Degenerate Cubic Surfaces and the Wallace-Simson-Theorem in Space

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Wallace-Simson - planar version

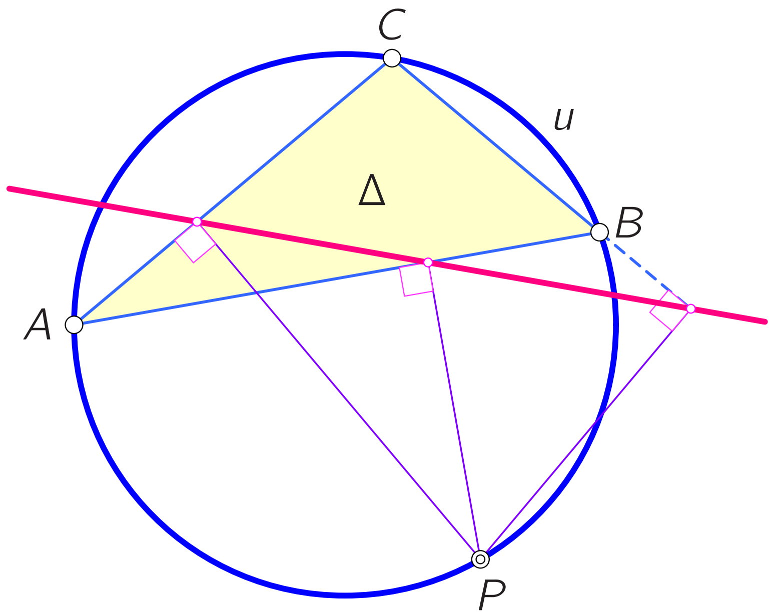
$\Delta = ABC$ ... triangle (in the Euclidean plane)

u ... Δ's circumcircle

$P \in u$ ... arbitrary point on u

Theorem:

The feet of the normals from P to Δ's sides are collinear if, and only if, P is chosen on u.



In the plane: The locus of such points P is never degenerate!

Wallace-Simson - spatial version

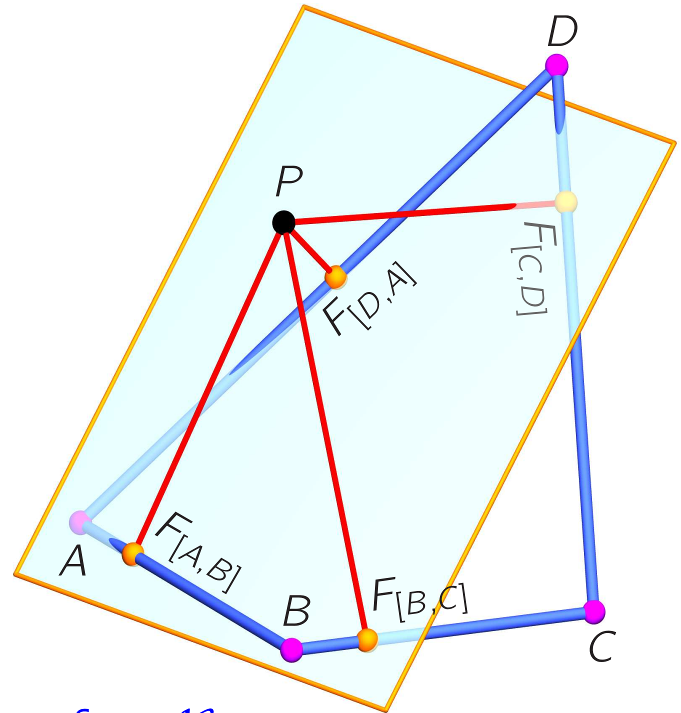
We replace the triangle by a skew quadrilateral $ABCD$ and ask for all points P such that the feet

$$F_{[A,B]},\ F_{[B,C]},\ F_{[C,D]},\ F_{[D,A]}$$

of the normals from P to the side lines

$$[A, B],\ [B, C],\ [C, D],\ [D, A]$$

are coplanar.



All points P with four coplanar feet lie on a cubic surface $\mathcal{K}$ passing through the vertices of the quadrilateral.

Are there **conditions on $ABCD$ such that $\mathcal{K}$ is degenerate**, *i.e.*, $\mathcal{K}$ splits into a plane and a quadric?

Wallace-Simson - spatial version - computation

coordinate vectors of the vertices A, B, C, D

$\mathbf{a}=(0,0,0), \mathbf{b}=(a,0,0), \mathbf{c}=(b,c,0), \mathbf{d}=(d,e,f)$

feet of normals from $P=\mathbf{x}$

$$F_{[A,B]}=\mathbf{b}\alpha,\ F_{[B,C]}=\mathbf{b}(1-\beta)+\mathbf{c}\beta,\ \ldots$$

with parameters $\alpha=\frac{\langle\mathbf{x},\mathbf{b}\rangle}{\|\mathbf{b}\|^2},\ \beta=\frac{\langle\mathbf{x}-\mathbf{b},\mathbf{c}-\mathbf{b}\rangle}{\|\mathbf{c}-\mathbf{b}\|^2},\ \ldots$

condition on four points $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ to lie in one plane

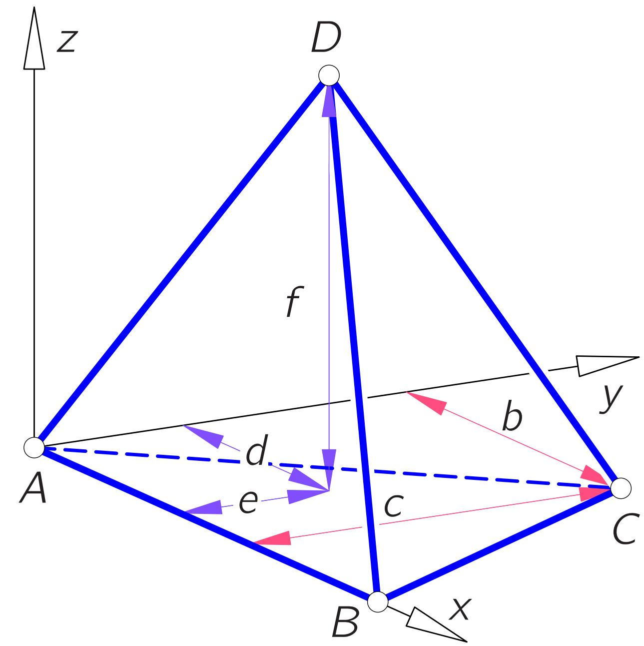
$\det(\mathbf{p},\mathbf{q},\mathbf{s})+\det(\mathbf{q},\mathbf{r},\mathbf{s})+\det(\mathbf{r},\mathbf{p},\mathbf{s})-\det(\mathbf{p},\mathbf{q},\mathbf{r})=0$

equation of $\mathcal{K}$, the locus of all $P=\mathbf{x}$ such that ...

