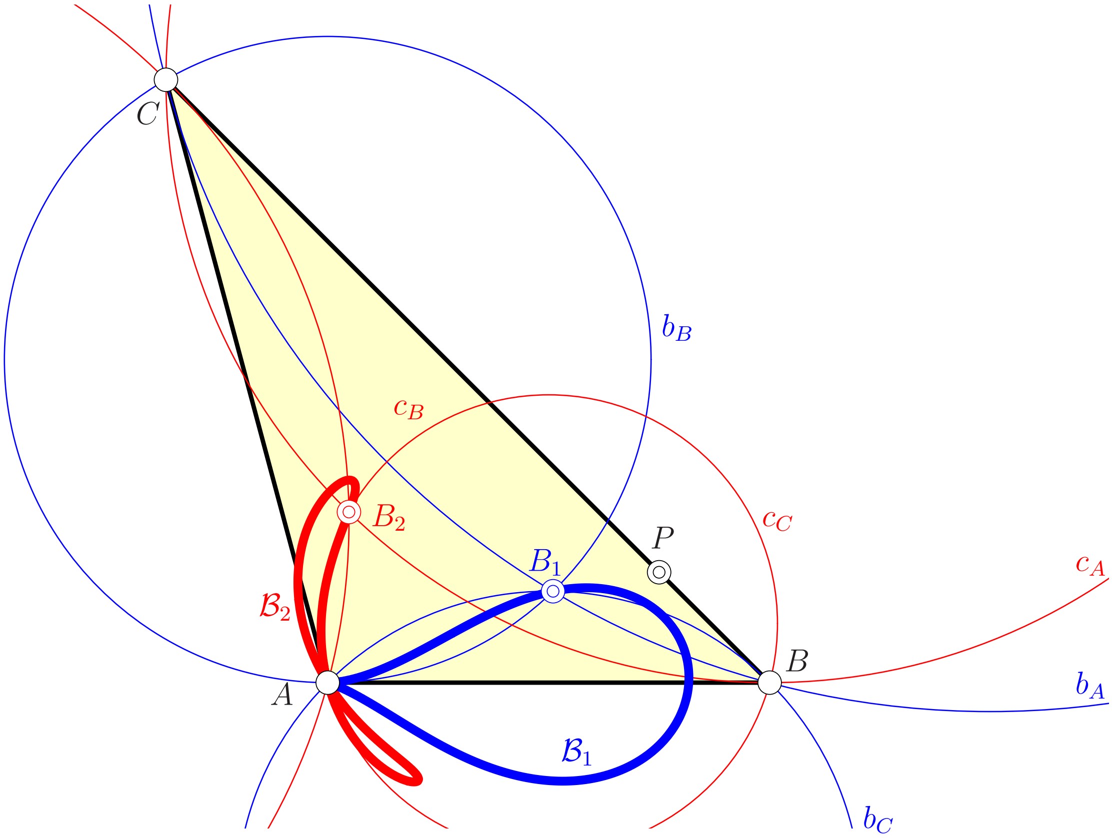

Figure 5: The loci $\mathcal{B}_1$ and $\mathcal{B}_2$ of the Brocard two points of all triangles in the family are quartic curves sharing the double point at A.

Hence, the orbits of the centers X_i with $i = 3, 6, 39, 182$ can now be parametrized and the computation of their orbits is straight forward. Surprisingly, we find the following result:

Theorem 3.4. *The locus of all Symmedian points of the triangles in the one-parameter family is an ellipse.*

Proof. The circumcenter X_3 is already determined (cf. (3)) and a parametrization of the orbits of the Brocard points B_1 and B_2 was computed prior to their implicit equations. Hence, the Brocard midpoint X_{39} which is the midpoint of B_1 and B_2 is well-defined. (An equation of the quartic curve parametrized by $X_{39}(\phi)$ can then be determined by eliminating the parameter ϕ. We skip this, because it will not deliver essentially new insight.) The circumcenter of X_3 and the two Brocard points B_1, B_2 equals the center X_{182} – the midpoint of the Brocard diameter –, which traces a quartic curve passing through the ideal points of the normals to $[A, B]$ and $[C, A]$. Now, the reflection of X_3 in X_{182} results in the Symmedian point:

$$X_6(\phi) = \left(\frac{d(\cos(2\alpha-\phi)+3\cos\phi)}{\cos(2\alpha-2\phi)-\cos 2\alpha+\cos 2\phi+3}, \frac{d(\sin(2\alpha-\phi)+\sin\phi)}{\cos(2\alpha-2\phi)-\cos 2\alpha+\cos 2\phi+3} \right),$$

