A triad of circles externally tangent to the nine-point circle and internally tangent to two sides of a triangle

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Abstract

We determine the three circles in the interior of an acute triangle Δ which touch the nine-point circle n from the outside and two sides of Δ from the inside. Some perspective triangles related to Δ and the three circles are found. A more general result on tangent triangles related to Apollonian configurations of circles leads to a specific result in the case of a special Apollonian configuration derived from the three circles in question. All constrcutions are linear once the excircles of the base triangle are constructed.

Key words: triangle, nine-point circle, Apollonian problem, tangent triangle, perspective triangles.

MSC 2010: 51N20

1 Prerequisites

Among the totality of ten circles touching the nine-point circle n and at least two sides of a triangle Δ we find the incircle i and the three excircles e_A, e_B, and e_C. The latter four circles touch all three sides of Δ.

In [1] those circles were determined that touch n from the inside and two sides of Δ. Naturally, these three circles lie entirely in the interior of Δ. The authors of [1] found that the centers of these circles are collinear. The line carrying these centers is the central line $\mathcal{L}_{4,10}$ joining the orthocenter X_4 with the Spieker center X_{10} of Δ. We use the symbols X_i and $\mathcal{L}_{i,j}$ in order to denote the i-th center and the central line joining the i-th and j-th center in the list of triangle centers given in [3, 4].

In the following, we assume that the triangle Δ is acute and its vertices are labelled with A, B, and C. Δ shall be refered to as the base triangle. A line joining two points $P \neq Q$ is denoted by $[P, Q]$. With $\overline{X\,Y}$ we shall denote the length of the line segment bounded by two points X and Y. We use homogeneous and exact trilinear coordinates of points with respect to the base triangle Δ. By $\cos A$ we denote the cosine of the interior angle of Δ at the vertex A.

Since cyclic symmetry plays an important role we define the following two operators: The function ζ applies to scalar and vector valued functions. If $f(a,b,c)$ is a function depending on the side lengths a, b, and c of Δ, then $\zeta(f(a,b,c)) = f(c,a,b)$. Applying ζ to a vector valued function means to apply it to each component of the vector. The second useful operator is denoted by σ and performs a cyclic shift of the coordinates of a vector. Thus, $\sigma(x_0,x_1,x_2) = (x_2,x_0,x_1)$. Note that the x_i may be functions of a, b, and c. Therefore, the expression $\zeta(\sigma(x_0,x_1,x_2))$ makes sense. Because $\zeta(\sigma(x_0,x_1,x_2)) = \sigma(\zeta(x_0),\zeta(x_1),\zeta(x_2)) = (\zeta(x_2),\zeta(x_0),\zeta(x_1))$ and $\sigma(\zeta(x_0,x_1,x_2)) = \zeta(x_2,x_0,x_1) = (\zeta(x_2),\zeta(x_0),\zeta(x_1))$ hold, ζ and σ commute.

In Section 2 we determine the centers and radii of the circles l_A, l_B, and l_C which are tangent to n (from the outside) and tangent to Δ's sides (from the inside). As byproducts we find some perspective triangles. Moreover, we shall give a new meaning to some triangle centers from Kimberling's list (see [4]) such as the points X_{181}, X_{429}, X_{442}, and X_{3822}. Section 3 is devoted to the Apollonian problem solved for the three circles l_A, l_B, l_C.

Finally, in Section 4 we show a general result on tangent triangles, *i.e.*, two triangles built by the common tangents of either l_k (with $k \in \{A,B,C\}$ and the two associated (conjugate) Apollonian circles of the given three circles. This leads to a result on two special tangent triangles.

At this point we shall remark that once the excircles are constructed the construction of all further points and circles done afterwards is linear. Points that are found on circles appear in any case as the intersection of a line with the carrier circle where one point of intersection is already known.