$$\mathcal{K}:\ \varepsilon_0-\varepsilon_1+\varepsilon_2-\varepsilon_3=0$$

ε_i ... i-th elementary symmetric function in $\alpha,\beta,\gamma,\delta$,

$A, B, C, D\in\mathcal{K}$



Suitable choice of the coordinate system simplifies the computation, but causes special cases.

A cubic surface instead of the circumcircle

Four points A, B, C, D define three different skew quadrilaterals

$ABCD$, $ABDC$, $ACBD$.

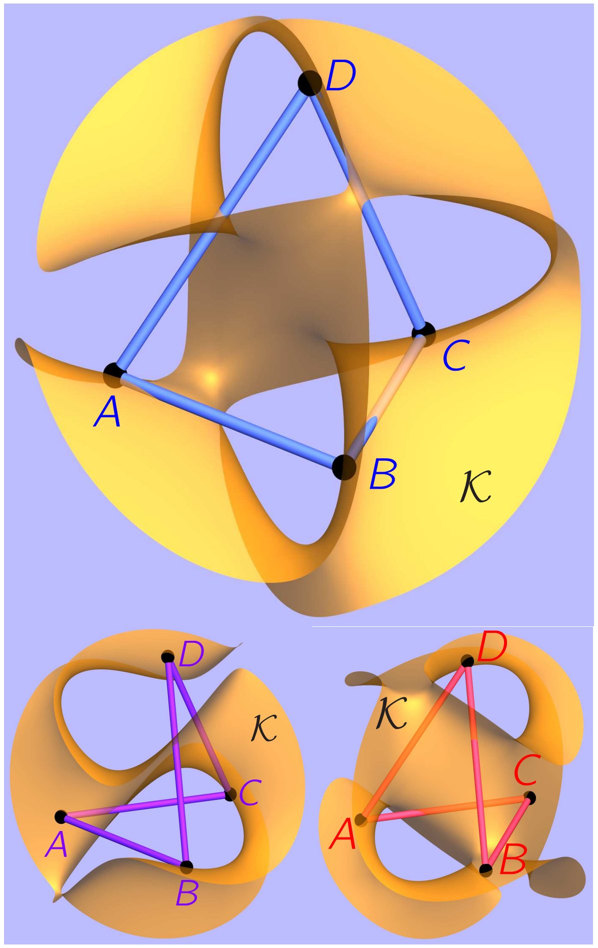
$\Longrightarrow$ There are three different cubic surfaces $\mathcal{K}$.

Degenerate surfaces $\mathcal{K}$ can be found by chosing the vertices of a regular tetrahedron.

Are these the only cases?

Is there a condition on $\mathcal{K}$ such that it degenerates?

How to find conditions on $ABCD$ such that $\mathcal{K}$ degenerates?

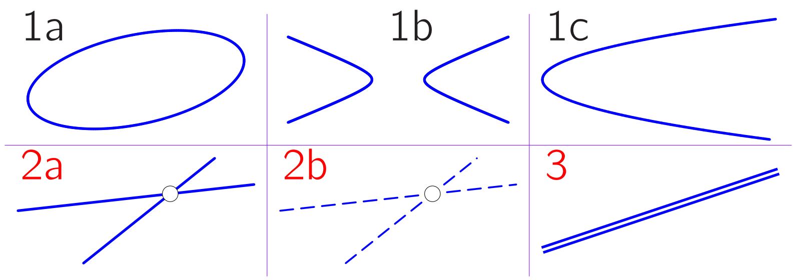


Degenerate conics and cubics in the plane

For conics there is one simple criterion for the degeneracy.

$C: \ \mathbf{x}^{\mathsf{T}} \mathbf{A} \mathbf{x} = 0$ is degenerate $\Longleftrightarrow \det \mathbf{A} = 0$

$\mathbf{x}$ . . . homogeneous coordinates, $\mathbf{A} \in \mathbb{C}^{3\times 3}$



cases 1a–c are regular, cases 2a,b with $\operatorname{rk} \mathbf{A} = 2$, case 3 with $\operatorname{rk} \mathbf{A} = 1$

There are different approaches to the degeneracy criterion $\det \mathbf{A} = 0$ for a conic:

A degenerate polarity leads to a degenerate conic or . . .

Degenerate conics and cubics in the plane

... assume that the degenerate conic is the union of two straight lines

$$L: \; l_0x_0 + l_1x_1 + l_2x_2 = 0 \text{ and } M: \; m_0x_0 + m_1x_1 + m_2x_2 = 0.$$

Then, compare the coefficients of all monomials $x^i y^j$ of $L \cdot M = 0$ and $C = 0$:

$$l_i m_i = a_{ii}, \quad i \in \{0, 1, 2\},$$
$$l_i m_j + l_j m_i = a_{ij}, \quad (i, j) \in \{(0, 1), (0, 2), (1, 2)\}.$$

Elimination of l_i, m_i yields only one condition

$$a_{00}a_{11}a_{22} + 2a_{01}a_{02}a_{12} - a_{00}a_{12}^2 - a_{01}^2a_{22} - a_{02}^2a_{11} = \det \mathbf{A} = 0.$$