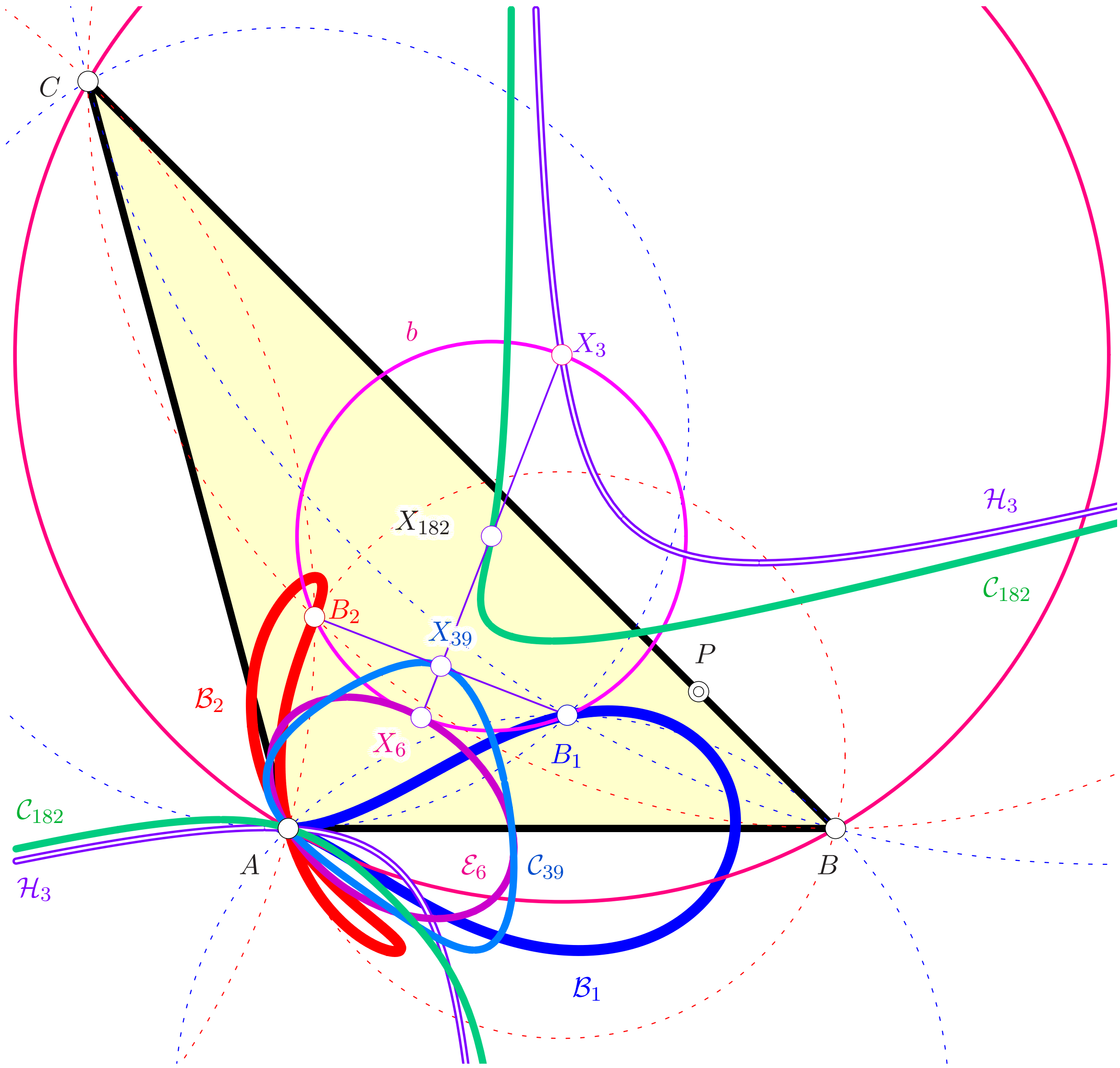

Figure 6: The Brocard points B_1 and B_2 move on their respective quartics. Meanwhile, the Brocard midpoint X_{39} and the center X_{182} of the Brocard circle trace their own quartics. The Symmedian point X_6 has an ellipse $\mathcal{E}_6$ for its orbit.

