Universal porisms and Yff conics

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Abstract
The Yff circumellipse and the Yff inellipse of a triangle allow for a poristic family of triangles (henceforth called Yff porism), since the initial triangle is already an interscribed triangle. Surprisingly, the Yff porism can be parametrized by means of rational functions, and thus, it delivers a porism in Universal Geometry. This also allows us to give explicit examples of poristic triangle families over finite fields. Considering the Yff inellipse and Yff circumellipse as the basis of an exponential pencil of conics, we can iterate the construction of the porism and find an infinite sequence (and thus infinitely many) nested rational triangle porisms over the real (and complex) number field or a finite closed chain of porisms in the case of a finite field.

Key words: Porism, inellipse, circumellipse, triangle, rational porism, rational parametrization, finite field, finite projective plane.

Sačetak
Yff cirkumelipsa i Yff inelipsa trokuta dopuštaju poroznu obitelj trokuta (odsada nazvanu Yff porizam), budući da je početni trokut već interkriniran trokut. Iznenađujuće, Yff porizam se može parametrizirati pomoću racionalnih funkcija, i stoga daje porizam u Univerzalnoj geometriji. To nam također omogućuje da damo eksplicitne primjere obitelji porističkih trokuta nad konačnim poljima. Uzimajući u obzir Yff inelipsu i Yff cirkumelipsu kao osnovu eksponencijalne olovke konika, možemo ponavljati konstrukciju porizma i pronaći beskonačan niz (a time i beskonačno mnogo) ugniježđeni racionalni trokutni porizmi nad realnim (i kompleksnim) poljem brojeva ili konačnim zatvorenim lancem porizmi u slučju konačnog polja.

Ključne riječi: Porizam, inelipsa, cirkumelipsa, trokut, racionalni porizam, racionalna parametrizacija, konačno polje, konačna projektivna ravnina.

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1 Introduction

Porisms have attracted the interest of geometers in the past years anew. An exhausting overview of this topic and newer as well as many classical results can be found in [3] and [4]. Certain subproblems, dealing with a detailed analysis of Chapple's porism (the most prominent one cf. [1]) were studied in [18] and inspired further investigations of orbits and invariants in relation to porisms in [5, 6, 7]. The isotropic version of Chapple's porism was investigated in [12]. Not only orbits of points and centers related with the moving triangles have gained interest. The closely related topic of billiards within ellipses, and in conics in general, was enriched with new results. To mention only a few, see for example [8] and a study on Poncelet grids in [21]. Projective invariants of Poncelet closure figures are presented in [23, 26], the motions induced by Poncelet closure figures are studied in [24], the diagonals in Poncelet grids are the subject of interest in [25], and focal billiards are described in [22].

In the vast majority, and especially in the case of Chapple's porism, explicit analytical descriptions of Poncelet triangle families involve algebraic (and by no means rational) expressions (cf. [6, 7, 18, 20]). This limits direct symbolic computations, and sometimes, even graphical representations. Even the (from the computational point of view) simple case of the isotropic Chapple porism studied in [12] needs square roots in order to describe the vertices of the moving triangles. Now, the question is near: Are there porisms that allow for rational (or, in terms of homogeneous coordinates, even polynomial) parametrizations? Such porisms would then also exist in *Universal Rational Trigonometry* (as defined in [28]) and would also be well-defined in planes over finite fields.

In this article, we shall present porisms that can be described by rational (or polynomial) functions. Sec. 2 is devoted to the basic setting and notations. In this section, it is further shown that the tritangent and the circumconic allow for certain porisms in general. Then, in Sec. 3 the parametrization of the poristic triangle families interscribed between Yff conics are derived and some examples in planes over finite fields are given in order to show different phenomena that can occur in various exotic planes. Sec. 4 shows how to find more such rational porisms based on the Yff porism. Finally and for the sake of completeness, Sec. 5 collects some results in the Euclidean plane.

2 Porisms interscribed between the Yff ellipses

In the plane of the initial triangle $\Delta = ABC$, we describe points and lines by homogeneous trilinear coordinates. Thus, the vertices of Δ have the coordinates

$$A = 1:0:0,\ B = 0:1:0,\ C = 0:0:1. \quad (1)$$

Circumconics, *i.e.*, conics which pass through all three vertices of Δ, are given by a homogeneous equation of the form

$$\mathcal{C}:\ pyz + qzx + rxy = 0, \quad (2)$$

where $p, q, r \in \mathbb{F} \setminus \{0\}$ and $\mathbb{F}$ is some commutative field. The conics $\mathcal{C}$ are always regular if neither of p, q, r vanishes, since $\det \mathrm{H}\,\mathcal{C} = 2pqr$, provided that $\operatorname{char}\mathbb{F} \neq 2$. Here, and in the following $\mathrm{H}\,\mathcal{Q}$ shall denote the Hessian matrix of a (trivariate) form $\mathcal{Q}$.

In the beginning, $\mathbb{F}$ shall be the real or complex number field. Later, we also consider finite fields $\mathbb{F}$ of order q, which we shall denote by $\mathrm{GF}\,(q)$. The order q can be a prime or a prime power. Projective planes of order q shall be denoted by $\mathrm{PG}(2, q)$.

A conic inscribed into Δ, or simply, an inconic of Δ touches all side lines of Δ. An inconic should rather be termed *tritangent conic*, since the contact points with the sides of Δ may also be exterior points. We use the term inconic or inscribed just as a simplification, though we know that such conics are not necessarily inscribed into Δ in the elementary geometric sense.

The inconics of Δ in the aforementioned sense can be given by equations of the form

$$\begin{aligned} \mathcal{D}:\ & l^2x^2 + m^2y^2 + n^2z^2 \\ & -2lmxy - 2mnyz - 2nlzx = 0, \end{aligned} \tag{3}$$

where $l, m, n \in \mathbb{F} \setminus \{0\}$. Note that the conics $\mathcal{D}$ are regular if neither of l, m, n vanishes, since $\det \mathrm{H}\,\mathcal{D} = 2l^2m^2n^2$, again provided that $\operatorname{char}\mathbb{F} \neq 2$. It is worth pointing at the characteristic of the underlying field as we shall see later.

The conics in the pencil spanned by $\mathcal{C}$ and $\mathcal{D}$ are called *Yff conics* (cf. [14]) among them, we also find the *permutation conics* (see [15, 19]). The existence of a poristic triangle family interscribed between $\mathcal{C}$ and $\mathcal{D}$ is obvious, since there exists already one interscribed triangle, namely Δ. We shall call this family the *Yff porism.*

Now, we shall return to the case of $\operatorname{char}\mathbb{F} = 0$ (*e.g.*, $\mathbb{F} \cong \mathbb{R}, \mathbb{C}$). Any two regular conics $\mathcal{C}$, $\mathcal{D}$ with equations (2) and (3) allow for poristic families of certain polygons:

Theorem 2.1. *The pair $(\mathcal{C}, \mathcal{D})$ of conics circumscribed to and inscribed into Δ allows for a poristic family of $3n$-gons for $n \in \mathbb{N} \setminus \{0\}$.*

Proof. The conics in the pencil spanned by $\mathcal{C} : \mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = 0$ and $\mathcal{D} : \mathbf{x}^{\mathrm{T}}\mathbf{D}\mathbf{x} = 0$ have the equations $\mathbf{x}^{\mathrm{T}}(t\mathbf{C} + \mathbf{D})\mathbf{x} = 0$. In order to apply the Cayley criterion (cf. [27, p. 432]), we expand $\sqrt{\det(\mathrm{H}(\mathrm{t}\mathbf{C} + \mathbf{D}))}$ in a power series $S(t) = a_0 + a_1t + a_2t^2 + \dots$. With the abbreviations $\lambda := lmn$, $\pi := pqr$, and $\omega := lp + mq + nr$, we find

$$\begin{aligned} S(t) = \mathrm{i}\sqrt{2}\Bigg(& 4\lambda - \omega t + \mathbf{0}\cdot t^2 - \frac{1}{8}\frac{\pi}{\lambda}t^3 \\ & -\frac{1}{32}\frac{\pi\omega}{\lambda^2}t^4 - \frac{1}{128}\frac{\pi\omega^2}{\lambda^3}t^5 + \dots\Bigg). \end{aligned}$$

Hence, $\delta_3 = a_2 = 0$, $\delta_4 = a_3 = \frac{\mathrm{i}\sqrt{2}}{8}\frac{\pi}{\lambda}$,

$$\delta_5 = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = -\tfrac{1}{32}\tfrac{\pi^2}{\lambda^2}, \quad \delta_6 = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0,$$

$$\delta_7 = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \end{vmatrix} = \tfrac{-\mathrm{i}\sqrt{2}}{2^{14}}\tfrac{\pi^4}{\lambda^5}, \delta_8 = \begin{vmatrix} a_3 & a_4 & a_5 \\ a_4 & a_5 & a_6 \\ a_5 & a_6 & a_7 \end{vmatrix} = \tfrac{-\mathrm{i}\sqrt{2}}{2^{20}}\tfrac{\pi^5}{\lambda^7},$$
$$\delta_9 = 0, \dots,$$

and $\delta_{3k} = 0$ for all $k \in \mathbb{N}$ which confirms the statement. □

The Cayley criterion uses the complex number field. However, the equations (2) and (3) can be considered as conics in projective planes over arbitrary commutative fields and the criterion can be used to test pairs of conics whether or not they allow for poristic polygons interscribed in between.

The square root of the cubic polynomial can be expanded in a power series $S(t)$ anyhow. In [2], the Cayley criterion was used in order to count possible cases of conic pairs allowing for triangle porisms. This does neither answer the question whether such porisms exist nor what they look like if they exist.

As is clear from the proof of Thm. 2.1, the pair $(\mathcal{C}, \mathcal{D})$ of conics will never allow for interscribed quadrilaterals according to the assumptions made on the coefficients in their equations.