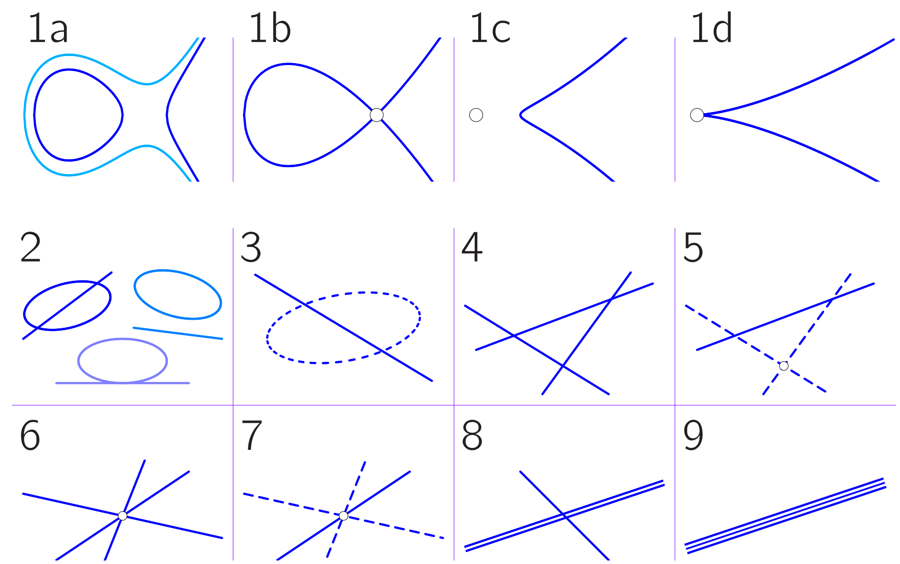
This is a very simple technique that applies to many algebraic problems.

Degenerate conics and cubics in the plane

In the case of planar cubics there are more different types of degenerate curves. $\Longrightarrow$ We cannot expect a single criterion for the degeneracy.

Case 1a: elliptic, cases 1b–d rational; all non-degenerate

Case 2–9: degenerate curves of degree (cycles of degree 3)



Important fact:

Each univariate cubic polynomial with real coefficients has at least one real root. $\Longleftrightarrow$
If a trivariate cubic polynomial (with real coefficients) factors, then there is at least one real factor of degree 1.

Degeneracy conditions for cubic surfaces

$$\mathcal{K}: \sum_{r+s+t\leq 3} k_{r,s,t} x^r y^s z^t = 0 \ldots \text{equation of the cubic surface}$$

assume $\mathcal{K}$ is degenerate $\Longrightarrow$ union of a plane $\mathcal{P}$ and something, say $\mathcal{Q}$, of degree 2

$$\mathcal{P}: \; l_0 + l_1 x + l_2 y + l_3 z = 0, \quad \mathcal{Q}: \sum_{r+s+t\leq 2} q_{r,s,t} x^r y^s z^t = 0$$

$$A \in \mathcal{K} \text{ and } \mathbf{a} = (0,0,0) \Longrightarrow k_{000} = 0$$

two cases to be treated separately (due to the proper choice of the coordinate system):

(A) $A \in \mathcal{P} \iff l_0 = 0$

(B) $A \in \mathcal{Q} \iff q_{000} = 0$

$$\mathcal{K} = \mathcal{P} \cup \mathcal{Q} \iff \sum_{r+s+t\leq 3} k_{r,s,t} x^r y^s z^t - (l_0 + l_1 x + l_2 y + l_3 z) \cdot \left(\sum_{r+s+t\leq 2} q_{i,j,k} x^i y^j z^k \right) = 0$$

collect the coefficients of monomials $x^r y^s z^t$, eliminate l_i and $q_{r,s,t}$

and take either case into account! $\Longrightarrow$

Degeneracy conditions for cubic surfaces - case (A)

$$k_{010}^3 k_{300} - k_{010}^2 k_{100} k_{210} + k_{010} k_{100}^2 k_{120} - k_{030} k_{100}^3 = 0,$$
$$k_{001}^3 k_{300} - k_{001}^2 k_{100} k_{201} + k_{001} k_{100}^2 k_{102} - k_{003} k_{100}^3 = 0,$$
$$k_{001}^3 k_{030} - k_{001}^2 k_{010} k_{021} + k_{001} k_{010}^2 k_{012} - k_{003} k_{010}^3 = 0,$$
$$k_{010}^2 k_{200} - k_{010} k_{100} k_{110} + k_{020} k_{100}^2 = 0,$$
$$k_{001}^2 k_{200} - k_{001} k_{100} k_{101} + k_{002} k_{100}^2 = 0,$$
$$k_{001}^2 k_{020} - k_{001} k_{010} k_{011} + k_{002} k_{010}^2 = 0,$$
$$-2k_{001} k_{010}^3 k_{300} + k_{00,1} k_{010}^2 k_{100} k_{210} - k_{001} k_{030} k_{100}^3 +$$
$$+ k_{010}^3 k_{100} k_{201} - k_{010}^2 k_{100}^2 k_{111} + k_{010} k_{021} k_{100}^3 = 0.$$

7 equations in 19 unknowns $k_{r,s,t}$ of degree ≤ 5

Degeneracy conditions for cubic surfaces - case (B)

$$[12, 116, [k_{010}, 6], [k_{020}, 6], [k_{030}, 4], [k_{100}, 6], [k_{120}, 4], [k_{200}, 6], [k_{210}, 4], [k_{300}, 4]],$$
$$[12, 116, [k_{001}, 6], [k_{002}, 6], [k_{003}, 4], [k_{100}, 6], [k_{102}, 4], [k_{200}, 6], [k_{201}, 4], [k_{300}, 4]],$$
$$[12, 116, [k_{001}, 6], [k_{002}, 6], [k_{003}, 4], [k_{010}, 6], [k_{012}, 4], [k_{020}, 6], [k_{021}, 4], [k_{030}, 4]],$$

$$[10, 84, [k_{010}, 6], [k_{020}, 2], [k_{030}, 2], [k_{100}, 6], [k_{110}, 4], [k_{200}, 6], [k_{210}, 4], [k_{300}, 4]],$$
$$[10, 84, [k_{001}, 6], [k_{002}, 2], [k_{003}, 2], [k_{100}, 6], [k_{101}, 4], [k_{200}, 6], [k_{201}, 4], [k_{300}, 4]],$$
$$[10, 84, [k_{001}, 6], [k_{002}, 2], [k_{003}, 2], [k_{010}, 6], [k_{011}, 4], [k_{020}, 6], [k_{021}, 4], [k_{030}, 4]],$$

$$[28, 23470, [k_{001}, 4], [k_{002}, 4], [k_{003}, 4], [k_{010}, 12], [k_{020}, 12], [k_{021}, 8], [k_{030}, 8], \ldots$$
$$\ldots, [k_{100}, 12], [k_{111}, 8], [k_{200}, 12], [k_{201}, 8], [k_{210}, 8], [k_{300}, 8]].$$

still 7 equations in 19 unknowns $k_{r,s,t}$, now of degree ≤ 28,

but one ugly equation with **23470 terms**!