where we have used the abbreviation $d = \xi \cos\phi + \eta \sin\phi$ once again. The points $X_6(\phi)$ lie on the conic

$$\begin{aligned} \mathcal{E}_6 :\ & 2(1-\cos 2\alpha)x^2 - 6\sin 2\alpha xy + 4(2+\cos 2\alpha)y^2 \\ & +(\xi(\cos 2\alpha - 1) + \eta \sin 2\alpha)x + (\xi \sin 2\alpha - \eta(3+\cos 2\alpha))y = 0, \end{aligned} \tag{9}$$

which is an ellipse independent of the choice of $\alpha \neq 0, \pi$ and for any admissible choice of $P = (\xi, \eta)$. □

The center of $\mathcal{E}_6$ equals

$$\frac{1}{14-2\cos(2\alpha)}\Big((\xi\cos(2\alpha)+\eta\sin(2\alpha)+5\xi),(\xi\sin(2\alpha)-\eta\cos(2\alpha)+3\eta)\Big).$$

If α is allowed to run through S^1, the latter is a parametrization of an ellipse e_6 with the equation

$$e_6:\ 8x^2-6\xi x+\xi^2+24y^2-10\eta y+\eta^2=0$$

centered at $\left(\frac{3}{8}\xi,\frac{5}{24}\eta\right)$ with principal axes lengths $\frac{1}{24}\sqrt{3\eta^2+9\xi^2}$, $\frac{1}{24}\sqrt{\eta^2+3\xi^2}$. The ellipses $\mathcal{E}_6$ pass through $A=(0,0)$ independent of the choice of α and envelop an elliptic quartic $\mathcal{H}_6$ if α traces the unit circle. Fig. 3.2 shows some ellipses as orbits of the Symmedian point for triangle pencils with triangle sides $[B,C]$ passing through P and various choices of $\alpha=\angle[C,A],[A,B]$.

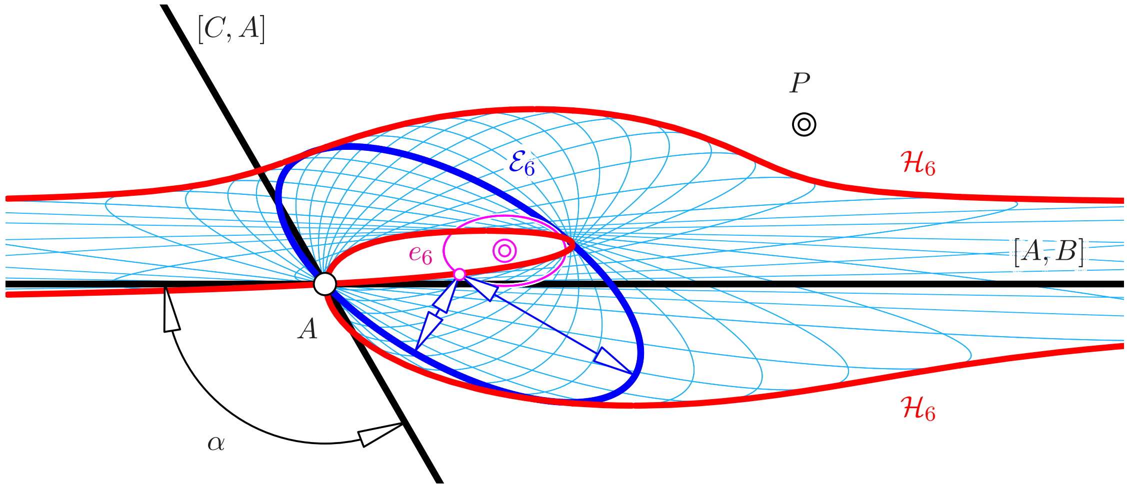

Figure 7: Some ellipses $\mathcal{E}_6$ and the envelope $\mathcal{H}_6$ of the ellipses $\mathcal{E}_6$. The quartic $\mathcal{H}_6$ is the boundary of the area of all possible Symmedian points X_6 of the two-parameter family of triangles (variable α and $[B,C]$ sweeps the pencil about P).