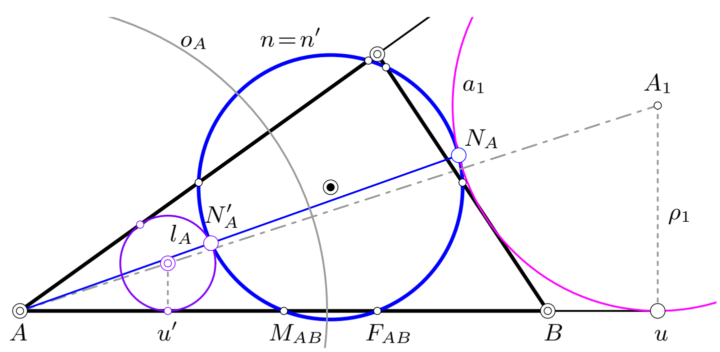

Figure 1: Construction of the circle l_A touching $[A,B]$, $[C,A]$, and n: The inverse of the excircle a_1 with respect to the ortho-circle o_A of n is the desired circle l_A.

2 Circles tangent to the nine-point circle from outside and tangent to two triangle sides

First, we determine the circle l_A that is tangent to the nine-point circle n from outside and to the sides $[A, B]$ and $[C, A]$. (The remaining two circles l_B and l_C can be found in an analogous way.)

We consider the inversion with respect to the ortho circle o_A of n. It is centered at A, intersects n twice at right angles, and maps n to itself. We determine o_A's radius ρ_{o_A} by applying the power theorem to the segments emanating from A to the midpoint M_{AB} of AB and to the pedal point F_{AB} of Δ's altitude from C to $[A, B]$, see Figure 1. Obviously, we have

$$\overline{AB} = c, \quad \overline{A\ M_{AB}} = \frac{c}{2}, \quad \overline{A\ F_{AB}} = b\cos A,$$

and thus,

$$\rho_{o_A}^2 = \frac{1}{2} bc\cos A.$$

The point of contact of the excircle a_1 and the line $[A, B]$ is at distance $u = \frac{1}{2}(a+b+c)$ from A. Its inverse in o_A is the point of contact of the circle l_A and $[A, B]$ which is at distance u' from A with

$$uu' = \rho_A^2 \quad \iff \quad u' = \frac{bc\cos A}{a+b+c} = \frac{b^2+c^2-a^2}{2(a+b+c)}.$$

This leads directly to the radius ρ_A and the center C_A of l_A. The radius reads

$$\rho_A = u' \tan\frac{A}{2} = \frac{b^2+c^2-a^2}{2(a+b+c)}\sqrt{\frac{(a-b+c)(a+b-c)}{(a+b+c)(-a+b+c)}}$$

and the actual trilinear coordinates of C_A are

$$C_A = (x_A : \rho_A : \rho_A).$$

The coordinate x_A equals the signed distance from C_A to $[B, C]$. Let W denote the intersection of the interior angle bisector through A with the side line $[B, C]$. Then, $x_A = \overline{C_A\ W}\sin\left(\frac{A}{2}+B\right)$ and $\overline{C_A\ W} = \overline{A\ W} - \overline{A\ C_A}$. With

$$\overline{A\ C_A} = \frac{\rho_A}{\sin\frac{A}{2}} \quad \text{and} \quad \overline{A\ W} = \frac{c\sin B}{\sin\left(\frac{A}{2}+B\right)}$$

we find

$$x_A = c\sin B - \frac{\rho_A \sin\left(\frac{A}{2}+B\right)}{\sin\frac{A}{2}},$$

or equivalently,

$$x_A = \frac{\tan\frac{A}{2}}{2a(a+b+c)}(-a^2 + a(b+c) + 2bc(b+c)).$$

The radii ρ_B and ρ_C of the circles l_B and l_C associated with the vertices B and C are $\rho_B = \zeta(\rho_A)$ and $\rho_C = \zeta(\rho_B)$. The respective centers C_B and C_C have actual trilinear coordinates

$$C_B = (\rho_B : y_B : \rho_B) \quad \text{and} \quad C_C = (\rho_C : \rho_C : z_C)$$

with $y_B = \zeta(x_A)$ and $z_C = \zeta(y_B)$. For an admissible (*i.e.*, acute) triangle the circles l_A, l_B, and l_C are shown in Figure 2.