Application to skew quadrilaterals

Conjecture:

If the tetrahedron $ABCD$ has no symmetries, shows no right angles between any pair of edges (whether skew or not), and has no pair of equally long edges, then none of the three cubic surfaces $\mathcal{K}$ associated with the three types of skew quadrilaterals ($ABCD$, $ABDC$, $ACBD$) degenerates.

Justification: (no proof, it's not a theorem!)

insert coefficients of $\mathcal{K}$'s equation (for any case) into the degeneracy conditions,

try to solve the emerging systems of equations ...

factors that are only vanishing if there are right angles or symmetries can be canceled ...

nothing useful remains ...

but we didn't get through all computations!

Algebraic and computational complexity

case (A)	case (B)
ABCD	
[5, 11, [a, 1], [b, 3], [c, 3], [d, 2], [e, 2], [f, 2]]	[39, 82424, [a, 12], [b, 21], [c, 23], [d, 24], [e, 24], [f, 18]]
[12, 651, [a, 4], [b, 9], [c, 9], [d, 6], [e, 9], [f, 8]]	[43, 110994, [a, 12], [b, 25], [c, 27], [d, 25], [e, 26], [f, 22]]
[15, 1561, [a, 5], [b, 10], [c, 10], [d, 9], [e, 11], [f, 10]]	**[$\leq$168, ?, [a, ?], [b, ?], [c, ?], [d, ?], [e, ?], [f, ?]]**
[16, 1979, [a, 6], [b, 10], [c, 10], [d, 9], [e, 12], [f, 12]]	
ABDC	
[12, 604, [a, 4], [b, 6], [c, 6], [d, 8], [e, 9], [f, 6]]	[39, 80604, [a, 13], [b, 24], [c, 24], [d, 21], [e, 23], [f, 16]]
[15, 1706, [a, 4], [b, 8], [c, 8], [d, 9], [e, 11], [f, 8]]	[43, 117506, [a, 13], [b, 25], [c, 26], [d, 25], [e, 27], [f, 20]]
[16, 2118, [a, 4], [b, 8], [c, 9], [d, 11], [e, 12], [f, 8]]	
ACBD	
[5, 25, [a, 2], [b, 3], [c, 2], [d, 3], [e, 2], [f, 2]]	[23, 6099, [a, 10], [b, 12], [c, 11], [d, 14], [e, 12], [f, 10]]
[7, 51, [a, 4], [b, 4], [c, 4], [d, 4], [e, 4], [f, 4]]	[27, 6775, [a, 9], [b, 15], [c, 12], [d, 16], [e, 14], [f, 14]]
[9, 111, [a, 5], [b, 5], [c, 4], [d, 5], [e, 4], [f, 4]]	[30, 11356, [a, 11], [b, 17], [c, 12], [d, 18], [e, 16], [f, 16]]
[11, 142, [a, 5], [b, 7], [c, 6], [d, 4], [e, 4], [f, 4]]	[30, 17065, [a, 12], [b, 16], [c, 16], [d, 19], [e, 16], [f, 14]]
[11, 331, [a, 6], [b, 6], [c, 5], [d, 6], [e, 5], [f, 4]]	[45, 103418, [a, 14], [b, 25], [c, 22], [d, 25], [e, 23], [f, 18]]
[16, 1094, [a, 7], [b, 8], [c, 8], [d, 8], [e, 8], [f, 6]]	[53, 186696, [a, 16], [b, 29], [c, 26], [d, 29], [e, 27], [f, 22]]
[23, 4112, [a, 9], [b, 12], [c, 11], [d, 11], [e, 11], [f, 8]]	**[$\leq$168, ?, [a, ?], [b, ?], [c, ?], [d, ?], [e, ?], [f, ?]]**

Reduced evaluated degeneracy conditions with non-vanishing factors canceled out.

Example 1 - one plane of symmetry

a$=(0,0,0)$, **b**$=(a,b,0)$, **c**$=(c,0,d)$, **d**$=(a,-b,0)$

one plane of symmetry: $\pi_3: \quad y=0$, provided that $c^2-2ac+d^2\neq 0$.