We can collect the latter results in:

Theorem 3.5. *The ellipses $\mathcal{E}_6$ as loci of the Symmedian point X_6 of all triangles with fixed α at A and side lines $[B,C]$ tracing the pencil about P envelop an elliptic quartic $\mathcal{H}_6$ with an ordinary double point at A and at the ideal point of $[A,B]$.*

Proof. We only have to determine an equation of the envelope of all $\mathcal{E}_6$ given in (9). This can be done in the same as in the case of the envelope of the circumcircles and we find

$$12y^2(x^2+4y^2)-4y(\eta x^2+3\xi\eta xy+10\eta y^2)+\eta^2x^2+6\xi\eta xy+(8\eta^2-\xi^2)y^2=0.$$

The ordinary double at A is obvious (no terms of degree lower than 2) and the double point the ideal point of $[A,B]$ has the two tangents $\eta+6y=0$ and $\eta-2y=0$. $\square$

Fig. 3.2 illustrates the contents of Thm. 3.5.

4 Traces of some more centers

There are some more triangle centers that can be reached with our analytical approach. The course of the incenter is rather unspectacular, since the pair of angle bisectors at A is fixed once α is chosen. The incenter X_1 allows for the analytical representation

$$X_1 = d\left(\frac{1+\cos\alpha}{\cos(\alpha-\phi)+\cos\phi+\sin\alpha}, \frac{\sin\alpha}{\cos(\alpha-\phi)+\cos\phi+\sin\alpha}\right). \tag{10}$$

Similarly, we can give the coordinates of the excenters. Note that the incenter and the excenter that lies on the exterior angle bisector through A interchange their roles as $[B,C]$ is rotating around P and forms the triangle "left" to A. This phenomenon frequently occurs when triangles smoothly change their shapes and orientations (or turn from acute to obtuse), see for example [1, 14].

The representation of the incenter given in (10) leads to the vertices $A_i \in [B,C]$ (cyclic) of the intouch triangle Δ_i

$$\begin{aligned} A_i &= \tfrac{d}{\cos(\alpha-\phi)+\cos\phi+\sin\alpha}\left(\cos\phi\sin\alpha+\cos\alpha+1, d\sin\alpha(1+\sin\alpha)\right), \\ B_i &= \tfrac{d(1+\cos\alpha)}{\cos(\alpha-\phi)+\cos\phi+\sin\alpha}\left(\cos\alpha,\sin\alpha\right), \quad C_i = \tfrac{d(1+\cos\alpha)}{\cos(\alpha-\phi)+\cos\phi+\sin\alpha}\left(1,0\right). \end{aligned} \tag{11}$$

Further, we shall give the coordinates of the excenter A_1 opposite to A

$$A_1 = \left(\frac{d(\cos(\alpha+\phi)+\cos\phi-\sin\alpha)-\eta\sin\alpha}{2\cos\phi(\cos\phi-\sin\alpha)}, \frac{-\sin\alpha(d(1+\sin\phi)+\eta)}{\cos\phi\cos(\alpha+\phi)-\sin\phi-\cos\alpha-1}\right)$$

and skip the other two because of the complexity of their coordinate representation, and moreover, because X_1 and A_1 together with the vertices (1) of Δ are sufficient in order to find the remaining excenters (if at all necessary).

4.1 Gergonne and Nagel point

The perspector of Δ and its intouch triangle Δ_i referred to as the Gergonne point X_7 (cf. [6, 7]). With (11), we can find a parametrization of the curve of Gergonne points corresponding to the triangles in $\mathcal{T}$. Then, we implicitize and find

$$\begin{aligned} \mathcal{C}_7:\; & y\Big(2ax+(a^2-1)y\Big)\Big(4x^2+6axy+(1+3a^2)y^2\Big) \\ & -4y\big(2ax+(a^2-1)y\big)\Big((a\eta+2\xi)x+(a\xi+(1+a^2)\eta)y\Big) \\ & -4\big(a^2\eta^2x^2-2a(\xi^2+a\xi\eta+\eta^2)xy+(\xi^2+(1-a^2)\eta^2)y^2\big)=0. \end{aligned}$$