3 Parametrizing the poristic family

3.1 Basic properties

Two conics that merely fulfill the Cayley criterion will not immediately lead to an explicit description of the poristic triangle family interscribed between them. In order to give an explicit example which later will even allow for a generalization, we choose $p=q=r=l=m=n=1$ which yields the Yff circumellipse $\mathcal{M}$ and the Yff inellipse $\mathcal{N}$.

The resulting two conics are indeed triangle conics because of the cyclic symmetry of their equations, and of course, they are Yff conics (cf. [14]). Their equations in terms of homogeneous trilinear coordinates do not depend on Euclidean notions such as the side lengths or the interior angles of the triangle Δ:

$$\begin{aligned} \mathcal{M} &: xy + yz + zx = 0, \\ \mathcal{N} &: x^2+y^2+z^2-2(xy+yz+zx)=0. \end{aligned} \tag{4}$$

From the elementary (affine) point of view, these conics are ellipses, the Yff ellipses. They span a pencil of the third kind considered as a pencil of the real or complex projective plane. The point $X_1 = 1 : 1 : 1$ is the common pole and the antiorthic axis $\mathcal{L}_1 = 1 : 1 : 1$ (or with the homogeneous equation $x+y+z=0$) is the common polar of all regular conics in the pencil (see Fig. 1). Here and in the following, the labelling of points (centers) and lines (central lines) related to triangles follows the labelling in [13, 16]. (Later, in Sec. 5, this will be of more significance.) We shall also use shorthand X_1 for the point defined by $1:1:1$ and the symbol $\mathcal{L}_1$ if we mean the line $x+y+z=0$ even if they are not the incenter and the antiorthic axis in the elementary geometric sense.

In $\mathrm{PG}(2,\mathbb{C})$, the conics $\mathcal{M}$ and $\mathcal{N}$ share the pair of complex conjugate points $1:\varepsilon:\varepsilon^2$ and $1:\varepsilon^2:\varepsilon$ on $\mathcal{L}_1$ (with ε being a non-trivial cube root of unity), hence $\mathcal{M} \cap \mathcal{N} = \emptyset$ in the real projective plane. However, in some finite planes, their intersection is not empty.

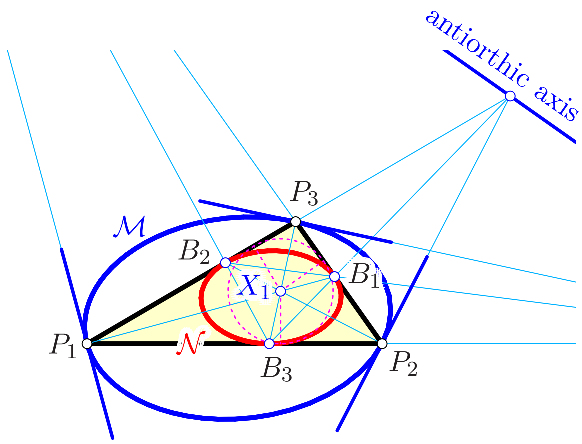

Figure 1: The Yff conics $\mathcal{M}$ and $\mathcal{N}$ in the real plane.

In order to describe the vertices P_1, P_2, P_3 of the triangles in the poristic family, we start with the homogeneous and polynomial parametrization of $\mathcal{M}$ given by

$$P_1 = -uv : u(u+v) : v(u+v),\ \ u:v \neq 0:0. \quad (5)$$

Note that the parametrization (5) of $\mathcal{M}$ makes sense over any field.

In order to find the remaining vertices P_2 and P_3, we determine the polar line of P_1 with respect to $\mathcal{N}$ which meets $\mathcal{N}$ in the contact points

$$\begin{aligned} B_2 &= u^2 : (u+v)^2 : v^2, \\ B_3 &= v^2 : u^2 : (u+v)^2, \end{aligned} \quad (6)$$

of the tangents from P_1 to $\mathcal{N}$. Note that the homogeneous representations of the contact points are also polynomial, *i.e.*, they do not involve square roots. This cannot be the case in CHAPPLE's porism (see, *e.g.*, [18, 27]). Now, we intersect the tangents $t_2 = [P_1, B_2]$ and $t_3 = [P_1, B_3]$ with the Yff circumellipse $\mathcal{M}$ and find the remaining vertices P_2, P_3 of the moving triangle as

$$\begin{aligned} P_2 &= v(u+v) : -uv : u(u+v), \\ P_3 &= u(u+v) : v(u+v) : -uv. \end{aligned} \quad (7)$$

For the sake of completeness, we determine the contact point B_1 of $[P_2, P_3]$ and $\mathcal{N}$, which has the homogeneous coordinates

$$B_1 = (u+v)^2 : v^2 : u^2. \quad (8)$$

By virtue of (6) and (8), we see that the homogeneous coordinate representation of B_1, B_2, and B_3 can be obtained from each other by applying cyclic shifts to the coordinate functions. The same holds true for the vertices P_1, P_2, and P_3 of the triangles in the poristic family. The fact that all coordinate functions of the vertices and the contact points are polynomial has the following consequence:

Theorem 3.1. *The Yff porism, that is the family of triangles interscribed between the Yff inellipse $\mathcal{N}$ and Yff circumellipse $\mathcal{M}$ given in* (4) *contains triangles whose vertices allow for rational parametrizations. The Yff porism is also well-defined over arbitrary fields $\mathbb{F}$ with positive characteristic not equal to 2.*

Proof. The rationality is obvious: The homogeneous coordinates of the vertices $P_i \in \mathcal{M}$ (5) and (7) as well as the contact points B_i (6) and (8) are polynomial. All polynomials are well-defined over any field. □

The case $\operatorname{char} \mathbb{F} = 2$ is excluded in Thm. 3.1 since the construction of the triangles in the poristic family uses the polar system of $\mathcal{N}$. In planes of characteristic 2, polarities are null polarities at the same time, and therefore, they are singular and all tangents of a conic pass through one point, the nucleus.

It makes sense to call the Yff porism a *unsiversal porism* in the sense of [28] for the vertices of the triangles (as well as the contact points) are given in terms of rational functions. Thus, they are defined over any finite field.

Depending on the characteristic of the underlying field $\mathbb{F}$, we can prove:

Theorem 3.2. *In any finite field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$, the Yff porism contains at most two degenerate triangles. In the case*

$\operatorname{char}\mathbb{F} = 3$*, the Yff porism contains a single degenerate triangle.*

Proof. The vertices (5) and (7) of the triangles in the Yff porism are at least collinear if, and only if, $\det(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0$ which is equivalent to

$$\delta^3 := (u^2 + uv + v^2)^3 = 0 \qquad (9)$$

(where $\mathbf{p}_i$ is a coordinate vector of P_i) vanishes for some parameter $u : v \neq 0 : 0$, *i.e.*, $\delta = u^2 + uv + v^2 = 0$. This is obviously a quadratic equation in $u : v$, and depending on the underlying field, it may have 0, 1, or two zeros.

If $\operatorname{char}\mathbb{F} = 3$, then $1 = -2$, and thus, $u^2 + uv + v^2 = u^2 - 2uv + v^2 = (u - v)^2$ which yields the single solution $u : v = 1 : 1$ (with multiplicity two). □

In Thm. 3.2, we did not explicitly state that the order of the underlying field is a prime, say p, different from a prime power p^k with $k \in \mathbb{N} \setminus \{0, 1\}$. It is clear that the number of degenerate triangles in an Yff porism will not exceed 2. If in $\mathrm{PG}(2, p)$ the Yff porism already has a degenerate triangle, then $\omega(x) = x^2 + x + 1$ already has at least one solution in $\mathrm{GF}(p)$ and $\omega(x)$ cannot be used for a quadratic field extension and no additional zeros will show up with a proper field extension.

We shall have a look at the following examples:
(1) If the underlying field $\mathbb{F}$ is a quadratic extension of $\mathrm{GF}(p)$, the number of zeros of δ may increase if $\mathbb{F} \cong \mathrm{GF}(p)[x]/(x^2 + x + 1)$. For example, there exists a unique quadratic extension of $\mathrm{GF}(2)$ in order to obtain $\mathrm{GF}(4)$, since x^2+x+1 is the only quadratic polynomial that has no zeros in $\mathrm{GF}(2)$. Hence, in $\mathrm{PG}(2, 4)$ the Yff porism contains two degenerate triangles, while in $\mathrm{PG}(2, 2)$ the Yff porisms consists of regular triangles only. We shall come back to $\mathrm{PG}(2, 4)$ in Sec. 3.2.3.

(2) The quadratic polynomial $x^2 + x + 1$ has a single zero of multiplicity two in $\mathrm{GF}(3)$. Therefore, it cannot be used for a quadratic extension of $\mathrm{GF}(3)$ in order to create $\mathrm{GF}(9)$. However, x^2+1 is suitable for the desired quadratic extension and its zeros are not zeros of δ from (9). Thus, in $\mathrm{GF}(9)$ the Yff porism still has a single degenerate triangle inherited from $\mathrm{GF}(3)$. For details, we refer to Sec. 3.2.6.

(3) The example of $\mathrm{GF}(5)$ is to show that there do exist quadratic field extensions so that the Yff porism shows both, degenerate and non-degenerate triangles. Both polynomials $\omega_1(x) = x^2+x+1$ and $\omega_2(x) = x^2+x+2$ have no zeros in $\mathrm{GF}(5)$. Since $\omega_1(x)$ is an inhomogeneous version of (9), the extension with $\omega_1(x)$ delivers two degenerate triangles, while the extension with $\omega_2(x)$ does not.

We will come back to field extensions and the thus created planes in Sec. 3.2.

The regularity condition for the contact triangle $B_1B_2B_3$ equals $2\delta = 0$. This shows again that the case $\operatorname{char}\mathbb{F} = 2$ plays an exceptional role.

