Algebraic Geometry and Geometric Modeling
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The geometry of flags

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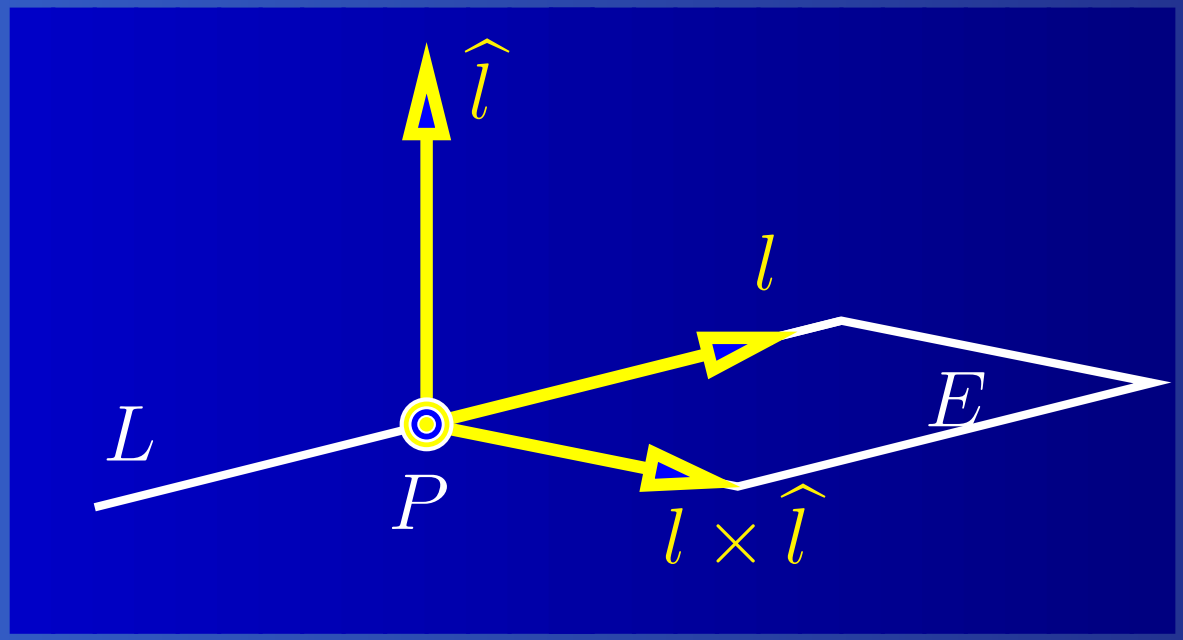
What is a flag?