The isotomic conjugate of X_7 is the Nagel point X_8 (cf. [6, 7]). In other words, the Nagel point is also the perspector of Δ and its extouch triangle Δ_e. This leads to a parametrization, and consequently, to the implicit equation

$$\begin{aligned} \mathcal{C}_8:\; & y(2ax-y)\Big(2ax+(a^2-1)y\Big)\Big(2a^3x-(3a^2+1)y\Big) \\ & -4a^2y\Big(2ax+(a^2-1)y\Big)\Big(a(2a\xi-\eta)x+((1+a^2)\eta-a\xi)y\Big) \\ & -4a^4\Big(a^2\eta^2x^2-2a(\xi^2+a\xi\eta+\eta^2)xy+(\xi^2+(1-a^2)\eta^2)y^2\Big)=0. \end{aligned}$$

Both curves $\mathcal{C}_7$ and $\mathcal{C}_8$ have an ordinary node at A, since A can be viewed as a "singular" triangle in $\mathcal{T}$. Fig. 4.4 shows the curves $\mathcal{C}_7$ and $\mathcal{C}_8$.

The quadratic factors in the inhomogeneous equations of $\mathcal{C}_7$ and $\mathcal{C}_8$ agree up to the constant factor a^4. Thus, the two quartics also share the tangents at the common double point A.

4.2 Mittenpunkt

The Mittenpunkt X_9 is the perspector of the medial triangle Δ_m and the excentral triangle Δ_e (see [6, 7]). With (1) and the excenters deduced from (10), we find a parametrization of the trace of the Mittenpunkt, and further, the equation

$$\begin{gathered}\mathcal{C}_9:\ y(1+a^2)\Big(2ax+(1+a^2)y\Big)\Big(4x^2+6axy+(1+3a^2)y^2\Big)\\ -4a^3\eta x^3-4a\Big(2(1+a^2)\xi+3a^3\eta\Big)x^2y-2\Big((1+a^2)(5a^2-2)\xi+a(6a^4+a^2+1)\eta\Big)xy^2\\ -2\Big(a(2a^2-1)(a^2+1)\xi-(2a^6+2a^2+1)\eta\Big)y^3\\ a^2\eta\Big(2a\xi+(a^2-1)\eta\Big)x^2+2a\Big((1+a^2)\xi^2+2a^3\xi\eta+(1+a^4)\eta^2\Big)xy\\ +\Big((a^4-1)\xi^2+2a^5\xi\eta+(a^6-1)\eta^2\Big)y^2=0.\end{gathered}$$

For a specific assumption on α, an example of the quartic curve housing all poses of the Mittenpunkt of the triangles in the family $\mathcal{T}$ is shown in Fig. 4.4.

For the very special choice of $a=-\frac{\xi}{\eta}$, *i.e.*, P is chosen on the exterior angle bisector at A, the double point of $\mathcal{C}_9$ at A becomes a tacnode with the tangent $\eta y+\xi x=0$ (orthogonal to $[A,P]$ passing through A).

4.3 de Longchamps point, Bevan point, Spieker point

As a point on the Euler line, the de Longchamps point X_{20} travels on a hyperbola (according to Thm. 2.1) with the equation

$$\begin{gathered}\mathcal{H}_{20}:\ 2\big(\cos(3\alpha)+7\cos\alpha\big)x^2+\big(3\sin(3\alpha)+7\sin\alpha\big)xy\\ -\big(\cos(3\alpha)-\cos\alpha\big)y^2+\big(\eta(\sin(3\alpha)-3\sin\alpha)+2\xi(\cos(3\alpha)-2\cos\alpha)\big)y\\ +\xi\big(\sin(3\alpha)-3\sin\alpha\big)x=0\end{gathered}$$

which is centered at

$$M_{20}=\left(\frac{1}{\sin\alpha}(\xi\sin\alpha-2\eta\cos\alpha),\frac{1}{\cos(2\alpha)-1}(2\xi\sin(2\alpha)-3\eta(\cos(2\alpha)-5)\right).$$

The equation of the hyperbola $\mathcal{H}_{20}$ can also be found by substituting $w=4$ into (6) and the corresponding implicit equation. The orbit of the centers of all $\mathcal{H}_{20}$ for varying angle α is the parabola with vertex $(\xi/2,(4\eta^2-\xi^2)/(4\eta))$, axis parallel to the y-axis, and the semi-latus rectum $\eta/2$.