Further, we can say:

Theorem 3.3. *If the Yff porism contains a degenerate triangle, then this triangle is a single point.*

Proof. According to Thm. 3.2, the matrix $\mathbf{P} := (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is singular if, and only if, (9) holds, *i.e.*, $\delta = 0$, or, likewise, $\operatorname{rk}\mathbf{P} \leq 2$. A triangle of the Yff porism becomes a single point if $\mathbf{P}$ is of rank 1 which is the case if, and only if, all 2×2 submatrices of $\mathbf{P}$ are singular. The determinants of the non-trivial 2×2 submatrices of $\mathbf{P}$ evaluate to one of the following polynomials (up to the coefficient -1, not playing a role even if $\operatorname{char}\mathbb{F} = 2$):

$$uv\delta, \quad u(u+v)\delta, \quad v(u+v)\delta,$$

which vanish all, if δ does. This is not the case if only one of the following is true $u : v = 1 : 0$, $u : v = 0 : 1$, or $u + v = 0$.

The only larger minor is already singular by assumption, and therefore, it does not have to be taken into account. □

In Thm. 3.3, a distinction of the underlying field is not necessary. Once $\delta = 0$, the Yff porism contains at least one degenerate triangle, no matter, if $\delta = 0$ is caused by a field extension or not.

Common points of $\mathcal{M}$ and $\mathcal{N}$ and degenerate triangles do not enter the scene independently:

Theorem 3.4. *A degenerate triangle in the Yff porism is necessarily a common point of $\mathcal{M}$ and $\mathcal{N}$, and* vice versa*.*

Proof. Assume that there is a parameter $u : v \neq 0 : 0$ (with $u, v \in \mathbb{F}$) such that (9) is annihilated. Then, according to Thm. 3.2, $P_1 = P_2 = P_3$, *i.e.*, a pose with a degenerate triangle is reached. Inserting (5) and (7) into (4), we see that both equations are also fulfilled.

On the other hand, common points of $\mathcal{M}$ and $\mathcal{N}$ can be found by eliminating one variable, say, *e.g.* z, from both equations in (4). This yields $(x^2 + xy + y^2)^2 = 0$, which is fulfilled by any of the parametrizations of (5) and (7) if, and only if, $u^2 + uv + v^2 = 0$, *i.e.*, the points P_i coincide. □

For the following 80 prime integers less than 1000 (which are in total 168), (9) has two solutions in $\mathrm{GF}(p)$:

$$\begin{gathered}7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127,\\ 139, 151, 157, 163, 181, 193, 199, 211, 223, 229, 241,\\ 271, 277, 283, 307, 313, 331, 337, 349, 367, 373, 379,\\ 397, 409, 421, 433, 439, 457, 463, 487, 499, 523, 541,\\ 547, 571, 577, 601, 607, 613, 619, 631, 643, 661, 673,\\ 691, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823,\\ 829, 853, 859, 877, 883, 907, 919, 937, 967, 991, 997.\end{gathered}$$

Thus, in $\mathrm{PG}(2, p)$ with one of the above p, the Yff porism contains two degenerate triangles and quadratic field extensions have to be constructed with a polynomial different from $x^2 + x + 1$ in any case. If $p = 3$, the polynomial $x^2 + x + 1$ is not suitable for a quadratic field extension, since then it is a full square. For any other $p \neq 3$ and not in the above list, a quadratic field extension with $x^2 + x + 1$ would add two degenerate triangles to the Yff porism.

Field extensions $\mathrm{GF}(p^k)$ with arbitrary $k > 2$ do not cause more degenerate triangles in the Yff porism as long as $x^2 + x + 1$ is not a divisor of the extension polynomial.

The chosen conics $\mathcal{M}$ and $\mathcal{N}$ with equation (4) have rather simple equations because of their relative position with respect to the underlying coordinate system. More general forms of rational and universal porisms can be obtained by applying collineations to $\mathcal{M}$, $\mathcal{N}$, and the family of interscribed triangles:

Theorem 3.5. *The totality of universal Yff porisms in a projective plane* $\mathrm{PG}(2, p)$ *can be obtained by applying the full group of regular projective transformations to the Yff porism determined by* $\mathcal{M}$ *and* $\mathcal{N}$*. In the projective plane* $\mathrm{PG}(2, p)$*, there exist* $p^3(p^3-1)(p^2-1)$ *collinear copies of the initial Yff porism.*

Proof. According to [9, p. 298], the number of 3×3 matrices $\mathbf{K}$ with entries from GF (p) and $\det \mathbf{K} = 1$ equals $p^3(p^3-1)(p^2-1)$. Since non-zero multiples of $\mathbf{K}$ describe the same collineation, it is admissible to normalize the transformation matrices such that their determinants are equal to unity. □

3.2 Examples of Yff porisms in small planes

In the following, we shall describe the universal porisms in some finite projective planes of low order, *i.e.*, in small planes. For details and basic information on finite projective planes, we refer to [11].

The points (1) appear as poses of the vertices P_1, P_2, P_3 of the triangles in the poristic family in any projective plane over any (finite) field. Since for any prime p (5) and (7) evaluate to multiples of the canonical basis vectors, we shall use the labels of the vertices of the initial triangle Δ for those poses of the points P_i.

The vertices P_i of the triangles have the coordinate representations $\mathbf{p}_i(u_0, v_0)$ with $i \in \{1, 2, 3\}$ and the homogeneous parameter $(u, v) \neq (0, 0)$ always traces the projective line $\mathrm{PG}(1, \mathbb{F})$, *i.e.*,

$$(u, v) \in \{(1_{\mathbb{F}}, 0_{\mathbb{F}}), (1_{\mathbb{F}}, 1_{\mathbb{F}}), \ldots, (0_{\mathbb{F}}, 1_{\mathbb{F}})\}.$$

Note that the parameter pairs are normalized, *i.e.*, the first coordinate is set to unity (except the last one) which can always be achieved. So, they are ordered numerically. This has no geometric meaning and is done just in order not to lose a point.

3.2.1 The minimal projective plane

The minimal projective plane is the unique projective plane with seven points and seven lines sometimes referred to as the Fano plane. (Despite not showing the Fano property: Here, the three diagonal points of a quadrilateral are collinear.) Its algebraic model is erected over GF (2). Although we have emphasized at several places that the case char $\mathbb{F} = 2$ has to be excluded or at least to be handled with care (polarities are null polarities at the same time), we find that the parametrizations (5) and (7) of the points P_i evaluate to meaningful expressions, whence we shall have a look at it.

There is only one triangle in the family: It is the standard triangle that plays its role in a threefold way and it is the only non-trivial triangle in this particular poristic family. We collect the triangles depending on the homogeneous parameter $u : v$ in a table:

$u : v$	$1 : 0$	$1 : 1$	$0 : 1$
triangle	BCA	ABC	CAB

Fig. 2 tries to illustrate the three poses of the moving triangle.

Note that in the minimal plane the conic $\mathcal{M}$ has a singular equation, since the determinant of the coefficient matrix vanishes:

$$\mathcal{M} : xy + yz + zx = \mathbf{x}^{\mathrm{T}} \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{=:\mathbf{M}} \mathbf{x} = 0,$$

hence $\det \mathbf{M} = 0$ and $\operatorname{rk} \mathbf{M} = 2$. Here, we shall point out that the usual way of extracting the coefficient matrix from a quadratic form fails: We cannot multiply coefficients by 2.

However, the three points on $\mathcal{M}$ are not collinear as should be the case with conics.

In comparison, $\mathcal{N}$ whose equation simplifies due to the speciality of the underlying field according to

$$\mathcal{N}: \; x^2 + y^2 + z^2 = 0$$

is regular, but contains the three collinear points

$$B_1 = 1:1:0, \quad B_2 = 1:0:1, \quad B_3 = 0:1:1.$$

They also lie on the line $\mathcal{L}_1$. Hence, $\mathcal{N}$ and $\mathcal{L}_1$ agree as sets of points. Note that there is no contact between the sides of Δ and $\mathcal{N}$, since all tangents of $\mathcal{N}$ pass through its nucleus.

3.2.2 The thirteen point plane

The unique projective plane of order three has thirteen points and lines. It can be modeled over GF (3). An incidence graph and the coordinatization that we use are shown in Fig. 3. In the thirteen point plane

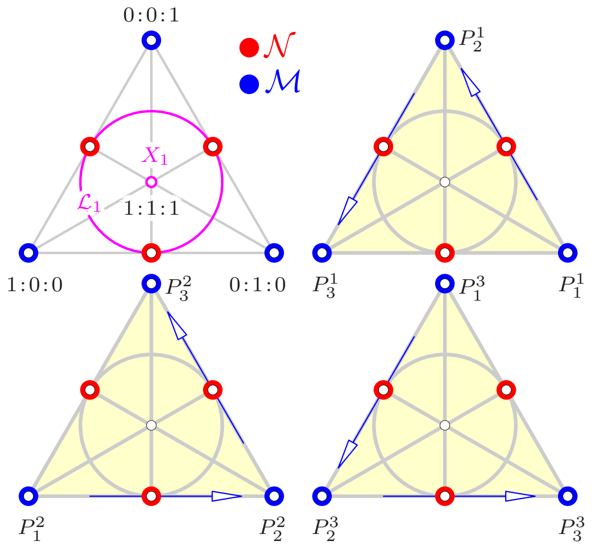

Figure 2: The projective plane PG(2, 2) with its seven points and lines and the two conics $\mathcal{M}$ and $\mathcal{N}$. A special feature can be observed here: The inconic $\mathcal{N} = \mathcal{L}_1$ consists of three collinear points. Superscripts denote the pose.

PG(2, 3), the two conics $\mathcal{M}$ and $\mathcal{N}$ are regular. They share precisely one point, *i.e.*, X_1 and have the line $\mathcal{L}_1$ as common tangent there.