$$\mathcal{K}(ABCD)=\mathcal{P}\cap\mathcal{Q}$$

with $\mathcal{P}=\pi_3$ and a quadric $\mathcal{Q}$ centered at

$$M=\frac{1}{2ad}\left(dl_1^2, 0, al_2^2-cl_1^2\right)\in\pi_3$$

with $l_1:=\overline{AB}=\overline{AD}$ and $l_2:=\overline{BC}=\overline{CD}$

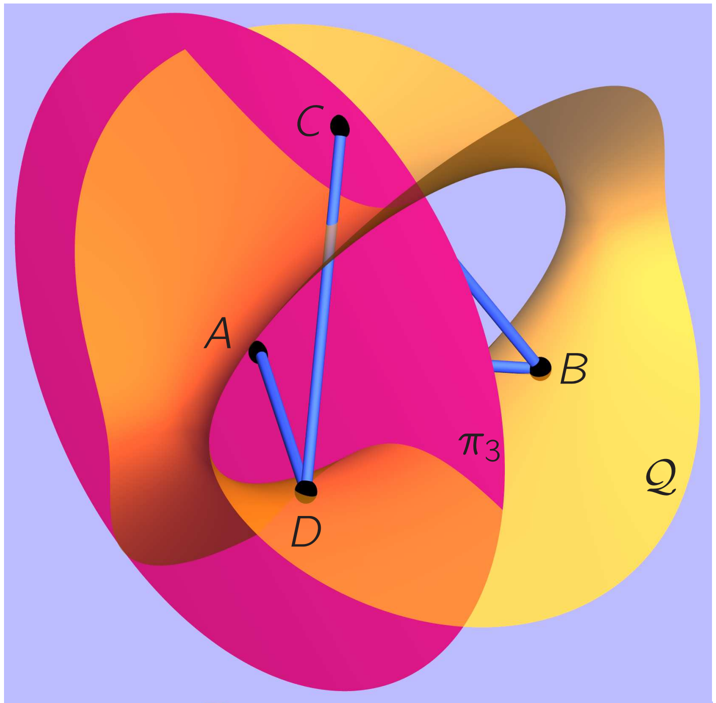
longest (real) axis parallel to $[B,D]$

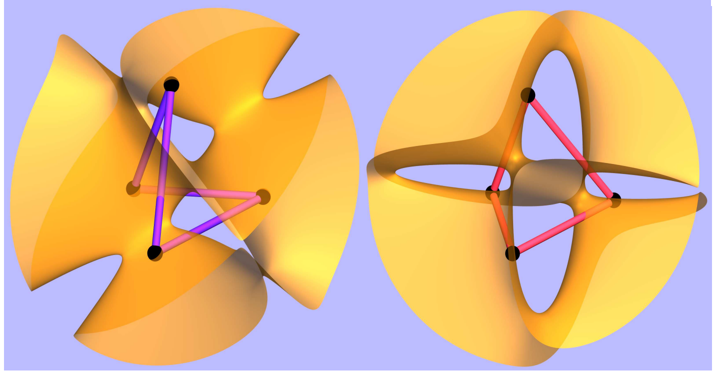
$\mathcal{Q}$, and thus, $\mathcal{K}$ degenerate further if $2ac-c^2-d^2=0$

$$\mathcal{Q}: \ (cz-dx)(cx+dz)=0$$

and $ABCD$ has got a second plane of symmetry, the bisector of the segment CD.

$\mathcal{K}(ABDC)$, $\mathcal{K}(ACBD)$ are not degenerate.





Example 2 - two planes of symmetry - one-parameter family of . . .

like before, but with $2ac - c^2 - d^2 = 0$ and $d \neq 0$

$\overline{AB}{=}\overline{BC}{=}\overline{CD}{=}\overline{DA}$, skew as long as $d \neq 0$

$ABCD$ has got four equally long edges.

C can be chosen on a circle

$$c{=}\left(\frac{2a}{1{+}t^2}, 0, \frac{2at}{1{+}t^2}\right)$$

$\mathcal{K}(ABCD)$ becomes the union of three planes:

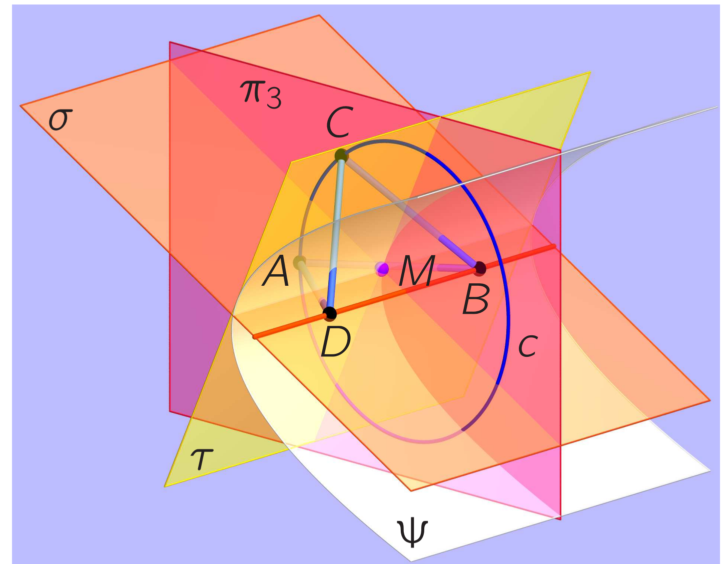
$y(x{+}tz{-}a)(2at(tx{-}z){-}(a^2{+}b^2)t^2{+}a^2{-}b^2){=}0.$

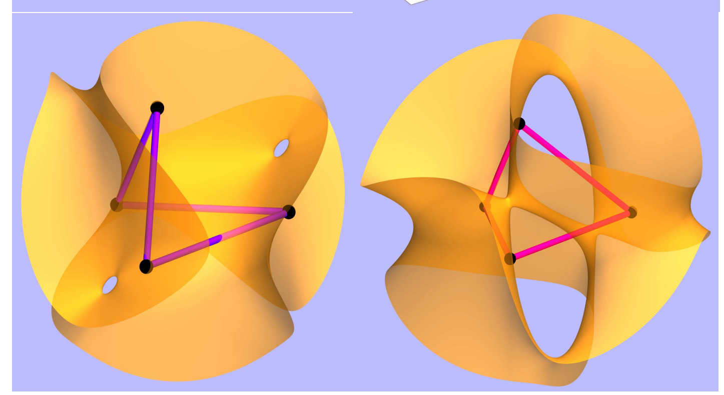
$\pi_3 : y{=}0$, $\sigma : x{+}tz{-}a{=}0$ . . . planes of symmetry

$2at(tx{-}z)\ldots{=}0$ . . . τ, tangent planes of a parabolic cylinder Ψ

$a = b \iff \Psi^\star$ becomes a pencil of planes.

$\mathcal{K}(ABDC)$, $\mathcal{K}(ACBD)$ are not degenerate.