We find the Spieker point X_{10} as the midpoint of the orthocenter X_4 and the Bevan point X_{40}, cf. [7]. Alternatively, but more intricate from the computational point of view, we could determine X_{10} as the incenter of the medial triangle Δ_m. According to [7], the Bevan point is the midpoint of the Nagel point X_8 and the de Longchamps point X_{20}. Hence, $X_{10} = \frac{1}{2}(X_4 + X_{40})$ and $X_{40} = \frac{1}{2}(X_8 + X_{20})$. Since $X_{20} = X_4(\Delta_a)$ (orthocenter of the anti-complementary triangle), X_{20} is the reflection of X_4 in X_3, and consequently, $X_{20} = 2X_3 - X_4$. Thus, $X_{10} = \frac{1}{2}(X_4 + \frac{1}{2}(X_8 + X_{20})) = \frac{1}{2}(X_4 + \frac{1}{2}(X_8 + 2X_3 - X_4)) = \frac{1}{4}(2X_3 - X_4 + X_8)$, which leads to a parametrization of the one-parameter family of Spieker points defined by the triangles in the triangle family $\mathcal{T}$.

The implicitization of the parametrization of the Spieker point $X_{10}(\phi)$ shows that it traces the sextic curve $\mathcal{C}_{10}$ with the equation

$$\begin{aligned}
\mathcal{C}_{10}:\ & 2^{10}(1+a^2)\Big(2ax + (a^2-1)y\Big)^2\Big(4x^2 + 6axy + (1+3a^2)y^2\Big) \\
& -2^8(1+a^2)y\Big(2ax + (a^2-1)y\Big)\Big(4a\eta(6a^2+5)x^3 \\
& +\big(4a(13a^2+15)\xi + 2(37a^4+23a^2-10)\eta\big)x^2y \\
& +\big((64a^4+56a^2-20)\xi + a(75a^4+52a^2-15)\eta\big)xy^2 \\
& +\big(a(25a^4+24a^2-5)\xi + (a^2-1)(25a^4+38a^2+15)\eta\big)y^3\Big) + \ldots = 0
\end{aligned}$$

up to constant coefficients.

If P is chosen on the exterior angle bisector at A, the ordinary node at A becomes a tacnode. The choice of $P \in [C, A]$ causes $\mathcal{C}_{10}$ split into a quartic curve and the repeated line $[C, A]$.

The sextic equation of the orbit $\mathcal{C}_{40}$ of the Bevan point X_{40} starts with

$$\begin{aligned}
\mathcal{C}_{40}:\ & \big((a^2-1)x - ay\big)\big(2(a^2-1)x - 3ay\big)\big((1+a^4)x + a(1-a^2)y\big) \\
& \cdot\big(2(a^4+a^2+1)x + a(1-a^2)y\big)\big(4x^2 + 6axy + (1+3a^2)y^2\big) + \ldots = 0,
\end{aligned}$$

where constant factors are cut out. The double point at A behaves in a way similar to that on $\mathcal{C}_7$, $\mathcal{C}_8$, and $\mathcal{C}_9$ depending on the choice of P.

4.4 Feuerbach point and its (X_1, X_5)-harmonic conjugate X_{12}

The Feuerbach point X_{11} is the point of contact of the nine-point circle and the incircle i of Δ. Since we have already found X_1, we also can give an equation of the incircle. Furthermore, the nine-point center X_5 is the midpoint of X_3X_4, the equation of the nine-point circle n (as the circumcircle of the medial triangle) is then also nearby. The computation of the one and only common point of i and n yields a parametrization of the family of all nine-point centers and the subsequent elimination of the parameter ϕ yields