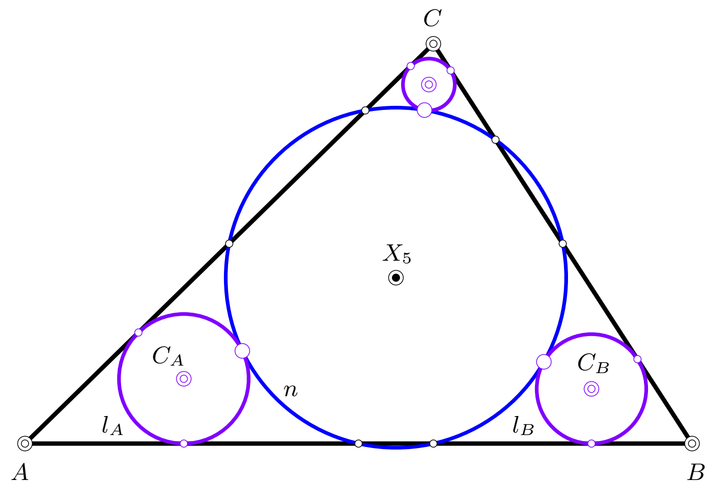

Figure 2: The three circles l_A, l_B, and l_C touching n from outside and touching Δ's sides.

It is not at all surprising that the base triangle Δ and its excentral triangle Δ_e is perspective to the triangle (C_A, C_B, C_C) built by the centers of the circles l_A, l_B, and l_C for the center of either l_k (with $k \in \{A, B, C\}$) lies on an interior angle bisector.

The excircle a_1 opposite to A touches the nine-point circle at

$$N_A = \left(-\sin^2\frac{B-C}{2} : \cos^2\frac{C-A}{2} : \cos^2\frac{A-B}{2}\right),$$

cf. [3]. The points N_B and N_C where a_1 and a_2 touch the nine-point circle are given by $N_B = \zeta(\sigma(N_A))$ and $N_C = \zeta(\sigma(N_B))$. Since these points lie collinear with the respective centers of inversion and the inverses with respect to o_A, o_B, and o_C we find

$$\begin{aligned} N'_A = \big((b^2 + ab + ac + c^2)(-b^3 - c^3 + cb^2 + bc^2 + a^2b + a^2c + 2abc)bc :& \\ (a + b - c)(c^2 - a^2 + b^2)(a + c)^2ac :& \\ (a - b + c)(c^2 - a^2 + b^2)(a + b)^2ab\big)\,,& \end{aligned} \tag{1}$$

and furthermore, $N'_B = \zeta(\sigma(N'_A))$ and $N'_C = \zeta(\sigma(N_B))$.

The triangle $\Delta'_F = (N'_A, N'_B, N'_C)$ of points of contact of n with l_A, l_B, l_C is perspective to the base triangle (A, B, C). This clear since the Feuerbach triangle Δ_F of Δ is perspective to the base triangle and the triplets (A, N_A, N'_A), (B, N_B, N'_B), and (C, N_C, N'_C) are triplets of collinear points. The common perspector of Δ, Δ_F, and Δ'_F is the center X_{12} which can easily be checked with the trilinear representation of all involved points. The trilinear coordinates of X_{12} can be found in [3, 4].

Now, we can easily verify the following:

Theorem 2.1.

1. *The orthic triangle Δ_o of Δ and Δ'_F are perspective with perspector X_{429}.*[1]

2. *The triangle Δ'_F is perspective to the medial triangle Δ_m of Δ with respect to the point X_{442}.*[2]

Proof. We use the trilinear representation of all involved points and show the linear dependency of lines with help of vanishing determinants. □

The three points of contact N'_A, N'_B, and N'_C of n and the three circles l_A, l_B, l_C define a new triangle Δ_{ti}, called the interior tangent triangle. Its side lines t_{i1}, t_{i2}, and t_{i3} are the common tangents of n and either l_k (with $k \in \{A, B, C\}$) at the points of contact N'_A, N'_B, and N'_C, respectively. The vertices of Δ_{ti} are denoted by T_{Ai}, T_{Bi}, and T_{Ci}. For example, we find $T_{Ai} = t_{i2} \cap t_{i3}$ with trilinear coordinates

$$\begin{aligned} T_{Ai} = (&-bc(b+c)^2(a^3+b^3+c^3-b^2c-bc^2) : \\ &: ca(a+c)(a^4-ca^3+ab^3+abc^2+ab^2c+ac^3-c^4+b^2c^2+b^3c-bc^3) : \\ &: ab(a+b)(a^4-a^3b+ab^3+abc^2+ab^2c+ac^3-b^4+b^2c^2-b^3c+bc^3), \end{aligned}$$

and then, $T_{Bi} = \zeta(\sigma(T_{Ai}))$, $T_{Ci} = \zeta(\sigma(T_{Bi}))$. Now we oberserve the following:

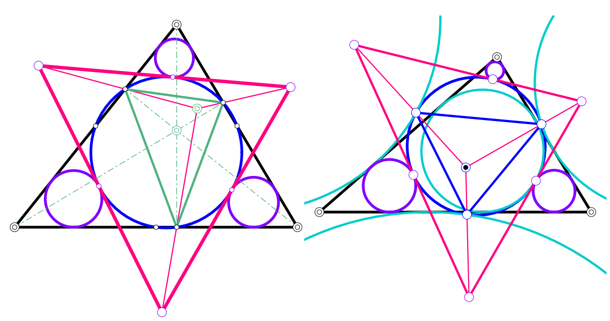

Figure 3: The inner tangent triangle Δ_{ti} is perspective to orthic triangle Δ_o (left), the Feuerbach triangle Δ_F (middle), and the medial triangle Δ_m (right), cf. Theorem 2.2.

Theorem 2.2.

1. *The inner tangent triangle Δ_{ti} is perspective to the orthic triangle Δ_o of the base triangle Δ.*

[1] X_{429} is the Euler X_{58}-5$^{\text{th}}$-substitution point, see [4].

[2] The center X_{442} is the complement of the Schiffler point X_{21}, see [4].

2. *The inner tangent triangle Δ_{ti} is perspective to the Feuerbach triangle Δ_F of the base triangle Δ.*

3. *The inner tangent triangle Δ_{ti} is perspective to the medial triangle Δ_m of the base triangle Δ.*

Proof. 1. First, we compute the trilinear coordinates of the vertices T_{Ai}, T_{Bi}, and T_{Ci} by intersecting the tangents of n at the respective points of contact. For example, we have $T_{Ai} = t_{N'_A} \cap t_{N'_B}$, $T_{Bi} = \zeta(\sigma(T_{Ai}))$, and $T_{Ci} = \zeta(\sigma(T_{Bi}))$. Then, it is elementary to verify that the lines $[T_{Ai}, O_A]$, $[T_{Bi}, O_B]$, and $[T_{Ci}, O_C]$ are concurrent in the innominate triangle center $P_1 = (\alpha_1 : \beta_1 : \gamma_1)$ with

$$\alpha_1 = bc(b+c)(a^3 + b^3 + c^2 - b^2c - bc^2)(b^3 + c^3 - a^2b - a^2c - abc),$$

cf. [**?**].

2. We use Eq. (1) in order to compute the coordinates of the lines $[N_A, T_{Ai}]$, $[N_B, T_{Bi}]$, and $[N_C, T_{Ci}]$. It is an elementary task to show the linear dependency of the respective coordinates. The perspector $P_2 = (\alpha_2 : \beta_2 : \gamma_2)$ of Δ_{ti} and Δ_F with

$$\begin{aligned}\alpha_2 = bc(b+c)^2(a^5 - a^3(b^2 + bc + c^2) + a^2(b+c)(b^2 - 3bc + c^2)\\ +bc(b-c)^2a - (b+c)(b-c)^2(b^2 - bc + c^2)\end{aligned}$$

is a triangle center of Δ which is not mentioned in [**?**, 4].

3. The perspectivity is shown in the usual way. The perspector $P_3 = (\alpha_3 : \beta_3 : \gamma_3)$ with

$$\alpha_3 = bc(b+c)(a^3 + b^3 + c^3 - b^2c - bc^2)(b^3 + c^3 - a^2b - a^2c + abc)$$

is a triangle center of the base triangle that does not show up in [4]. □

Figure 3 shows the inner tangent triangle of the circles l_k (with $k \in \{A, B, C\}$) and the perspective triangles Δ_o (left), Δ_F (in the middle), and Δ_m (right) as described in Theorem 2.2.