Example 3 - axial symmetry

skew $ABCD$ with axial symmetry: $\mathbf{a}=(0,0,0)$, $\mathbf{b}=(a+c,\,b+d,\,0)$, $\mathbf{c}=(a,b,h)$, $\mathbf{d}=(c,d,h)$

All cubics are degenerate: $\mathcal{K}_i = \mathcal{P}_i \cup \mathcal{Q}_i$ with

$$\mathcal{P}_1 : d=2y, \quad \mathcal{P}_2 : a=2x, \quad \mathcal{P}_3 : h=2z$$

and concentric quadrics with center $\frac{1}{2}(a,d,h)$

$\mathcal{Q}_1 :\; a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{D}_{ah}(d-y)y + h^2\mathrm{S}_{ad}(z-h)z = 0,$

$\mathcal{Q}_2 :\; a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{S}_{ah}(d-y)y + h^2\mathrm{S}_{ad}(z-h)z = 0,$

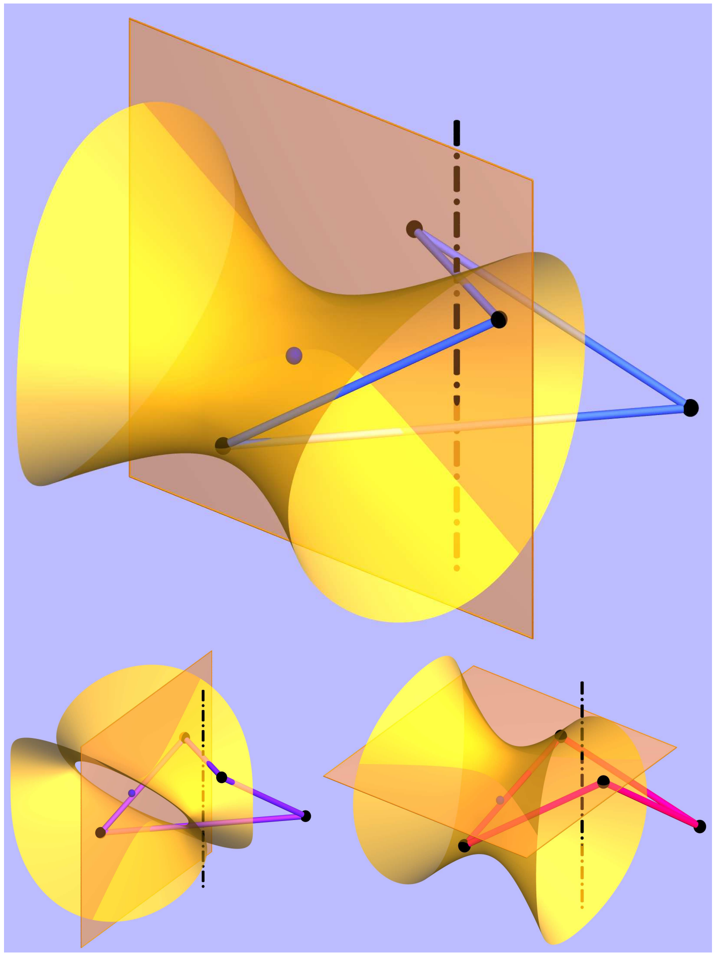
$\mathcal{Q}_3 :\; a^2\mathrm{S}_{dh}(a-x)x + d^2\mathrm{S}_{ah}(y-d)y + h^2\mathrm{D}_{ad}(h-z)z = 0,$

with $\mathrm{S}_{uv} := u^2+v^2$, $\mathrm{D}_{uv} := u^2-v^2$.

b and c do not show up!

Quadrics degenerate if additionally

$$a = h \text{ or } d = h \text{ or } a = d.$$



Examples 4, 5, 6 - orthoschemes, corners of cuboids, regular tetrahedron

Orthoschemes:

3 subsequent orthogonal edges

vertices can be given by

a$=(0,0,0)$, **b**$=(a,0,0)$, **c**$=(a,b,0)$, **d**$=(a,b,c)$

$\Longrightarrow$ degenerate $\mathcal{K}$ if either $a=\pm b$ or $a=\pm c$

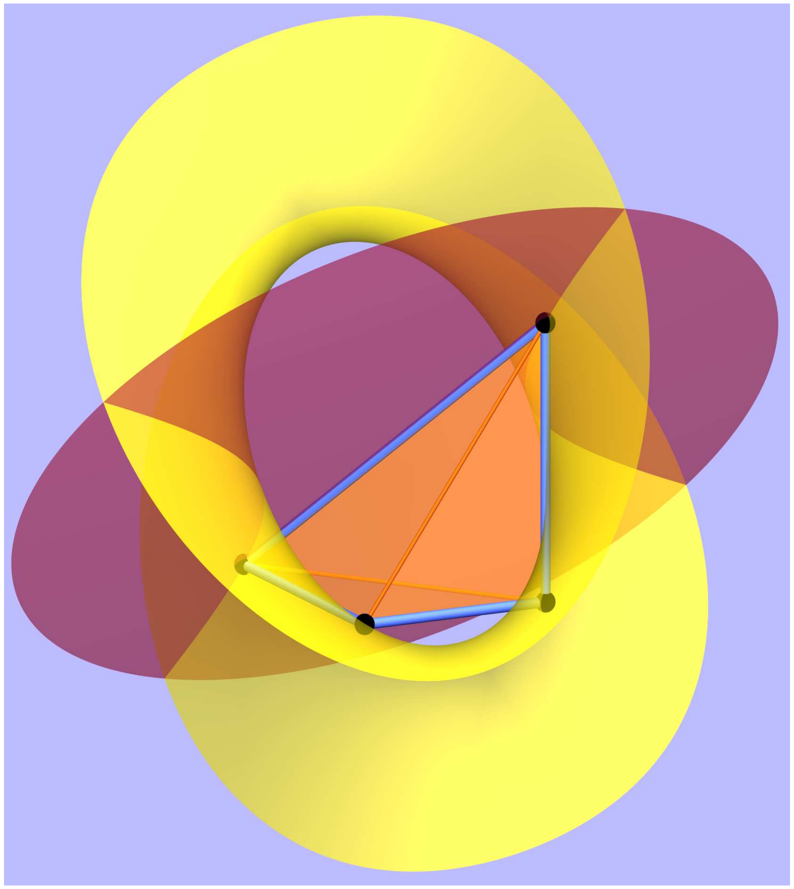
$(x\mp z)(by^2+axy\pm ayz\mp bxz-(a^2+b^2)y)=0$

corner of cuboids: degenerate $\mathcal{K}$ if

there is at least one plane of symmetry

regular tetrahedrons:

$\mathcal{K}$ is the union of three planes of symmetry.