Again, the standard triangle plays a threefold role for the parameter values

$$u : v = 1 : 0, \quad u : v = 1 : 2, \quad u : v = 0 : 1.$$

According to Thm. 3.2, the Yff porism in PG(2, 3) contains a single degenerate triangle corresponding to δ's single (double) root $u : v = 1 : 1$. The degenerate triangle equals the point X_1, which is the only point of intersection of $\mathcal{M}$ and $\mathcal{N}$. This holds true in any plane over GF (3^k) with positive k.

Fig. 4 illustrates the position of $\mathcal{M}$ and $\mathcal{N}$ in PG(2, 3) relative to each other. Fig. 5 is given in order to illustrate the poristic family in PG(2, 3).

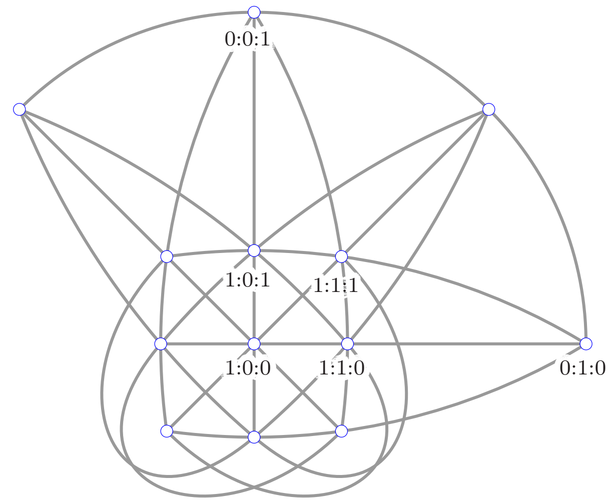

Figure 3: The plane of order 3 with 13 points and lines is isomorphic to the projective plane PG(2, 3).

In order to track the triangle while it moves through the poristic family, we collect the different poses of $P_1P_2P_3$ in the following table:

$u\!:\!v$	1:0	1:1	1:2	0:1
triangle	BCA	1:1:1	ABC	CAB

.

3.2.3 The projective plane of order 4

The projective plane PG(2, 4) of order 4 consists of 21 points and lines. The underlying field GF (4) is obtained from GF (2) by the unique quadratic field extension GF (4) = GF (2)[x]/(x^2+x+1). This clearly shows that the Yff porism in PG(2, 4) contains two degenerate triangles (since the polynomial used for the extension is an inhomogeneous version of δ from (9)). However, PG(2, 4) inherits all properties from PG(2, 2) including the singularity and regularity of conics. Only the order of the un-

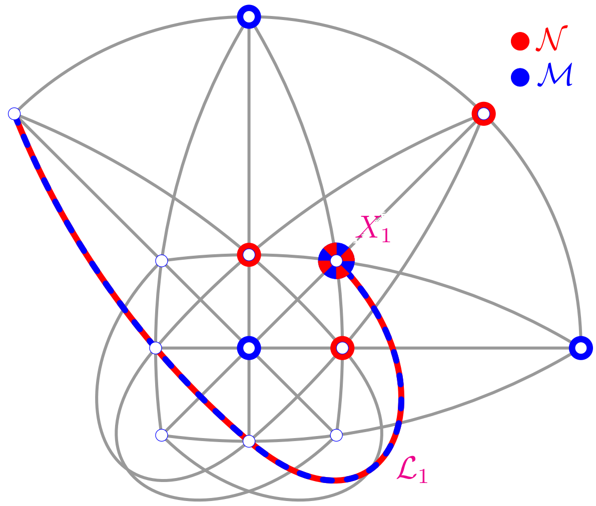

Figure 4: Both Yff conics consist of four non-collinear points, they intersect in X_1 and share the tangent $\mathcal{L}_1$ there.

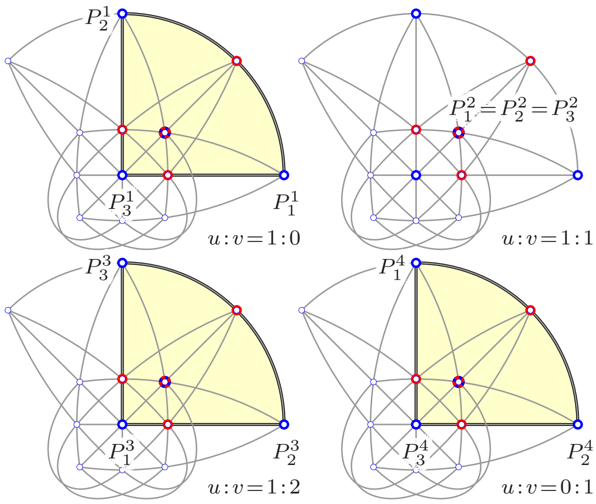

Figure 5: The Yff porism in the 13 point plane shows a single degenerate triangle corresponding to $u : v = 1 : 1$ and all three vertices fall into the point $1 : 1 : 1$ (see top right).

derlying field changes, not so the characteristic.

The field GF (4) is obtained from GF (2) by the quadratic field extension with the *only* quadratic polynom $\omega(x)=x^2+x+1$ that has no zeros over GF (2). If we label the elements of GF (4) by $\{0, 1, a, 1+a\}$, then we compute modulo 2 and simplify sums and products according to $a^2+a+1=0$. Fig. 6 shows an incidence diagram of PG$(2,4)$ with the coordinatization used in this section.

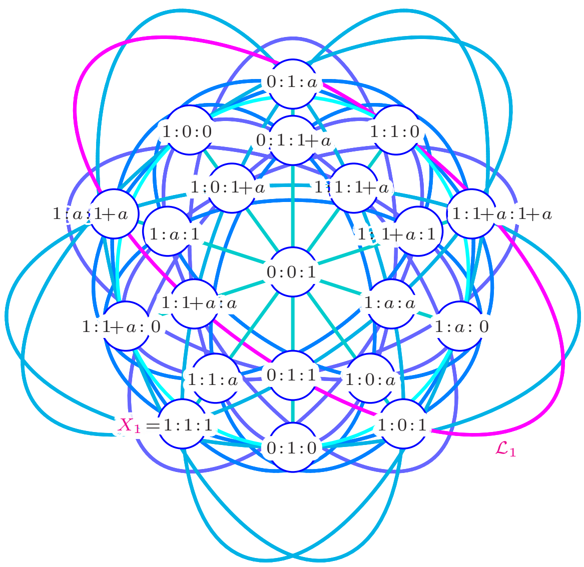

Figure 6: The 21 point plane modeled over GF (4) contains Fano-subplanes, and therefore, it doesn't have the Fano property: The diagonal points of quadrilaterals are collinear mirroring the property of the algebraic model based on the field GF (4) whose characteristic equals 2.

The conic $\mathcal{M}$ has a singular equation in PG$(2,4)$ (like in the case of GF (2) and for the same reasons) and consists of the five points

$$0:1:0,\quad 1:0:0,\quad 1:a:1+a,\\ 1:1+a:a,\quad 0:0:1,$$

of which no three are collinear, while the conic $\mathcal{N}$ has a regular equation and consists of the five collinear points

$$1:1:0,\quad 1:0:1,\quad 0:1:1,\\ 1:a:1+a,\quad 1:1+a:a.$$

Obviously, the two Yff conics intersect in

$$S_1=1:a:1+a \quad\text{and}\quad S_2=1:1+a:a.$$

The diagram in Fig. 7 illustrates the relative position of the two conics $\mathcal{M}$ and $\mathcal{N}$ in PG$(2,4)$.

The quadratic form $\delta=u^2+uv+v^2$ equals ω if we substitute $u=1$ and $v=x$. Hence, the two new elements in the extension GF (4) of GF (2) are zeros of δ. Thus, in GF (4), the Yff porism contains two degenerate triangles. The parameters $u:v=1:a$ and $u:v=1:1+a$ deliver the two degenerate triangles in the Yff porism, which coincide with the points S_1 and S_2 (the intersections of $\mathcal{M}$ and $\mathcal{N}$).

By virtue of (5) and (7), we find the vertices of the triangles in the poristic family as

$u:v$	$1:0$	$1:1$	$1:a$
triangle	BCA	ABC	$1:a:1+a$
$u:v$	$1:1+a$	$0:1$	
triangle	$1:1+a:a$	CAB	

.

3.2.4 The planes over GF (5) and GF (7)

We shall treat the two planes over GF (5) and GF (7) simultaneously which helps simplifying the comparison. In Figs. 8 and 9,

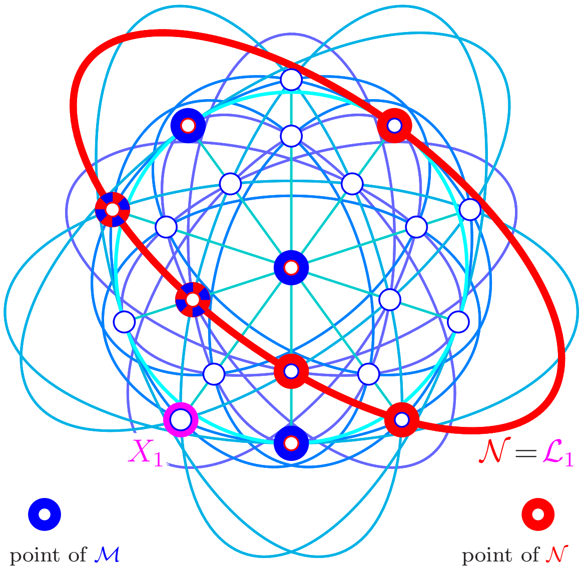

Figure 7: In PG(2, 4), the *circumconic* $\mathcal{M}$ is regular, while the *inconic* $\mathcal{N}$ consists of five collinear points. Further, $\mathcal{M}$ and $\mathcal{N}$ share two points playing the role of the degenerate triangles in the poristic family.