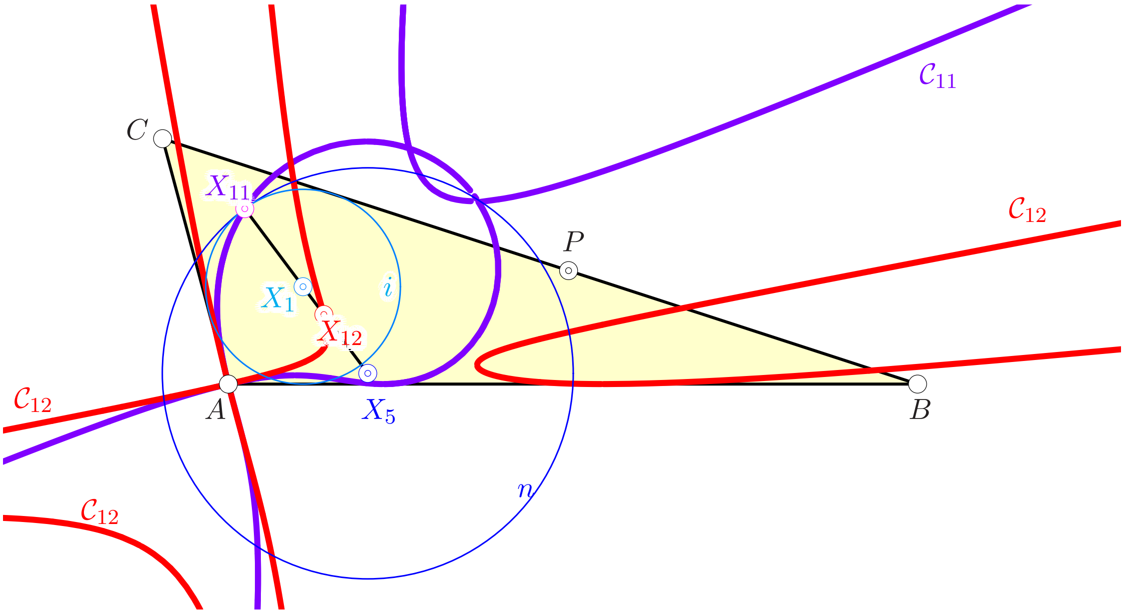

Figure 8: The trace of the Feuerbach point X_{11} is a quartic curve. The same holds true for X_{12}, the harmonic conjugate of X_{11} with respect to (X_1, X_5).

an equation of the nine-point orbit as

$$\begin{gathered}\mathcal{C}_{11}: \ (x^2+y^2)\big(8a^3x+(3a^4-6a^2-1)y\big)\big((1+3a^2)y-2ax\big)\\ +8a^3(2a\xi+(1-a^2)\eta)x^3-4a((5a^4+2a^2+1)\xi+2a(a^2-1)(a^2-2)\eta)x^2y\\ -2((a^2-1)(3a^4+1)\xi+4a(4a^4+a^2-1)\eta)xy^2\\ +(4a(1-a^2)(3a^2+1)\xi-2(9a^6-17a^4-a^2+1)\eta)y^3\\ -(4a^4\xi^2-4a^3(a^2-1)\xi\eta+a^2(a^2-1)^2\eta^2)x^2\\ +(2a(2a^4+1)\xi^2+2a^2(2a^4-2a^2+3)\xi\eta+2a(7a^4-4a^2+1)\eta^2)xy\\ +((3a^2+1)(a^2-1)\xi^2+12a^3(a^2-1)\xi\eta+(9a^6-19a^4+7a^2-1)\eta^2)y^2=0,\end{gathered}$$

which is a circular quartic curve. The curve $\mathcal{C}_{11}$ has three ordinary double points: at A, and further, at

$$\frac{1}{\eta\sin\frac{\alpha}{2}+\xi\cos\frac{\alpha}{2}}\left(\cos\frac{\alpha}{2},\sin\frac{\alpha}{2}\right) \text{ and } \left(\frac{\xi\cos\alpha-\eta\sin\alpha-2\xi}{2\cos\alpha-3},\frac{3\eta\cos\alpha-\xi\sin\alpha-2\eta}{2\cos\alpha-3}\right).$$

The harmonic conjugate of X_{11} with respect to X_1 and X_{12} is known as the center X_{12}. We shall not write down its implicit equation due to its length. However, the curve $\mathcal{C}_{12}$ is a rational quartic with an ordinary double point at A and two further ordinary double points.

5 Conclusion and future work

The synthetic in approach [12] is by far the most elegant approach. Nevertheless, in some cases it is limited and the algebraic approach can succeed then. This requires a proper