Let $L_{k,AB}$ be the point of contact of the circle l_k ($k \in \{A, B, C\}$) with the line $[A, B]$. With analogous symbols we denote all the other points of contact. Since $[A, B]$ is a common tangent of l_A and l_B, the midpoint P_{AB} of $L_{A,AB}$ and $L_{2,AB}$ has equal power to both circles l_A and l_B. Now we are able to show:

Lemma 2.1. *The lines $[T_{Ai}, P_{BC}]$, $[T_{Bi}, P_{CA}]$, and $[T_{Ci}, P_{AB}]$ are the radical lines of the three circles l_A, l_B, and l_C.*

Proof. We only have to show that the line $r_A := [T_{Ai}, P_{BC}]$ is orthogonal to the line $[C_B, C_C]$ joining the centers of l_A and l_B. For that purpose we compute the trilinear coordinates of the points P_{jk} of equal power with respect to l_j and l_k on the side line $[j, k]$ of Δ where $(j, k) \in \{(A, B), (B, C), (C, A)\}$.

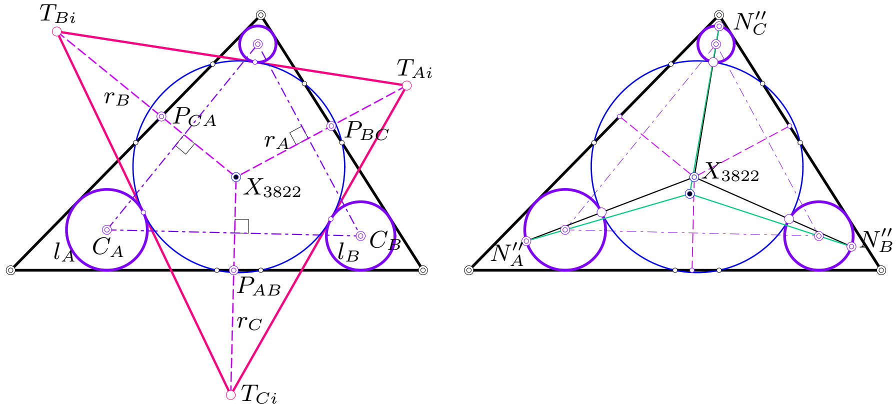

Figure 4: The center X_{3822} is the radical center of the three circles l_A, l_B, l_C (left). The outer Apollonian circle of l_A, l_B, and l_C touches at the points N''_A, N''_B, and N''_C (right).

Note that P_{jk} is the midpoint of $L_{j,rs}$ and $L_{k,rs}$ on the side $[r, s]$ with $(r, s) \in \{(A, B), (B, C), (C, A)\}$. We find

$$\begin{aligned} P_{AB} &= (b(a^2 - b^2 + c^2 + ac + bc) : a(b^2 + c^2 - a^2 + ac + bc) : 0), \\ P_{BC} &= (0 : c(a^2 + b^2 - c^2 + ab + ac) : b(a^2 - b^2 - c^2 - ab - ac)), \\ P_{CA} &= (c(a^2 + b^2 - c^2 + ab + bc) : 0 : a(b^2 + c^2 - a^2 + ab + bc)). \end{aligned}$$

Then, we use a formula given in [3, p. 31] in order to characterize orthogonal lines and show that r_A is orthogonal to $[C_B, C_C]$. Since r_A contains P_{BC} it is the radical line of l_A and l_B. In the same way we proceed for the other radical lines. $\square$

Figure 4 (left) shows the radical lines of the circles l_k ($k \in \{A, B, C\}$) together with the radical center.

As a consequence of Lemma 2.1, the point $R = [T_{Ai}, P_{BC}] \cap [T_{Bi}, P_{CA}]$ is the radical center of the three circles l_A, l_B, l_C. It turns out that the following holds:

Theorem 2.3. *The radical center of the circles l_A, l_B, l_C is the triangle center X_{3822}.*

Proof. Compute the point $[T_{C_i}, P_{AB}] \cap [T_{A_i}, P_{BC}]$, show that it is a center which is incident with $[T_{B_i}, P_{CA}]$. Then, compare with the trilinear representation given in [4]. $\square$

3 The outer Apollonian circle of l_A, l_B, and l_C

The three circle l_A, l_B, and l_C define eight tritangent circles, *i.e.*, their Apollonian circles. In any case and independent of the shape of the base triangle Δ, the Apollonian circles are eight different circles.