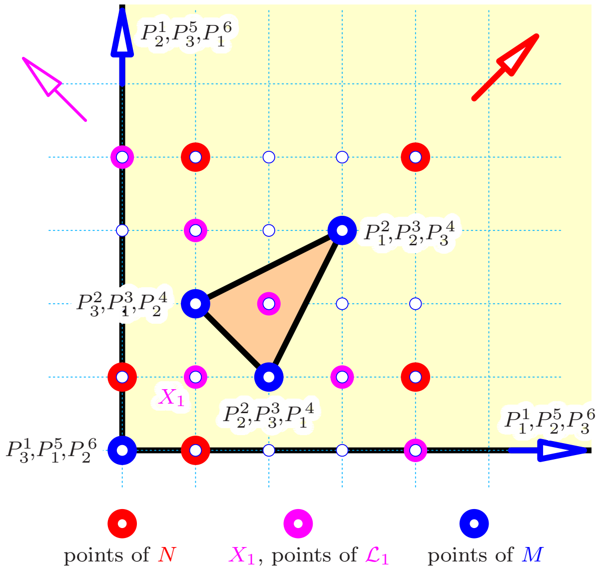

Figure 8: The Yff porism in an affine part of the plane PG(2, 5): $\mathcal{M} \cap \mathcal{N} = \emptyset$ and the Yff porism contains only non-degenerate triangles each playing a threefold role.

we have illustrated an affine version of the respective Yff porisms.

In both planes PG(2, 5) / PG(2, 7), both Yff conics have regular equations and consist of 6 / 8 points where no three of them are collinear. While $\mathcal{M}$ and $\mathcal{N}$ do not intersect in PG(2, 5), they share the two points

$$S_1 = 1 : 2 : 4 \quad \text{and} \quad S_2 = 1 : 4 : 2$$

in PG(2, 7). Further, $\mathcal{M}$ and $\mathcal{N}$ share the tangents S_1 and S_2. Thus, in PG(2, 7), the pencil spanned by $\mathcal{M}$ and $\mathcal{N}$ resembles a pencil of the third kind as we know it from the case $\mathbb{F} = \mathbb{C}$.

The two intersection points S_1 and S_2 of $\mathcal{M}$ and $\mathcal{N}$ in PG(2, 7) serve as the degenerate triangles in the Yff porism. On the contrary, degenerate triangles are missing in the Yff porism over GF (5).

3.2.5 A cubic field extension: the projective plane over GF (8)

The field GF (8) shall be constructed from GF (2) by the cubic field extension with the roots of $\omega(x) = x^3 + x + 1$ which is irreducible in GF (2). This means computations are performed modulo 2 and modulo ω. The elements of GF (8) shall be denoted by

$$\{0, 1, a, 1+a, a^2, 1+a^2, a+a^2, 1+a+a^2\}.$$

The triangle $P_1P_2P_3$ with the parametrization (5) and (7) reaches the following poses while the homogeneous parameter $u : v$ tra-

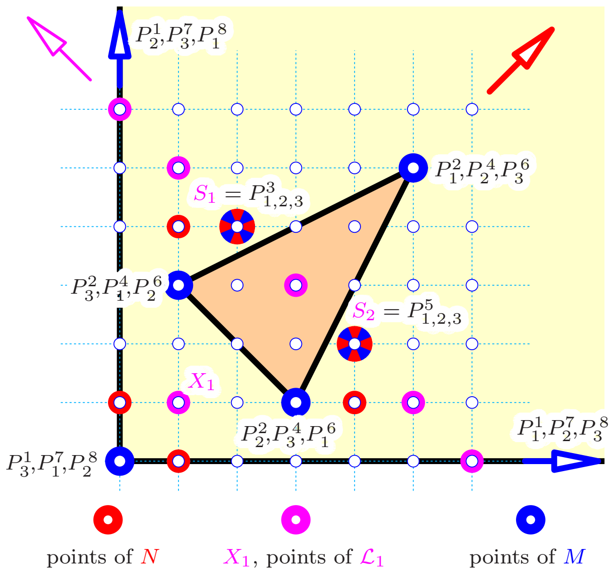

Figure 9: The Yff porism in an affine part of the plane PG$(2,7)$: $\mathcal{M}\cap\mathcal{N} = \{S_1, S_2\}$ and the two degenerate triangles fall into the points S_1 and S_2.

verses PG$(1,8)$:

$u\!:\!v$	$1\!:\!0$	$1\!:\!1$	$1\!:\!a$
triangle	BCA	ABC	$R_1S_1T_1$
$u\!:\!v$	$1\!:\!1\!+\!a$	$1\!:\!a^2$	$1\!:\!1\!+\!a^2$
triangle	$R_2S_2T_2$	$S_1T_1R_1$	$T_2R_2S_2$
$u\!:\!v$	$1\!:\!a\!+\!a^2$	$1\!:\!1\!+\!a\!+\!a^2$	$0\!:\!1$
triangle	$T_1R_1S_1$	$S_2T_2R_2$	CAB

,

where we have set

$$R_1 = 1\!:\!a^2\!:\!1\!+\!a,\quad S_1 = 1\!:\!a\!:\!1\!+\!a\!+\!a^2,\quad T_1 = 1\!:\!a\!+\!a^2\!:\!1\!+\!a^2,$$
$$R_2 = 1\!:\!1\!+\!a\!+\!a^2\!:\!a,\quad S_2 = 1\!:\!1\!+\!a\!:\!a^2,\quad T_2 = 1\!:\!1\!+\!a^2\!:\!a+a^2.$$

There is no degenerate triangle in the Yff porism in PG$(2,8)$, since (9) has no zeros in GF(8). The poristic orbit of the triangle ABC splits into three suborbits and the *moving* triangle changes the orbits more or less irregularly, for it is not possible to establish an ordering in GF(8).

3.2.6 The only Desarguesian plane of order 9

Among the four non-isomorphic projective planes of order 9, only the plane PG$(2,9)$ is Desarguesian. Because of the commutativity of GF(9), the projective plane PG$(2,9)$ is also Pappian, and thus, the study of conics makes sense there (cf. [27]), whence it makes sense to consider this particular plane of order 9. (Note that any Pappian plane is Desarguesian, but on the contrary, not any Desarguesian plane is Pappian, cf. [27]).

The field GF(3) shall be extended to GF(9) by adding the roots of $\omega(x) = x^2+1$ (which do not exist in GF(3)). It is well-known that any other quadratic polynomial (without zeros in GF(3)) leads to an isomorphic copy of the field of order 9. We label the nine elements of GF(9) by

$$\{0, 1, 2, a, 1+a, 2+a, 2a, 1+2a, 2+2a\}$$

and calculate modulo 3 and modulo ω. Then, (5) and (7) yield the following triangles:

$u\!:\!v$	$1\!:\!0$	$1\!:\!1$	$1\!:\!2$	$1\!:\!a$
triangle	BAC	$1\!:\!1\!:\!1$	ACB	$R_1S_1T_1$
$u\!:\!v$	$1\!:\!1\!+\!a$	$2\!+\!a$	$2a$	$1\!+\!2a$
triangle	$S_1T_1R_1$	$R_2S_2T_2$	$T_1R_1S_1$	$T_2R_2S_2$
$u\!:\!v$	$2\!+\!2a$	$0\!:\!1$		
triangle	$S_2T_2R_2$	CAB		

,

where we have used the abbreviations

$$R_1 = 1:2+a:2+2a, \quad S_1 = 1:a:1+a,$$
$$T_1 = 1:1+2a:2a,$$
$$R_2 = 1:2a:1+2a, \quad S_2 = 1:1+a:a,$$
$$T_2 = 1:2+2a:2+a,$$

There is only one degenerate triangle in the poristic family. The degenerate triangle corresponds to the parameter $u : v = 1 : 1$ and is inherited from GF (3), and as such, rather a feature of GF (3), than of GF (9). However, $\operatorname{char} \mathrm{GF}(9) = \operatorname{char} \mathrm{GF}(3) = 3$.

4 More universal porsims

4.1 The tangent triangle

For the case of CHAPPLE's porism (triangles with common incircle and circumcircle), it is shown that the vertices of the tangent triangle Δ_t, *i.e.*, the triangle of tangents to the circumcircle at Δ's vertices, move on an ellipse which is of course traced thrice, while Δ traverses the poristic family (see [18]).

For the tangent triangle of $\mathcal{M}$ in the Yff porism, we can show:

Lemma 4.1. *In any projective plane* $\mathrm{PG}(2, \mathbb{F})$ *with* $\operatorname{char} \mathbb{F} \neq 2$*, the vertices of the tangent triangle of* $\mathcal{M}$ *trace a single conic* $\mathcal{T}$*, while the initial triangle traverses the Yff porism.* $\mathcal{T}$ *is the image of* $\mathcal{N}$ *under the harmonic homology with center* X_1 *and axis* $\mathcal{L}_1$.

Proof. When determining the vertices T_i of the tangent triangle Δ_t, we observe that the tangents t_i of $\mathcal{M}$ at P_i have homogeneous coordinate vectors proportional to those of B_i given in (8) and (6). Then, for example, $T_1 = t_2 \cap t_3$ is given by

$$T_1 = u^2 + 3uv + v^2 : -u^2 - uv + v^2 : u^2 - uv - v^2$$

and the others are obtained by permuting the coordinate functions of the latter. Now, the implicitization of the parametrization of T_1 yields

$$\mathcal{T} : \sum_{\text{cyclic}} x^2 + 3yz = 0 \qquad (10)$$

and confirms that T_1 runs on a conic. It is easily verified that the points T_2 and T_3 also trace $\mathcal{T}$ while Δ and Δ_t move through the Yff porism.

