Variations on Frégier's Theorem

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Abstract. We show that Frégier's theorem allows for some generalizations within the framework of projective geometry. This also contains the notion of non-Euclidean versions of Frégier's theorem and Frégier conics. Two different generalizations of Frégier's theorem shall be considered: (1) The involution of right angles is replaced with an arbitrary involution induced by some polarity. (2) The envelopes of chords of a conic whose endpoints are assigned in a projective mapping (not involutive) are conics.

Keywords: Frégier's theorem, Frégier conic, involution, polarity, projective mapping, pencil of conics.

1 Introduction

Let c be a conic and assume that P is a point on c. Then, let (g, g') be a pair of lines through P such that g and g' are orthogonal (in the Euclidean sense). Further, let $Q = c \cap g \setminus \{P\}$ and $Q' = c \cap g' \setminus \{P\}$. Now, Frégier's theorem states (cf. [1, 2]):

The chords $[Q, Q']$ pass through a single point F (the Frégier point of P with respect to[1] *c) independent of the choice of g.*

Generalizations to non-Euclidean Frégier conics have been studied in [4]. Quadratic transformations based on Frégier's constructions are defined and investigated in [5, 6] along with higher dimensional analogues. The fact that the ordinary Euclidean Frégier conics are the only conic shaped generalized offsets to conics is shown in [3]. Especially in the latter article, the right angle which is a substantial ingredient for Frégier's theorem was replaced with an arbitrary fixed angle ϕ. It turned out that the thus defined chords $[Q, Q']$ envelop conics (the generalized Frégier conics) c_ϕ assigned to the point P and w.r.t. c. If the angle ϕ traces the open interval $]0, \frac{\pi}{2}[$, then the generalized Frégier conics c_ϕ trace a pencil of conics of the third kind (to which c also belongs, see [3]) and the ordinary Frégier point is the only real point of a singular conic in the Frégier pencil.

In the following, we generalize Frégier's theorem by replacing fixed angles at first by an involutive mapping induced by some polarity in Sec. 2. There are six different types of such Frégier constructions depending on whether the polarity is elliptic or hyperbolic, and then, since a hyperbolic polarity is always that of a conic d, we have to distinguish between the five different types of pencils spanned by c and d. For special assumptions on the polarity, this yields the non-Euclidean notion (cf. [4]) of Frégier conics

[1] We shall write w.r.t. short hand for *with respect to*.

as by-catch. This immediately raises the question: What happens if we look at the chords of a conic c whose endpoints are assigned in an arbitrary projective mapping acting on c? This will be described in Sec. 3.

2 Polarities instead of the right angle

The right angles appearing in Frégier's construction as well as the constant angles in the Frégier variant described in [3] induce special projective mappings on the underlying regular conic c with the polarity γ. Now, we may assume that in the pencil around the pivot point $P \in c$ an arbitrary involution is acting. The involution shall be induced by an arbitrary polarity δ which assigns to each line $g \ni P$ a unique line $g' \ni P$ by

$$g' = [\delta^\star(g), P],$$

where $\delta^\star$ is the adjoint mapping of the polarity δ. The lines g and g' intersect c in P and each in a further point $Q \in g$ and $Q' \in g'$. Now, we can state and prove:

Theorem 2.1. *Let c be a regular conic, let P be a point on c, and let further δ be a regular polarity (different from that w.r.t. c). Now, consider the projective and involutive mapping α in the pencil of lines around P that sends each line g to the line $g' = [P, \delta^\star(g)]$. If now Q and Q' are defined as above, then the chords $[Q, Q']$ pass through a single point F (the generalized Frégier point of P w.r.t. c).*

Proof. The generalized Frégier point F is simply the center of the involution α lifted to the conic c (see [2]). □