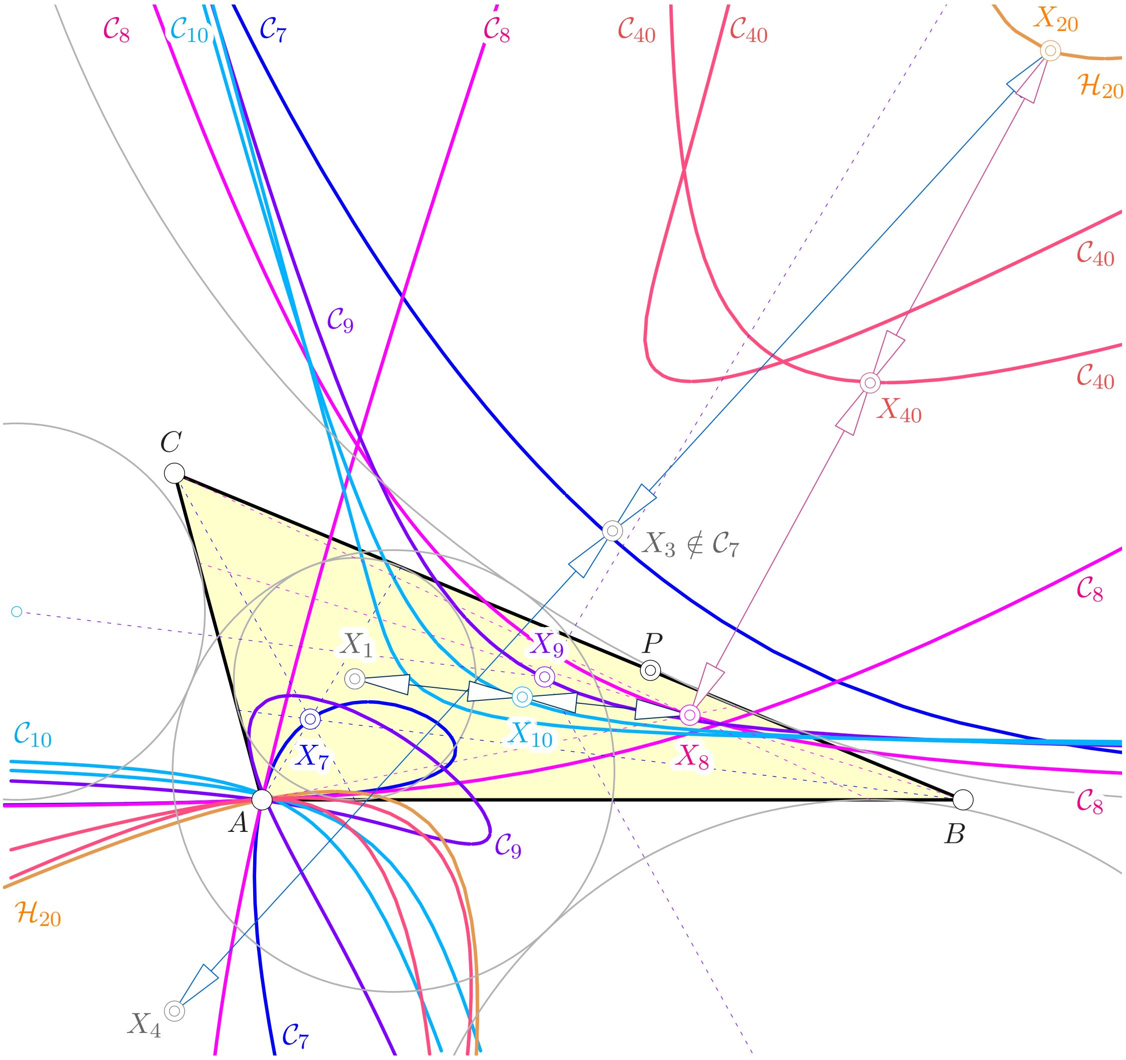

Figure 9: The orbits of X_7, X_8, and X_9 (Gergonne and Nagel point, Mittenpunkt) are quartic curves. The curve $\mathcal{C}_{10}$ is a sextic housing all poses of the Spieker point X_{10}, while X_{40} moves on a quartic $\mathcal{C}_{40}$ and the trace of the de Longchamps point X_{20} is a hyperbola $\mathcal{H}_{20}$ according to Thm. 2.2.

Ansatz, *i.e.*, a suitable way to parametrize the moving and changing objects. We will not claim that there is a unique *Ansatz* that does the job.

We have seen that triangle centers on the Euler line move on hyperbolae while the triangles vary in the family $\mathcal{T}$. Although, the Symmedian point X_6 is (in general) not located on the Euler line, it moves on an ellipse. It is so far the only point off the Euler line we know to move on a conic, and even on an ellipse. It remains unclear whether there are some more centers behaving that way.

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