We can check that $\operatorname{cr}(P_i, B_i, X_1, T_i) = -1$ for all $i \in \{1, 2, 3\}$. Furthermore, by the initial construction, it is elementary to verify that $[B_i, B_j] \cap t_j = [P_i, P_j] \cap t_k \in \mathcal{L}_1$ for $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Hence, the harmonic homology $\rho_{X_1, \mathcal{L}_1}$ (with center X_1 and axis $\mathcal{L}_1$) sends B_i to T_i and maps the respective tangents of $\mathcal{N}$ and $\mathcal{T}$ onto each other. ☐

Lem. 4.1 is in particular valid in the real and in the complex projective plane. In planes of characteristic two (or equivalently, in planes over fields of characteristic two), the construction of the tangent conic must fail for two reasons:
(1) The tangents of a conic pass through a single point (the nucleus). If the characteristic of the underlying plane (or field) equals 3, we have $\mathcal{T} : \sum_{\text{cyclic}} x^2 = 0$.
(2) In the planes of characteristic 2 there are no harmonic homologies.

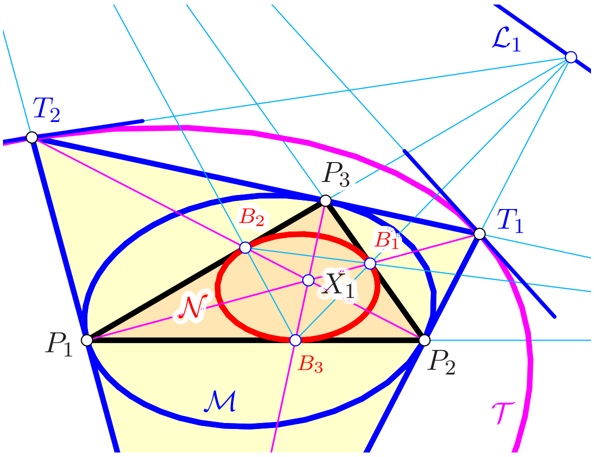

Figure 10: Iterating the Yff porism can yield infinitely many universal porisms.

Fig. 10 illustrates the geometric relations between pairs of subsequent conics and the orbits of tangent triangles in the Yff pencil.

If $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{T}$ denote the coordinate matrices of the homogeneous equations of the conics $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{T}$, then $\mathbf{T} = 5\cdot\mathbf{M}+2\cdot\mathbf{N}$. Hence, $\mathcal{T}$ is also an Yff conic (cf. [14]).

4.2 A linear family of matrices

The coefficient matrices of the conics $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{T}$ belong to the special family of matrices that can be parametrized by

$$\mathbf{L}(p,q) = \begin{pmatrix} p & q & q \\ q & p & q \\ q & q & p \end{pmatrix}, \tag{11}$$

with entries $p, q \neq 0$ from some commutative field $\mathbb{F}$. The linear one-parameter family of matrices (11) forms a straight line in the eight-dimensional projective space of 3×3 matrices. Note that the coefficient matrices of these triangle conics do also not depend on Euclidean notions such as the side lengths of Δ.

For the sake of simplicity (and since non-zero multiples do not count), we shall assume $\gcd(p,q)=1$ (in the underlying field under consideration). The matrices $\mathbf{L}(p,q)$ are regular if, and only if,

$$p^3 - 3pq^2 + 2q^3 = (p-q)^2(p+2q) \neq 0,$$

i.e., $p\!:\!q \neq 1\!:\!1$ and $p\!:\!q \neq -2\!:\!1$. The two singular matrices in the family (11) are $\mathbf{L}(1,1)$ and $\mathbf{L}(-2,1)$. While $\operatorname{rk}\mathbf{L}(1,1) = 1$ and $\ker\mathbf{L}(1,1) = [(1,-1,0),(1,0,-1)]$, we have $\operatorname{rk}\mathbf{L}(-2,1)=2$ and $\ker\mathbf{L}(-2,1)=(1,1,1)$.

The regular matrices (11) form a commutative group, since the multiplication obeys the rule

$$\mathbf{L}(p_1,q_1)\cdot\mathbf{L}(p_2,q_2) = \\ = \mathbf{L}(p_1p_2+2q_1q_2,\, p_1q_2+p_2q_1+q_1q_2),$$

the inverses are

$$\mathbf{L}(p,q)^{-1} = \mathbf{L}\,(p+q,-q)\,/(p^2+pq-2q^2)$$

(provided that $p^2+pq-2q^2 \neq 0$, *i.e.*, $\mathbf{L}(p,q)$ is regular), and $\mathbf{L}(1,0)$ is the neutral element.

4.3 More tangent triangles

The coefficient matrices of $\mathcal{M}$ and $\mathcal{N}$ of the Yff conics in (4) are $\mathbf{M}=\mathbf{L}(0,1)$ and $\mathbf{N}=\mathbf{L}(1,-1)$. For the coefficient matrix of $\mathcal{T}$ from (10), we find $\mathbf{T}=\mathbf{L}(2,3)$. Moreover, the respective matrices in (4) satisfy

$$\mathbf{M}\cdot\mathbf{N} = -2\mathbf{I}_3 \tag{12}$$

as long as $\operatorname{char}\mathbb{F} \neq 2$. Further, we can easily verify that

$$\mathbf{T} = \mathbf{M}(\mathbf{N}^{-1}\mathbf{M})^1 = -\frac{1}{2}\mathbf{M}^3. \tag{13}$$

Since $\mathcal{T}$ is the polar conic of the dual conic $\mathcal{N}^\star$ (the set of tangents of $\mathcal{N}$) with respect to $\mathcal{M}$, $\mathcal{T}$ is a successor of $\mathcal{N}$ and $\mathcal{M}$ in the exponential pencil of conics spanned by $\mathcal{N}$ and $\mathcal{M}$ (cf. [10]).

According to [10], the conics in the exponential pencil spanned by two regular conics $\mathcal{C}_0 : \mathbf{x}^{\mathrm{T}}\mathbf{C}_0\mathbf{x} = 0$ and $\mathcal{C}_1 : \mathbf{x}^{\mathrm{T}}\mathbf{C}_1\mathbf{x} = 0$ have equations of the form $\mathbf{x}^{\mathrm{T}}\mathbf{C}(t)\mathbf{x} = 0$ with

$$\mathbf{C}(t) = \mathbf{C}_1 \cdot \left(\mathbf{C}_0^{-1} \cdot \mathbf{C}_1\right)^{t-1}, \quad t \in \mathbb{F}.$$

Again, the case $\operatorname{char}\mathbb{F} = 2$ has to be excluded, since there $\mathbf{M}$ is singular. In [10], the coordinate t in the exponential pencil was assumed to be real. By virtue of Lem. 4.1, $t \in \mathbb{F}$ it makes sense.

With $\mathbf{C}_0 = \mathbf{N}$ and $\mathbf{C}_1 = \mathbf{M}$ and by virtue of (12), the coefficient matrices of the conics' equations in the thus defined exponential pencil are

$$\mathbf{C}(t) = (-\tfrac{1}{2})^{t-1}\mathbf{M}^{2t-1} = (-2)^u\mathbf{N}^{1-2u}, \tag{14}$$

where $t + u = 1$. Here, it is more obvious that fields $\mathbb{F}$ with $\operatorname{char}\mathbb{F} = 2$ do not play a role.

Defining the matrices $\mathbf{C}_i$ the other way around, *i.e.*, $\mathbf{C}_1 = \mathbf{N}$ and $\mathbf{C}_0 = \mathbf{M}$, means to trace the pencil of conics in the opposite direction.

In the case $\mathbb{F} \cong \mathbb{R}$, we can describe the limit conics in the exponential pencil: For $t \to \infty$, the matrices $\mathbf{C}(t)$ converge towards the singular matrix $\mathbf{L}(1, 1)$ which describes $\mathcal{L}_1$ as a repeated line. The limit $t \to -\infty$ yields $\mathbf{L}(2, -1)$ corresponding to X_1 as the intersection of the pair of complex conjugate tangents

$$\sum_{\text{cyclic}} x^2 - yz = (x + \varepsilon y + \varepsilon^2 z)(x + \varepsilon^2 y + \varepsilon z) = 0,$$

(where ε is a non-trivial third root of unity) common to all conics in the linear and the exponential pencil (in the case of $\mathbb{F} = \mathbb{C}$). Obviously, $\mathbf{C}(2)$ evaluates to the coefficient matrix of the conic $\mathcal{T}$ in (10), *i.e.*, $\mathbf{C}(2) = -\frac{1}{2}\mathbf{L}(2, 3)$.

We can repeat the construction of the tangent triangle now applied to $\mathcal{T}$ and the triangles $T_1T_2T_3$ and, by virtue of Lem. 4.1, we can state:

Theorem 4.1. *Any pair of subsequent conics in the exponential pencil spanned by $\mathcal{N}$ and $\mathcal{M}$ allows for a universal porism of $3n$-gons, provided there are sufficiently many points and conics in the plane under consideration and the exponential parametrization is evaluated only at integers.*

We can also give a more synthetic generation of the sequence of pairs of conics allowing for rational porisms:

Theorem 4.2. *Any conic in the exponential pencil together with its tangent triangle (including the contact points) is the image of its pre-predecessor under a harmonic homology with center X_1 and axis $\mathcal{L}_1$.*

4.4 The chain of universal porisms

The triangle $B_1B_2B_3$ is perspective with $P_1P_2P_3$ and both are perspective with $T_1T_2T_3$. The common perspector for any pair of triangles is the point X_1. Any two out of the three triangles share the perspectrix, the line $\mathcal{L}_1$.

According to Thm. 4.1, this is true for any pair of triangles in the infinite chain of contact and tangent triangles. In this way, we find infinitely many nested Desargues configurations with the perspector X_1 and perspectrix $\mathcal{L}_1$. The same holds true for the contact triangle. These Desargues configurations are more special than those constructed in [17].