Problems

- evaluation of degeneracy conditions
- Are the degeneracy conditions sufficient?
- Are degeneracy conditions invariants of quadrilaterals (provided that no special coordinate system is chosen)?
- Homogeneous coordinates in three-space not used, because of computational complexity.
- Is there a relation to Cayley's hyperdeterminant for quaternery cubic forms?
- Is it really necessary to consider $ABDC$ and $ACBD$ if $ABCD$ is already checked?

Thank You For Your Attention!

Non-vanishing factors - redundant factors cut out

$ABCD$ skew	if $f \neq 0$
different vertices	if
$A \neq B$	$a \neq 0$
$A \neq C$	$b^2 + c^2 \neq 0$
$A \neq D$	$d^2 + e^2 + f^2 \neq 0$
$B \neq C$	$(a-b)^2 + c^2 \neq 0$
$B \neq D$	$(a-d)^2 + e^2 + f^2 \neq 0$
$C \neq D$	$(b-d)^2 + (c-e)^2 + f^2 \neq 0$
surface normals ore not $\mathbf{o}$	if
$\mathbf{n}_{ABC}$	$c \neq 0$
$\mathbf{n}_{ABD}$	$e^2 + f^2 \neq 0$
$\mathbf{n}_{ACD}$	$b^2(e^2+f^2) - 2bcde + c^2(d^2+f^2) \neq 0$
$\mathbf{n}_{BCD}$	$(ac+be)^2+(ae+cd)^2+(a^2+b^2+c^2)f^2$ $-2ab(e^2+f^2)-2c(a^2e+acd+bde)\neq 0$
no two edges are of the same length	if
$\overline{AB} \neq \overline{AC}$	$a^2 - b^2 - c^2 \neq 0$
$\overline{AB} \neq \overline{AD}$	$a^2 - d^2 - e^2 - f^2 \neq 0$
$\overline{AB} \neq \overline{BC}$	$2ab - b^2 - c^2 \neq 0$
$\overline{AB} \neq \overline{BD}$	$2ad - d^2 - e^2 - f^2 \neq 0$
$\overline{AB} \neq \overline{CD}$	$(b-d)^2 + (c-e)^2 + f^2 - a^2 \neq 0$
$\overline{AC} \neq \overline{AD}$	$b^2 + c^2 - d^2 - e^2 - f^2 \neq 0$
$\overline{AC} \neq \overline{BC}$	$2b - a \neq 0$
$\overline{AC} \neq \overline{BD}$	$b^2 + c^2 - (d-a)^2 - e^2 - f^2 \neq 0$
$\overline{AC} \neq \overline{CD}$	$2bd + 2ce - d^2 - e^2 - f^2 \neq 0$
$\overline{AD} \neq \overline{BC}$	$(b-a)^2 + c^2 - d^2 - e^2 - f^2 \neq 0$
$\overline{AD} \neq \overline{BD}$	$2d - a \neq 0$
$\overline{AD} \neq \overline{CD}$	$(2d-b)b + (2e-c)c \neq 0$
$\overline{BC} \neq \overline{BD}$	$2a(d-b) + b^2 + c^2 - d^2 - e^2 - f^2 \neq 0$
$\overline{BC} \neq \overline{CD}$	$(a-2b)a-(d-2b)d-(e-2c)e-f^2\neq 0$
$\overline{BD} \neq \overline{CD}$	$(a-2d)a-(b-2d)b-(c-2e)c\neq 0$

edge not orthogonal to face	if
$[A,B] \neq \perp\mathbf{n}_{ACD}$	$b^2e^2 + b^2f^2 - 2bcde + c^2d^2 \neq 0$
$[A,B] \neq \perp\mathbf{n}_{BCD}$	$a^2c^2 - 2a^2ce + a^2e^2 + a^2f^2 + 2abce$ $-2abe^2 - 2abf^2 - 2ac^2d + 2acde + b^2e^2$ $+b^2f^2 - 2bcde + c^2d^2 \neq 0$
$[A,C] \neq \perp\mathbf{n}_{ABD}$	$b^2e^2 + b^2f^2 + c^2e^2 \neq 0$
$[A,C] \neq \perp\mathbf{n}_{BCD}$	$a^2b^2c^2 - 2a^2b^2ce + a^2b^2e^2 + a^2b^2f^2 + a^2c^4$ $+a^2c^2e^2 + 2ab^3ce - 2ab^3e^2 - 2a^2c^3e - 2ab^3f^2$ $-2ab^2c^2d + 2ab^2cde + 2abc^3e - 2abc^2e^2 - 2abc^2f^2$ $-2ac^4d + 2ac^3de + b^4e^2 + b^4f^2 - 2b^3cde + b^2c^2d^2$ $+b^2c^2e^2 + 2b^2c^2f^2 - 2bc^3de + c^4d^2 + c^4f^2 \neq 0$
$[A,D] \neq \perp\mathbf{n}_{ABC}$	$d^2 + e^2 \neq 0$
$[A,D] \neq \perp\mathbf{n}_{BCD}$	$a^2c^2d^2 + a^2c^2e^2 - 2a^2cd^2e - 2a^2ce^3 - 2a^2cef^2$ $+a^2d^2e^2 + a^2d^2f^2 + a^2e^4 + 2a^2e^2f^2 + a^2f^4$ $+2abcd^2e + 2abce^3 + 2abcef^2 - 2abd^2e^2$ $-2abd^2f^2 - 2abe^4 - 4abe^2f^2 - 2abf^4 - 2ac^2d^3$ $-2ac^2de^2 - 2ac^2df^2 + 2acd^3e + 2acde^3 + 2acdef^2$ $+b^2d^2e^2 + b^2d^2f^2 + b^2e^4 + 2b^2e^2f^2 + b^2f^4$ $-2bcd^3e - 2bcde^3 - 2bcdef^2 + c^2d^4 + c^2d^2e^2$ $+2c^2d^2f^2 + c^2e^2f^2 + c^2f^4 \neq 0$
$[B,C] \neq \perp\mathbf{n}_{ACD}$	$a^2e^2 + a^2f^2 - 2abe^2 - 2abf^2 + b^2e^2 + b^2f^2 + c^2e^2 \neq 0$
$[B,C] \neq \perp\mathbf{n}_{ACD}$	$a^2b^2e^2 + a^2b^2f^2 - 2a^2bcde + a^2c^2d^2 - 2ab^3e^2 - 2ab^3f^2$ $+4ab^2cde - 2abc^2d^2 - 2abc^2f^2 + b^4e^2 + b^4f^2 - 2b^3cde$ $+b^2c^2d^2 + b^2c^2e^2 + 2b^2c^2f^2 - 2bc^3de + c^4d^2 + c^4f^2 \neq 0$
$[B,D] \neq \perp\mathbf{n}_{ABC}$	$(a-d)^2 + e^2 \neq 0$
$[B,D] \neq \perp\mathbf{n}_{ACD}$	$a^2b^2e^2 + a^2b^2f^2 - 2a^2bcde + a^2c^2d^2 - 2ab^2de^2$ $-2ab^2df^2 + 4abcd^2e - 2ac^2d^3 - 2ac^2df^2 + b^2d^2e^2$ $+b^2d^2f^2 + b^2e^4 + 2b^2e^2f^2 + b^2f^4 - 2bcd^3e - 2bcde^3$ $-2bcdef^2 + c^2d^4 + c^2d^2e^2 + 2c^2d^2f^2+c^2e^2f^2+c^2f^4\neq 0$
$[C,D] \neq \perp\mathbf{n}_{ABC}$	$(b-d)^2 + (c-e)^2 \neq 0$
$[C,D] \neq \perp\mathbf{n}_{ABD}$	$b^2e^2 + b^2f^2 - 2bde^2 - 2bdf^2 + c^2e^2 - 2ce^3 - 2cef^2$ $+d^2e^2 + d^2f^2 + e^4 + 2e^2f^2 + f^4 \neq 0$