Now that we have found the radical center of the circles l_k (cf. Lemma 2.1 and Theorem 2.3) we can give a simple, and infact, linear construction for that Apollonian circle that encloses all three circles l_i since the Apollonian circle that is enclosed by the l_ks is the nine-point circle n.

We follow the construction given by Gergonne (see [2]): Assume we are given three circles l_k (with $k \in \{A, B, C\}$). Determine one axis a of similarity (*i.e.*, the line joining two centers of similarity from different circles). Find the poles A_k of a with respect to l_k. Then, the points of contact of the tritangent circles on either l_k are constructed as the intersections of the lines joining A_k with the radical center R of the l_ks.

Consequently, we do not have to determine the axis a of similarity to l_A, l_B, and l_C which would be the polar line of X_{48} with regard to the base triangle Δ. The line a carries the three exterior centers of similarity of the l_is, and thus, $a = [S_{AB}, S_{BC}]$ with $S_{AB} = [A, B] \cap [C_A, C_B]$ and $S_{BC} = [B, C] \cap [C_B, C_C]$. Since one solution (associated to a) of the Apollonian problem is already known (namely n together with the points N'_A, N'_B, N'_C of contact, we just determine the points N''_A, N''_B, N''_C of contact of the second solution m (the outer one) as $N''_A = \{[X_{3822}, N'_A] \cap l_A\} \setminus \{N'_A\}$ and similarly for N''_B and N''_C. The construction of the latter three points is linear and shown in Figure 4.

For example, N''_A is given by its homogeneous trilinear coordinates as

$$\begin{gathered} N''_A = ((b+c)^2 a^5 + (b+c)(c^2+6bc+b^2)a^4 + \\ -(b^2-8bc+c^2)(b+c)^2 a^3 - (b+c)(c^4+2bc^3-14b^2c^2+2b^3c+b^4)a^2 + \\ -4bc(b^2-bc-c^2)(b^2+bc-c^2)a - 4b^2c^2(b+c)(b-c)^2 : \\ : ab(a+c)^2(a+b-c)(-a^2+b^2+c^2) : \\ : ac(a+b)^2(a-b+c)(-a^2+b^2+c^2)) \end{gathered}$$

from which we obtain $N''_B = \zeta(\sigma(N''_A))$ and $N''_C = \zeta(\sigma(N''_B))$.

The center M of the outer Apollonian circle m can be found as $[C_A, N''_A] \cap [C_B, N''_B]$ and its trilinear coordinates are $(\alpha_M : \beta_M : \gamma_M)$ with center function

$$\begin{gathered} \alpha_M = 2(b+c)a^5 + (b+c)^2 a^4 - (b+c)(3b^2-2bc+3c^2)a^3 \\ -(b^4+4b^3c+4b^2c^2+4bc^3+c^4)a^2 + (b+c)(b^4-2b^3c-2bc^3+c^4)a \\ +2bc(b-c)^2(b+c)^2 \end{gathered}$$

and $\beta_M = \zeta(\alpha_M)$ and $\gamma_M = \zeta(\beta_M)$. Obviously, the point M is a center of the base triangle Δ. It is not yet mentioned in [4].

We can state the following:

Theorem 3.1. *The points M, X_{3822}, and X_5 are collinear.*

Proof. The points M and X_5 are the centers of the two Apollonian circles n and m (the outer one) to l_A, l_B, and l_C for the special choice of the axis of similarity, namely a. These two solutions are known to be inverse with respect to a circle about the radical center of the given circles. Thus, the radical center X_{3822} is collinear with M and X_5.