Subsequent contact and tangent triangles are mapped to their successors by means of a harmonic homology. Further, the coefficients of equations of all conics in the pencil contain only elements of the underlying field and (especially in the Euclidean case) do not depend on the triangle's side lengths, we have:

Theorem 4.3. *Independent of the underlying field* $\mathbb{F}$ *(with* $\operatorname{char}\mathbb{F} \neq 2$*), the exponential pencil of conics spanned by* $\mathcal{N}$ *and* $\mathcal{M}$ *contains at most as many poristic* $3n$*-gon families as there are pairs of subsequent conics in the exponential pencil of conics.*

The number of poristic triangle families in the Yff pencil is equal to the number of points on a line if $\mathbb{F} \cong \mathbb{R}, \mathbb{C}$. In the case of finite fields we have to distinguish several cases:

(1) If $\operatorname{char}\mathbb{F} = 2$, the Yff pencil consists of three conics with the coefficient matrices $\mathbf{N} = \mathbf{L}(1,0)$, $\mathbf{M} = \mathbf{L}(0,1)$, $\mathbf{T} = \mathbf{L}(1,1)$ of rank 3, 2, 1. The *conic* $\mathcal{T}$ is the repeated line $\mathcal{L}_1$, which agrees with $\mathcal{N}$ as point set. In PG(2, 2), tangent conics do not exits, to be more precise the tangent triangle of $\mathcal{M}$ collapses to a point as can be seen by evaluating the parametrization of $\mathcal{T}$ given prior to (10). So, there is only a triad of triangles forming the one and only poristic family. (as explained in Section 3.2.1).

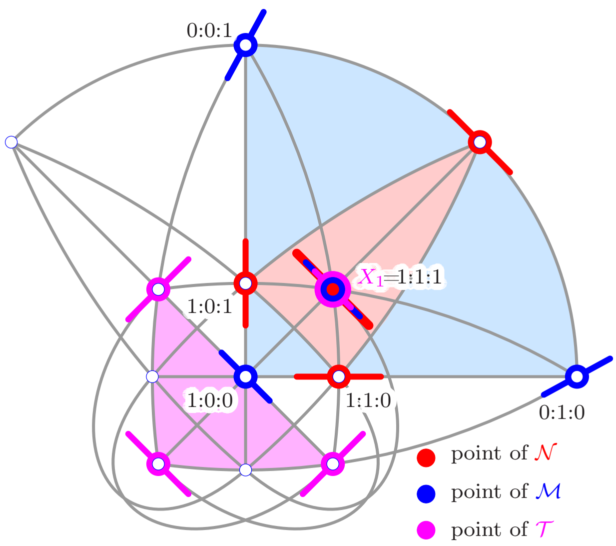

Figure 11: The chain of Yff porisms in PG(2, 3) contains three poristic triangle families: (i) between $\mathcal{N}$ and $\mathcal{M}$, (ii) between $\mathcal{M}$ and $\mathcal{T}$, and (iii) between $\mathcal{T}$ and $\mathcal{N}$. The point X_1 plays the role of the degenerate triangle in each porism. The line $\mathcal{L}_1$ (with multiplicity two) is a degenerate conic in the discrete exponential pencil.

(2) If $\operatorname{char}\mathbb{F} = 3$, the Yff pencil contains the four conics defined by the coefficient matrices $\mathbf{N} = \mathbf{L}(1,2)$, $\mathbf{M} = \mathbf{L}(0,1)$, $\mathbf{T} = \mathbf{L}(1,0)$, and further $\mathbf{L}(1,1)$. The latter is of rank 1 and corresponds to the repeated line $\mathcal{L}_1$ as a singular conic. The tangent triangle

of $\mathcal{M}$ moves on $\mathcal{T}$ and passes one instant of degeneracy, which is again the point X_1 corresponding to $u : v = 1 : 1$. Since there are only 3 regular conics in the pencil, we find only 3 different Yff porisms in PG$(2, 3)$ (up to collineations). Fig. 11 shows one particular pose in each of the three nested poristic families.

(3) Let us now assume that char $\mathbb{F} \neq 2, 3$ is a prime p. Then it is rather elementary to verify that the coefficient matrices of the conics in the pencil are matrices proportional to

$$\mathbf{L}\left(\frac{1}{6}(4^k - 4), \frac{1}{6}(4^k + 2)\right), \quad k \in \mathbb{F}.$$

(Note the denominator.) The base $4_{\mathbb{F}}$ can never be a generator of the multiplicative group in some GF (p), for odd powers of $2_{\mathbb{F}}$ cannot be reached. Hence, the number of conics in the Yff pencil is at most $\frac{1}{2}(p-1)$.

In Tab. 1, we have collected those primes $17 \leq p \leq 2011$ for which in PG$(2, p)$ the number y of conics in the discrete exponential pencil and of Yff porisms is less than $\frac{1}{2}(p-1)$.

In the case of a prime $p = F(k) = 2^{2^k} + 1$, we have observed that $y(p) = 2^k$ if $k \in \{1, 2, 3, 4\}$. Unfortunately, the projective plane of order $F(0) = 3$ does not fit. Since no prime $F(k)$ with $k > 5$ is known (as to October 2024), it is therefore also unclear, whether there do exist further finite projective planes of Fermat prime order $F(k)$ that allow for only 2^k Yff porisms.

p	17	31	41	43	73	89	97	109	113
y	4	5	10	7	9	11	24	18	14
p	127	137	151	157	193	223	229	233	241
y	7	34	15	26	48	37	29	12	38
p	251	257	277	281	283	307	313	331	337
y	25	8	47	35	47	51	78	15	21
p	353	397	401	409	431	433	439	449	457
y	44	22	100	102	43	36	73	112	38
p	499	521	569	571	577	593	601	617	631
y	83	130	142	57	72	74	25	77	45
p	641	643	673	683	691	727	733	739	761
y	42	107	24	11	115	121	122	123	190
p	769	809	811	827	881	911	919	929	937
y	192	202	135	214	55	91	153	232	117
p	953	971	977	997	1009	1013	1021	1033	1049
y	34	97	244	166	252	46	170	129	131
p	1051	1069	1093	1097	1103	1129	1153	1163	1181
y	175	170	182	137	29	282	144	83	118
p	1193	1201	1217	1249	1289	1297	1321	1327	1361
y	149	150	76	78	161	324	30	221	340
p	1399	1409	1423	1429	1433	1459	1471	1481	1489
y	233	352	237	42	179	243	245	185	372
p	1553	1579	1597	1601	1609	1613	1627	1657	1697
y	97	263	266	200	201	26	271	46	424
p	1699	1709	1721	1723	1753	1777	1789	1801	1811
y	283	122	215	287	73	37	298	25	181
p	1831	1873	1889	1913	1933	1993	1999	2003	2011
y	305	468	236	239	322	498	333	143	201

Table 1: Orders p of planes PG$(2, p)$ in which the number y of subsequent regular conics and porisms in the discrete exponential Yff pencil is less than $\frac{1}{2}(p-1)$.

Applying Thm. 3.5 to all results in this section leads to all possible variants of chains of rational Yff porisms in finite planes. Thms. 4.1, 4.2, and 4.3 remain valid if we apply any regular projective transformation to $\mathcal{N}$, $\mathcal{M}$, and the interscribed triangle family.

5 Some Euclidean properties of the Yff porism

For the sake of completeness, we shall end the study of the Yff pencil and Yff porism by adding some results concerning the Euclidean plane. We restrict ourselves to a small number of moving points.

5.1 Some central orbits

With the parametrization of the triangles in the poristic family from (5) and (7) we can immediately determine the orbits of triangle centers. Many of them turn out to be centers which still does not answer the question raised in [5] why there are so many elliptic orbits. Of course, since each triangle vertex takes the role of any other vertex, the orbits of centers are traced three times by the corresponding center. We can state:

Theorem 5.1. *The orbits of the triangle centers X_i of a triangle Δ traversing the Yff porism in the Euclidean plane with Kimberling indices $i = \{2, 3, 4, 5, 6\}$ are ellipses.*

Proof. The centroid X_2 is the harmonic conjugate of the ideal line $\omega := a : b : c$, *i.e.*, the pole of ω with respect to Δ. Hence, we obtain a parametrization of the one-parameter family of centroids with triangle center function

$$\begin{gathered} bc(u^6+v^6)+3bcuv(u^4+v^4)\\ +(b-c)(b+c-2a)(v^4-u^4)uv\\ -(3a^2-8ab+2ac+b^2-4c^2)u^4v^2\\ -(3a^2+2ab-8ac-4b^2+c^2)u^2v^4\\ -(6a^2-6ab-6ac-3b^2+5bc-3c^2)v^3u^3, \end{gathered}$$

i.e., the second and third coordinate function can be obtained by cyclically replacing a, b, c, while we keep the parameters u and v in their place.