Non-vanishing factors - redundant factors cut out

angle not right	if
$\sphericalangle(A, B.C)$	$a - b \neq 0$
$\sphericalangle(B, C, A)$	$ab - b^2 - c^2 \neq 0$
$\sphericalangle(C, A, B)$	$b \neq 0$
$\sphericalangle(A, B, D)$	$a - d \neq 0$
$\sphericalangle(B, D, A)$	$ad - d^2 - e^2 - f^2 \neq 0$
$\sphericalangle(D, A, B)$	$d \neq 0$
$\sphericalangle(A, C, D)$	$(b - d)b + (c - e)c \neq 0$
$\sphericalangle(C, D, A)$	$(d - b)d + (e - c)e + f^2 \neq 0$
$\sphericalangle(D, A, C)$	$bd + ce \neq 0$
$\sphericalangle(B, C, D)$	$(b - a)(b - d) + (c - e)c \neq 0$
$\sphericalangle(C, D, B)$	$ab - ad - bd - ce + d^2 + e^2 + f^2 \neq 0$
$\sphericalangle(D, B, C)$	$a^2 - ab - ad + bd + ce \neq 0$
$\sphericalangle([A, B], [C, D])$	$b - d \neq 0$
$\sphericalangle([A, C], [B, D])$	$ab - bd - ce \neq 0$
$\sphericalangle([A, D], [B, C])$	$ad - bd - ce \neq 0$
$\sphericalangle(\mathbf{n}_{ABC}, \mathbf{n}_{ABD})$	$e \neq 0$
$\sphericalangle(\mathbf{n}_{ABC}, \mathbf{n}_{ACD})$	$be - cd \neq 0$
$\sphericalangle(\mathbf{n}_{ABC}, \mathbf{n}_{BCD})$	$b(e^2 + f^2) - cde \neq 0$
$\sphericalangle(\mathbf{n}_{ABD}, \mathbf{n}_{ACD})$	$ac - ae + be - cd \neq 0$
$\sphericalangle(\mathbf{n}_{ABD}, \mathbf{n}_{BCD})$	$ace - ae^2 - af^2 + be^2 + bf^2 - cde \neq 0$
$\sphericalangle(\mathbf{n}_{ACD}, \mathbf{n}_{BCD})$	$abce - abe^2 - abf^2 - ac^2d + acde + b^2e^2$ $+b^2f^2 - 2bcde + c^2d^2 + c^2f^2 \neq 0$

line not in bisector	if
$[A, B] \notin \sigma_{CD}$	$(d^2+e^2+f^2-b^2-c^2)^2+4a^2(b-d)^2 \neq 0$
$[A, C] \notin \sigma_{BD}$	$(d^2+e^2+f^2-a^2)^2+4(ab-bd-ce)^2 \neq 0$
$[A, D] \notin \sigma_{BC}$	$(b^2+c^2-a^2)^2+4(ad-bd-ce)^2 \neq 0$
$[B, C] \notin \sigma_{AD}$	$(d^2+e^2+f^2-2ad)^2+4((b-a)d+ce)^2 \neq 0$
$[B, D] \notin \sigma_{AC}$	$(b^2+c^2-2ab)^2+4((d-a)b+ce)^2 \neq 0$
$[C, D] \notin \sigma_{AB}$	$(a-2b)^2+4(b-d)^2 \neq 0$

not kollinear	if
ABC	$c \neq 0$
ABD	$e^2 + f^2 \neq 0$
ACD	$f^2(b^2 + c^2) + (cd - be)^2 \neq 0$
BCD	$(ac - ae + be - cd)^2 + f^2((b - a)^2 + c^2 + a^2) \neq 0$

point not in face (coplanarity)	if
$A \notin [B, C, D]$	$acf \neq 0$
$B \notin [A, C, D]$	$acf \neq 0$
$C \notin [A, B, D]$	$cf \neq 0$
$D \notin [A, B, C]$	$f \neq 0$

Some polynomial factors appear at least twice.

Since $a, b, c, d, e, f \neq 0$, any sum of squares of a, b, c, d, e, f is not equal to zero and can be canceled. $\Longrightarrow 2^6 - 7 - 3$ more factors to be canceled.

Related work

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[4] P. Pech: *On the Wallace-Simson Theorem and its Generalizations.* J. Geometry Graphics **9**/2 (2005), 141–153.

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