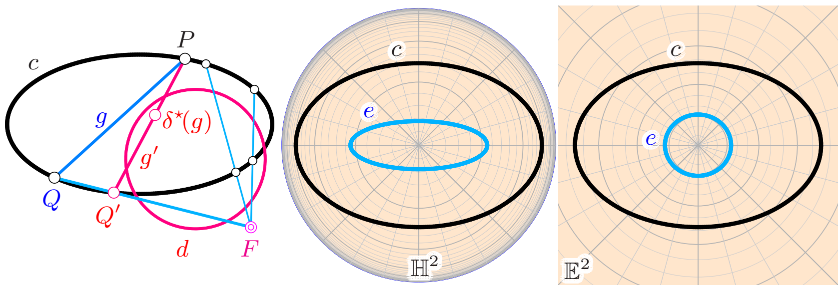

Fig. 1: Left: A Frégier point F defined by means of a generic involution induced by a polarity w.r.t. a conic d. Center and right: Frégier conics e of a conic c in a hyperbolic and an elliptic plane.

The polarity δ may either be that of a conic d with real points (hyperbolic polarity) or that of an empty conic (elliptic polarity). The case of the elliptic polarity causes no special cases since there no real self-conjugate points w.r.t. δ. In the case of the polarity w.r.t. a real conic d, we have to distinguish between five cases, according to the type of pencil spanned by c and d. We are able to show:

Theorem 2.2. *The construction of the generalized Frégier conic from Thm. 2.1 does not depend on the type of pencil spanned by c and d.*

Proof. We just lay down what is necessary in order to give a synthetic proof. First, it means no restriction to assume that c is given by $x_0x_2 - x_1^2 = 0$. The polarity δ shall be that of a conic d that spans a pencil of 1., ..., 5. kind with c. Therefore, we can assume that the equations of the conics d in the pencil of type are given by $d = c + \lambda s_i$ (we identify the conic with its equation), where s_i is the equation of a singular conic in the pencil of the i-th kind and can be chosen as: $s_1 :\ x_1(px_0 - (1+p)x_1 + x_2)$, $s_2 :\ x_2(x_2 - x_0)$, $s_3 :\ x_1^2 = 0$, $s_4 :\ x_2(x_1 - x_2) = 0$, and $s_5 :\ x_2^2 = 0$. In any case, the pivot point P can be given by $1 : t : t^2$ (with $t \neq 0, 1, \infty$). In s_1, $p \neq 0, 1, \infty$ guarantees that c and s_1 really span a pencil of the 1. kind. Further, $Q = 1 : u : u^2$ (with $u \neq p, t, 0, 1, \infty$). For a special but proper choice of μ, we obtain a regular conic d in the pencil, and thus, a polarity δ. Then, $g = [P, Q]$ and we are able to compute the pole $\delta^\star(g)$ of g w.r.t. all conics in the pencil (variable $\lambda \neq 0$). Finally, $g' = [P, \delta^\star(g)]$ and $Q' = g' \cap c \setminus \{P\}$. Then, we can show that $[Q, Q']$ passes through a point F (independent of Q, *i.e.*, the parameter u). For variable pivot point P (*i.e.*, variable t), the points F trace a conic, the generalized Frégier conic e of c w.r.t. δ. It can be shown that e passes through the base points of the pencil only if they are at least of multiplicity two by intersecting e and c. ☐

Fig. 2 shows a generalized Frégier conic e (red) of a conic c (blue) w.r.t. to the polarity δ of a regular conic d (magenta) which, together with c, spans a pencil of the first, second, third, fourth, or fifth kind.

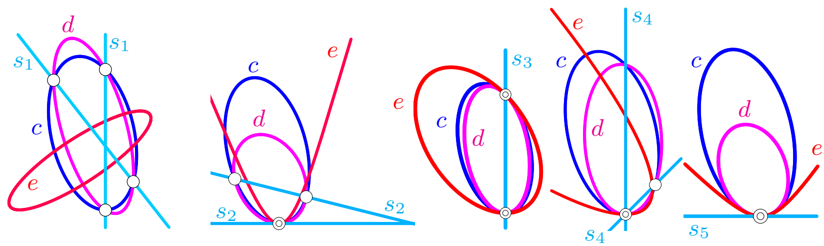

Fig. 2: The generalized Frégier conics e according to Thm. 2.1 pass through base points of the pencil $\lambda c + \mu d$ only if these are at least two-fold.