Implicitization of the latter parametrization yields the quadratic trivariate form

$$\begin{gathered} \mathcal{O}_2: \sum_{\text{cyclic}} a(a+b+c)x^2\\ -(2a(b+c)-b^2+bc-c^2)yz=0, \end{gathered}$$

which is the equation of a conic centered at the yet unknown triangle center with the generating trilinear center function

$$\begin{gathered} 2(b+c)a^2-(3b^2+5bc+3c^2)a\\ +(b+c)(b^2-3bc+c^2). \end{gathered}$$

The orbits of the circumcenter, the orthocenter, the nine-point center, and the symmedian point are determined in the same way, once a parametrization of the respective centers is known. The circumcenter X_3 is the center of the circumircle $\mathcal{U}$ of $\Delta = P_1P_2P_3$ with the equation

$$\begin{gathered} \mathcal{U}: \sum_{\text{cyclic}} x^2auv(u+v)\Big(bc(b-c)(u^3-v^3)\\ +\big((b+c)a^2-abc-b^3+2b^2c-bc^2-c^3\big)u^2v\\ +\big((b+c)a^2-abc-b^3-b^2c+2bc^2-c^3\big)uv^2\Big)\\ +\Big(a^2bc(u^2+v^2)(u^4-u^2v^2+v^4)\\ +a\big((b+c)a^2+3abc-(b-c)(b^2+c^2)\big)u^5v\\ -a\big((b+c)a^2-3abc-(b-c)(b^2+c^2)\big)uv^5\\ -\big(a^4-(3b-2c)a^3-2(b^2+c^2)a^2\\ +(3b^3-4b^2c+bc^2-2c^3)a+(b^2-c^2)^2\big)u^4v^2\\ -\big(a^4+(2b-3c)a^3-2(b^2+c^2)a^2\\ -(2b^3-b^2c+4bc^2-3c^3)a+(b^2-c^2)^2\big)u^2v^4\\ -\big(+2a^4-(b+c)a^3-(4b^2-5bc+4c^2)a^2\\ (c+b)(b^2-4bc+c^2)a+2(b^2-c^2)^2\big)u^3v^3\Big)yz. \end{gathered}$$

Now, it is rather elementary to determine the center of $\mathcal{U}$ as the pole of ω with respect to $\mathcal{U}$. We omit writing down the rather lengthy parametrization of the circumcenters. The elimination of the homogeneous parameter $u : v \neq 0 : 0$ yields the equation of an ellipse centered at the unknown triangle center

$$\begin{gathered}C_3 = a\big((b+c)a^4 - (2b^2+5bc+2c^2)a^3 + \\ 4bc(b+c)a^2 + \\ (2b^4+b^3c-4b^2c^2+bc^3+2c^4)a \\ -(b-c)^2(b+c)^3\big) ::\end{gathered}$$

The orthocenter X_4 is the intersection of Δ's altitudes. Note that in terms of trilinear coordinates, the homogeneous trilinear coordinates of the altitudes are found as

$$\mathbf{h}_k = \mathbf{p}_k \times \mathbf{G}(\mathbf{p}_i \times \mathbf{p}_j)$$

with $(i,j,k) \in J_3$, where

$$J_3 := \{(1,2,3),(2,3,1),(3,1,2)\},$$

and $\mathbf{p}_i$ are homogeneous coordinate vectors of the vertices P_i, and $\mathbf{G}$ is the singular 3×3 matrix

$$\mathbf{G} = \begin{pmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos B & 1 \end{pmatrix}$$

ruling the orthogonality in Δ's plane (cf. [13, p. 29]). We recall that the cosines of Δ's interior angles can be expressed in terms of its side lengths as rational functions:

$$\cos A = (b^2+c^2-a^2)/(2bc) \quad \text{(cyclic)}.$$

The intersection of any two altitudes results in a rather lengthy homogeneous parametrization of the orthocenter's trace, which after implicitization, again results in a conic $\mathcal{C}_4$ with the equation

$$\begin{gathered}\mathcal{C}_4 : \sum_{\text{cyclic}} x^2 a(a^2-b^2-c^2) \\ \big(a^3-(b+c)a^2-(b^2+c^2)a+(b+c)(b-c)^2\big) \\ -\big((b^2+bc+c^2)a^4 \\ -2(b^2-c^2)^2a^2+(b^2-bc+c^2)(b^2-c^2)^2\big)yz=0\end{gathered}$$

centered at the yet unknown triangle center with the trilinear center function

$$\begin{gathered}C_4 = (b^2-bc+c^2)a^4 - 2(b+c)(b^2-bc+c^2)a^3 \\ +2bc(b-c)^2a^2 \\ +2(b+c)(b^2-bc+c^2)(b-c)^2a \\ -(b^2-3bc+c^2)(b^2-c^2)^2 :: .\end{gathered}$$

In order to verify the statement for X_5, we recall that the nine-point center is the circumcenter of the medial triangle $\Delta_m = M_1M_2M_3$ with M_i being the midpoint of the segments M_jM_k (again with $(i,j,k) \in J_3$). Note that the midpoint of the segment M_jM_k is the harmonic conjugate of the ideal point of the line $[M_j, M_k]$ with respect to M_j and M_k.

The symmedian point X_6 is the perspector of Δ and its tangent triangle $\Delta_t = T_1T_2T_3$ whose vertices T_i are the intersections of the tangents t_j and t_k of the circumcircle $\mathcal{U}$ at P_j and P_k. □

In a similar way, we can show that X_{75} (the isotomic conjugate of X_1) traces the conic

$$\begin{gathered}\mathcal{C}_{75} : \sum_{\text{cyclic}} x^2a^2(a^2+b^2+c^2) - \big((b^2+c^2)a^2 \\ +2bc(b+c)a - bc(b^2-bc+2c^2)\big)yz = 0\end{gathered}$$

which is centered at $X_9 = b+c-a ::$.

Triangle centers of the initial triangle that lie on $\mathcal{M}$ trace $\mathcal{M}$ three times. These are

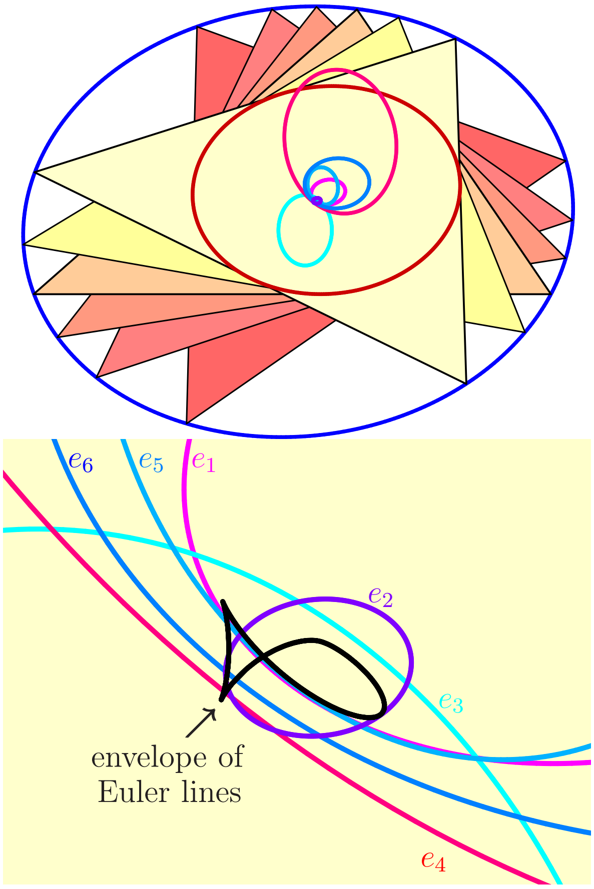

Figure 12: Top: Some triangles in the Euclidean Yff porism and the traces of the triangle centers $X_1, \ldots, X_6$.
Bottom: Close-up of the top figure including the envelope of the Euler lines.

the 124 triangle centers X_i with Kimberling indices (< 63000)

$$i \in \{88, 100, 162, 190, 651, 653, 655, 658, 660, 662, 673, 771, 799, 823, 897, 1156, 1492, 1821, 2349, 2580, 2581, 3257, 4598, 4599, 4604, 4606, 4607, 8052, 20332, 23707, 24625, 27834, 29059, 32680, 34085, 34234, 36083 - 36102, 37128 - 37143, 37202 - 37223, 38340, 40110, 43069, 43192, 43757 - 43764, 45875, 46116 - 46122, 55321, 55325, 55328, 55331, 60055 - 60057, 61240, 62535\}.$$

The same holds true for the inconic $\mathcal{N}$, where the following 24 triangle centers with Kimberling numbers (< 63000) orbit three times:

$$i \in \{244, 678, 2310, 2632, 2638, 2643, 3248, 4094, 4117, 10501, 24012, 41211, 42074 - 42084, 52302$$

Further, 204 centers lie on the antiorthic axis $\mathcal{L}_1 : x + y + z = 0$ which have the following Kimberling indices (< 63000):

$$i \in \{44, 649, 650, 652, 654, 656, 657, 659, 661, 672, 770, 798, 822, 851, 896, 899, 910, 1155, 1491, 1575, 1635, 1755, 2173, 2182, 2183, 2225, 2227 - 2240, 2243 - 2247, 2252 - 2254, 2265, 2272, 2290, 2312 - 2315, 2348, 2483, 2484, 2503, 2509, 2511, 2515, 2516, 2522, 2526, 2578, 2579, 2590, 2591, 2600, 2610, 2624, 2630, 2631, 2635, 2637, 2641, 2642, 3000, 3013, 3287, 3330, 3768, 4394, 4724, 4782, 4784, 4790, 4813, 4893, 4979, 7655, 7659, 8043, 8061, 9356, 9360, 9393, 9404, 9508, 9511, 10495, 13401, 14298, 14299, 14300, 15586, 17410, 17418, 17420, 18116, 20331, 20979, 21127, 21894, 22108, 22443, 23503, 24533, 24750, 25143, 29357, 29361, 30600, 38472, 39690, 40109, 40137, 40338, 44151, 44319, 45877, 45881 - 45886, 46380 - 46393, 47777, 47810, 47811, 47826 - 47828, 47842, 48019 - 48033, 48160, 48162, 48193, 48194, 48213, 48226, 48244, 48544, 48572, 50328, 50335, 50336, 50349, 50350, 50358, 50359, 50454, 50455, 50505, 50525, 53300, 54258, 54277, 54278, 55216, 57164, 58288, 58374, 58773, 58842\}.$$

Fig. 12 (top) shows the orbits of $X_1, \ldots, X_6$ in the Euclidean Yff porism. The bottom of Fig. 12 shows a close-up of the central orbits together with the envelope of the Euler lines, which is a sextic curve.

6 Final remarks

The construction of further Yff porisms by means of tracing the exponential pencil (14) or by applying harmonic homologies to existing pairs of conics leaves open, whether field extensions enrich the family of Yff porisms or not. The number of points, lines, and even conics in the extended plane is definitely raised. This does not necessarily mean that the exponential pencil contains more members. In principle, the parametrization of the exponential pencil

(14) can be evaluated at any value taken from the underlying field. However, it is unclear if matrix powers evaluate to meaningful matrices if we insert elements from the extension of a finite field, since discrete logarithms evaluate only to integers.

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