One could also show that the trilinear coordinate vectors of M, X_{3822}, and X_5 are linearly dependent. $\square$

The triangle $\Delta_{ce} = (N''_A, N''_B, N''_C)$ of contact points of the exterior Apollonian circle has a perspective colleague:

Theorem 3.2. *The triangle Δ_{ce} is perspective to the base triangle Δ with the Apollonius point X_{181}*[3] *for its perspector.*

Proof. This is easily verified by using the trilinear representation of the vertices of Δ_{ce} and Δ. $\square$

4 Tangent triangles related to an Apollonian configuration

The circles n and m are two particular but associated (conjugate) solutions of the Apollonian problem to given circles l_k. We show the following remarkable result which applies to any Apollonian configuration.

Lemma 4.1. *Let e and i be the exterior and interior Apollonian circles to three circles c_1, c_2, and c_3. Further, let Δ_e and Δ_i be the two triangles built by the common tangents of e and c_k, or i and c_k (with $k \in \{1, 2, 3\}$ respectively, and call them exterior and interior tangent triangles.*

Then, the tangent triangles Δ_e and Δ_i are perspective with respect to the radical center R and the perspectrix is the axis a of similarity of c_1, c_2, and c_3.

Proof. We follow Gergonne's way of constructing Apollonian cirlces to three given circles [2]. Therefore, we first construct the axis a of similarity of the given circles c_1, c_2, and c_3 and the poles A_k of a with regard to c_k. According to Gergonne, the points of contact of any Apollonian circle are the points of intersection of the lines $[R, A_k]$ with c_k, for any $k \in \{1, 2, 3\}$.

Since conjugacy with respect to a conic is symmetric, the pole of the line $[C^e_k, C^i_k]$ joining the exterior point of contact C^e_k and interior point of contact C^i_k of the

[3]Let w denote the Apollonian circle that encloses the three excircles e_A, e_B, e_C of the triangle Δ. The triangle of contact points $w \cap e_A$, $w \cap e_B$, $w \cap e_C$ is perspective to the base triangle Δ and the perspector is the point X_{181}, cf. [3, 4]. Further, X_{181} is the external center of similarity of the incircle and Apollonius circle w.