2.1 Non-Euclidean versions

We assume that the polarity w.r.t. to c and the polarity δ have a common polar triangle, *i.e.*, the corresponding bilinear forms can be diagonalized simultaneously. Thus, we can assume that c is given by $x_1^2/a^2+x_2^2/b^2 = x_0^2$ (with $a, b \neq 0$ and $a \neq b$). The case $b^2 < 0$ needs a separate discussion. In the hyperbolic case, δ's self-conjugate points can be given by $\omega: \ x_1^2 + x_2^2 = x_0^2$. The conic ω can be viewed as the absolute conic of hyperbolic geometry. Therefore, for any point $P = 1 + t^2 : a(1 - t^2) : 2bt$ (with $t \in \mathbb{R} \cup \{\infty\}$), the Frégier point

$$F_h = (a^2b^2-a^2-b^2)(1+t^2) : a(a^2b^2+a^2-b^2)(t^2-1) : -2bt(a^2b^2-a^2+b^2)$$

in the sense of hyperbolic geometry traces the *hyperbolic Frégier conic*

$$x_0^2 = \frac{(a^2b^2-a^2-b^2)^2}{a^2(a^2b^2+a^2-b^2)^2}x_1^2 + \frac{(a^2b^2-a^2-b^2)^2}{b^2(a^2b^2-a^2+b^2)^2}x_2^2.$$

The hyperbolic Frégier conic is regular if, and only if, $(a^2b^2-a^2+b^2)(a^2b^2+a^2-b^2)(a^2b^2-a^2-b^2) \neq 0$. This leads to a three-branched variety of singular hyperbolic Frégier conics which are studied in detail in [4].

The conic $\omega: \ x_0^2 + x_1^2 + x_2^2 = 0$ is empty (over the real numbers) and can serve as the absolute conic of elliptic geometry. Then, the point $P = 1 + t^2 : a(1 - t^2) : 2bt$ on the conic $c: \ x_1^2/a^2 + x_2^2/b^2 = x_0^2$ defines the *elliptic Frégier point*

$$F_e = (a^2b^2+a^2+b^2)(1+t^2) : a(a^2b^2-a^2+b^2)(t^2-1) : -2bt(a^2b^2+a^2-b^2)$$

which traces the *elliptic Frégier conic*

$$x_0^2 = \frac{(a^2b^2+a^2+b^2)^2}{a^2(a^2b^2-a^2+b^2)^2}x_1^2 + \frac{(a^2b^2+a^2+b^2)^2}{b^2(a^2b^2+a^2-b^2)^2}x_2^2.$$

The elliptic Frégier conic is regular if, and only if, $(a^2b^2-a^2+b^2)(a^2b^2+a^2-b^2) \neq 0$. (The factor $a^2b^2+a^2+b^2$ cannot vanish under the above made assumptions.) However, in elliptic geometry, the singular Frégier conics of a given conic can only be arranged in two groups (for details see [4]). Fig. 1 shows the generalized Frégier conic (in the sense of Thm. 2.1) for a hyperbolic (center) and an elliptic polarity (right). The curves e can be viewed as the Frégier conics of the conic c in the hyperbolic and elliptic plane.

2.2 Euclidean and pseudo-Euclidean Frégier conics

A singular polarity, *i.e.*, an involutive mapping on a straight line l can also be the basis of the Frégier construction. If l is chosen as the line at infinity, then the involutive mapping $\alpha: l \to l$ can either be hyperbolic or elliptic. In the first case, we can consider this Frégier construction as the *pseudo-Euclidean* version, while in the second case, α acting on l can serve as the absolute polarity of Euclidean geometry which leads to the well-known *Euclidean* version.

3 Arbitrary (non-involutive) projective mappings

In what follows, we shall replace the involutive projective mapping α acting on c with an arbitrary projective mapping $\beta:\ c \to c$. Such a mapping is uniquely defined by prescribing three pairs of assigned points, *i.e.*, three by two points $A, A', B, B', C, C' \in c$ with $A' = \alpha(A)'$, $B' = \alpha(B)$, and $C' = \alpha(C)$. We can show the following result:

Theorem 3.1. *The chords $[X, \alpha(X)]$ of c joining each point with its projective image envelop a conic f which spans with c a pencil of conics of the third kind if α is elliptic or hyperbolic. In the case of a parabolic projectivity α, the conics c and f span a pencil of the fifth kind,* i.e., *they hyperosculate each other.*

Proof. It means no loss of generality to assume that c is given by the homogeneous equation $x_0x_2 - x_2^2 = 0$. Further, we can assume that $A = 1:0:0$, $B = 1:1:1$, $C = 0:0:1$ and $A' = 1:u:u^2$, $B' = 1:v:v^2$, $C' = 1:w:w^2$ (with $u, v, w \neq 0, 1, \infty$, $u \neq v \neq w \neq u$). Then, the axis a of the projectivity α (which contains the points $[A, B'] \cap [A', B]$, $[A, C'] \cap [A', C]$, $[B, C'] \cap [B', C]$) has the homogeneous coordinates $a = u\overline{u} : w\overline{w} - \overline{u} : -\overline{w}$, where $\overline{u} = v - w$, $\overline{v} = w - u$, $\overline{w} = u - v$. Hence, a point $X = 1 : t : t^2$ (with $t \neq u, v, w, 0, 1, \infty$) is mapped to

$$X' = (t\overline{w} + \overline{u})^2 : (t\overline{w} + \overline{u})(tw\overline{w} + u\overline{u}) : (tw\overline{w} + u\overline{u})^2.$$

The chords $s = [X, X']$ with homogeneous coordinates

$$s = t(tw\overline{w} + u\overline{u}) : -t^2\overline{w} - t(w\overline{w} + \overline{u}) - u\overline{u} : t\overline{w} + \overline{u}$$

envelop the conic

$$e : \boldsymbol{x}^{\mathrm{T}} \begin{pmatrix} u^2\overline{u}^2 & u\overline{u}(w\overline{w} - \overline{u}) & -\overline{uw}(\overline{v} + w) \\ u\overline{u}(w\overline{w} - \overline{u}) & \overline{w}^2((u + \overline{w})^2 + 4u - 4\overline{w} + 4) & -\overline{w}(w\overline{w} - \overline{u}) \\ -\overline{uw}(\overline{v} + w) & -\overline{w}(w\overline{w} - \overline{u}) & \overline{w}^2 \end{pmatrix} \boldsymbol{x} = 0,$$

The conics c and e span a pencil of the third kind with the repeated line a as a singular conic in the pencil. The common points of a and c are the fixed points of α. The projectivity α is parabolic if, and only if, a is tangent to c, and then, c and e hyperosculate each other, *i.e.*, they span a pencil of the fifth kind. □

Figure 3 shows the three possible cases: an elliptic projectivity (left), a hyperbolic projectivity (in the middle), and a parabolic projectivity where p_α touches c and e (which are hyperosculating at the common point).

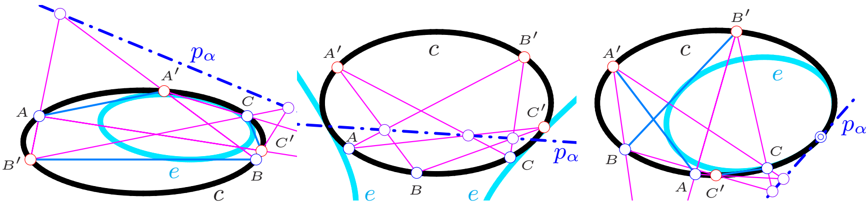

Fig. 3: The chords of c envelop a conic e provided that the endpoints are assigned in a projective mapping α. If α is elliptic or hyperbolic, the pencil spanned by c and e is of the third kind. A pencil of the fifth kind is obtained if α is parabolic.

4 Conclusion

We have shown two variations of Frégier's theorem. Both can be formulated in terms of projective geometry. The mathematical approach towards these generalizations are formulated in terms of polynomial equations and rational parametrizations. At no instant, extensions or assumptions on characteristic of underlying fields are necessary which makes the computations possible within the framework of finite fields. Hence, these results are universal in the sense of [7].

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