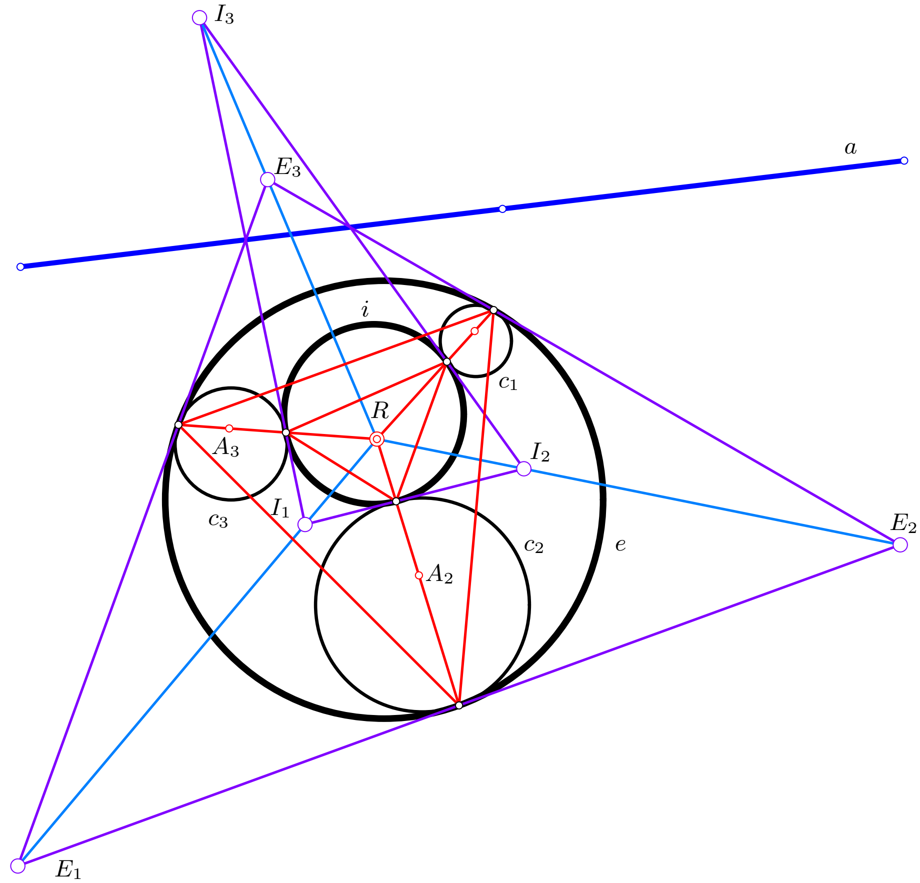

Figure 5: The tangent triangles $\Delta_i = (I_1, I_2, I_3)$ and $\Delta_e = (E_1, E_2, E_3)$ of the interior Apollonian circle i and exterior Apollonian circles to three given circles c_1, c_2, and c_3 are perspective. The perspector equals the radical center R of c_1, c_2, and c_3. The perspectrix of Δ_i and Δ_e is the axis of similarity of the triplet of circles.

k-th circle with the exterior and interior Apollonian circle e and i lies on a. Consequently, any pair of tangents of a circle c_k (consisting of a tangent to e and a tangent to i) intersects in a point on a. This shows that corresponding sides of Δ_e and Δ_i intersect in points of a and the two triangles are perspective to the line a. According to Desargues theorem they are also perspective to a point. It is clear from the construction that this point, *i.e.*, the perspector is the radical center of the circles. $\square$

To be more precise, we shall replace the phrase *interior and exterior Apollonian circle* by *a pair of conjugate Apollonian circles* since these come along as the solutions of a quadratic equation.

Figure 5 illustrates the contents of Lemma 4.1 for a special choice of the axis of similarity. However, the Lemma holds for any choice of axis of similarity,

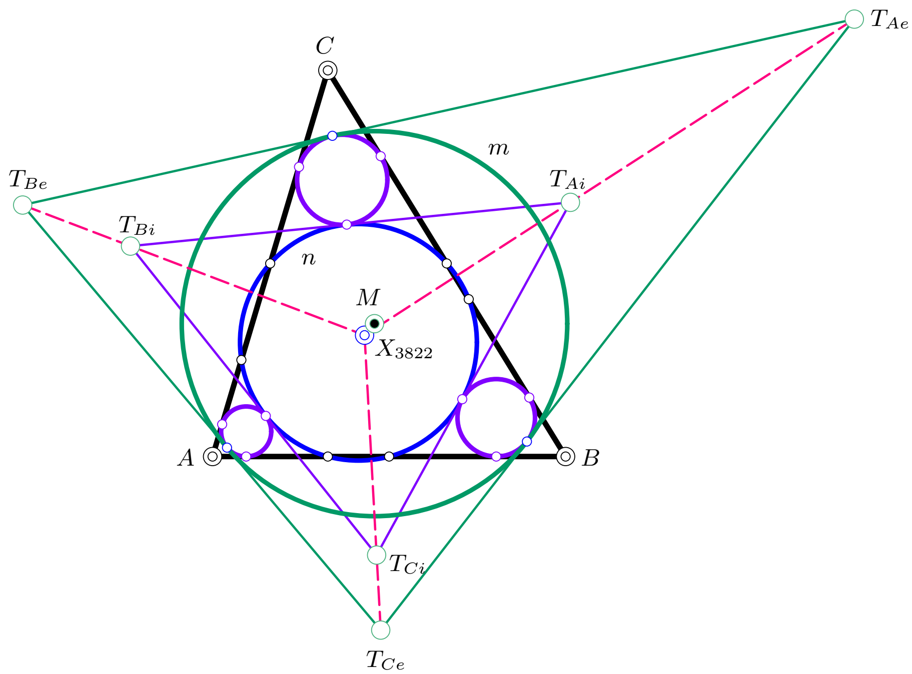

Figure 6: The interior tangent triangle Δ_{ti} is perspective to the exterior tangent triangle Δ_{te} with the radical center X_{3822} of l_i for their perspector.

and thus there are up to four pairs of perspective tangent triangles related to a complete Apollonian configuration.

As a consequence of Lemma 4.1 we have the following result (illustrated in Figure 6):

Theorem 4.1. *The interior tangent triangle Δ_{ti} of l_k and the exterior tangent triangle Δ_{te} of l_k (with $k \in \{A, B, C\}$ are perspective with perspector X_{3822}.*

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