flag $\mathcal{F} = (P, L, E)$ with $P \in L \subset E$

flag = coordinate system

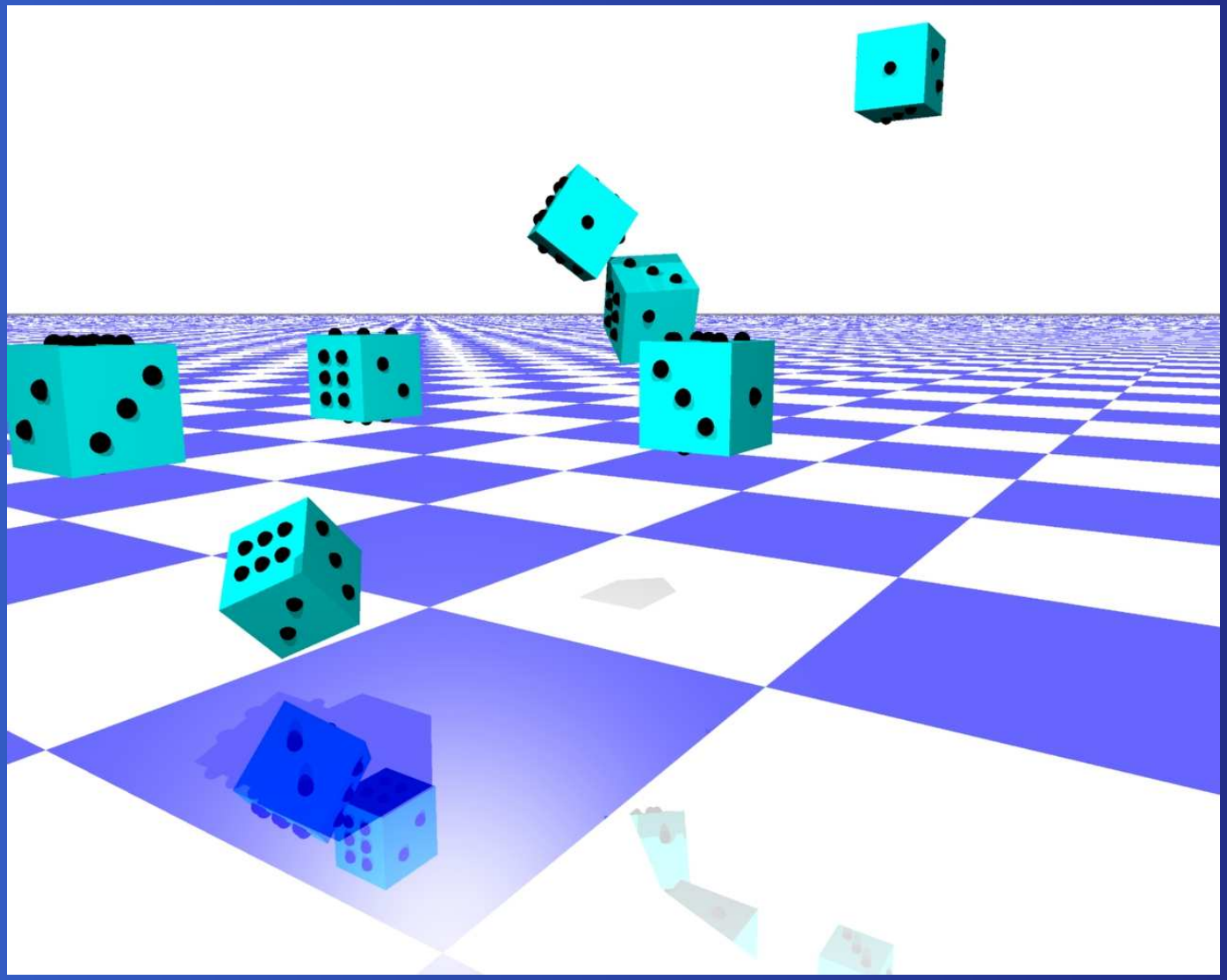
Motivation

manifold of flags M_8^6 = group of Euclidean motions in $\mathbb{R}^3$

until now studied only from the view point of projective geometry: M_8^6 = intersection of Segre-Variety $S_{3,5,3} \subset \mathbb{P}^{95}$ with a $\mathbb{P}^{57}$

applications:
motion planning, subdivison motions

Motivation



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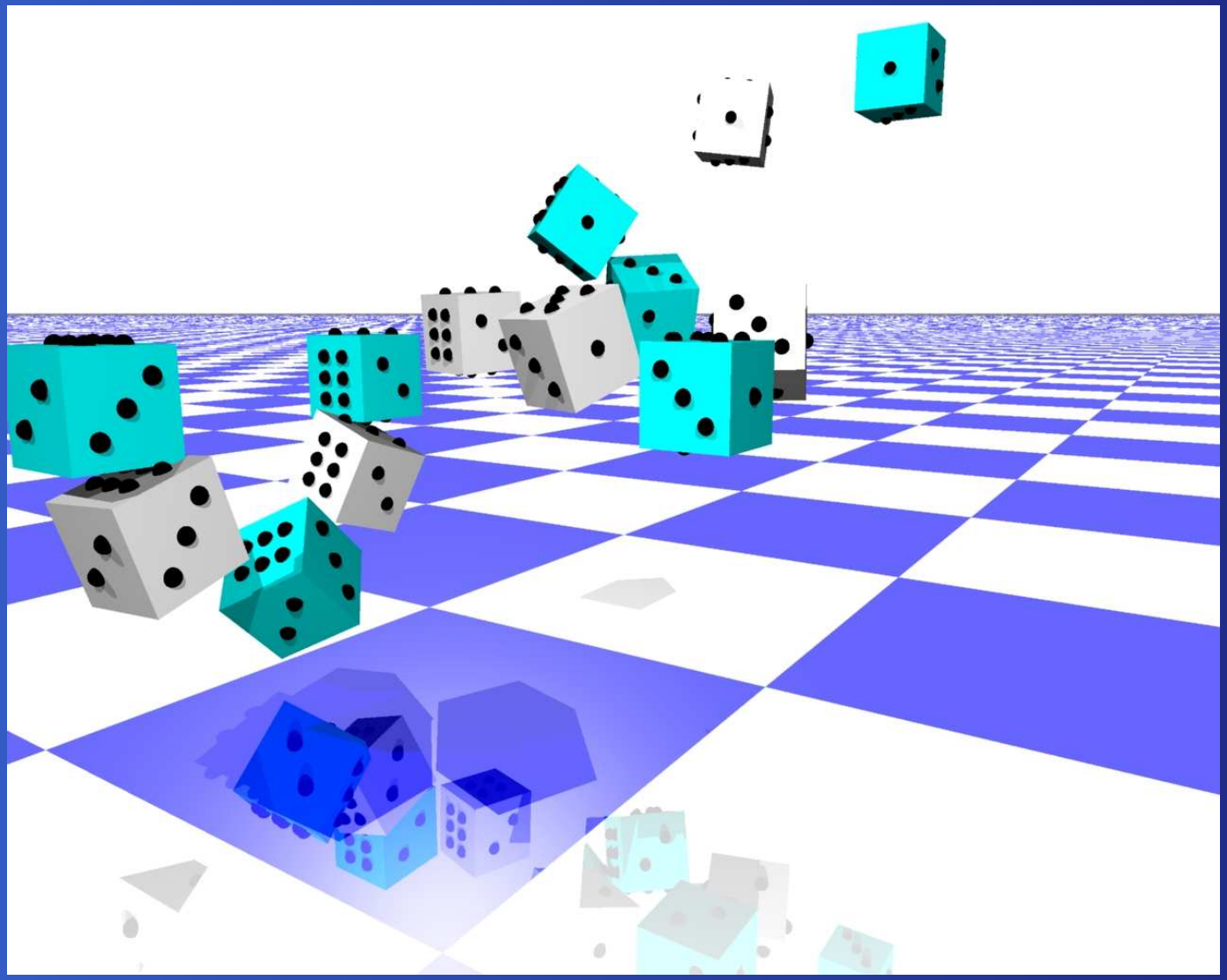
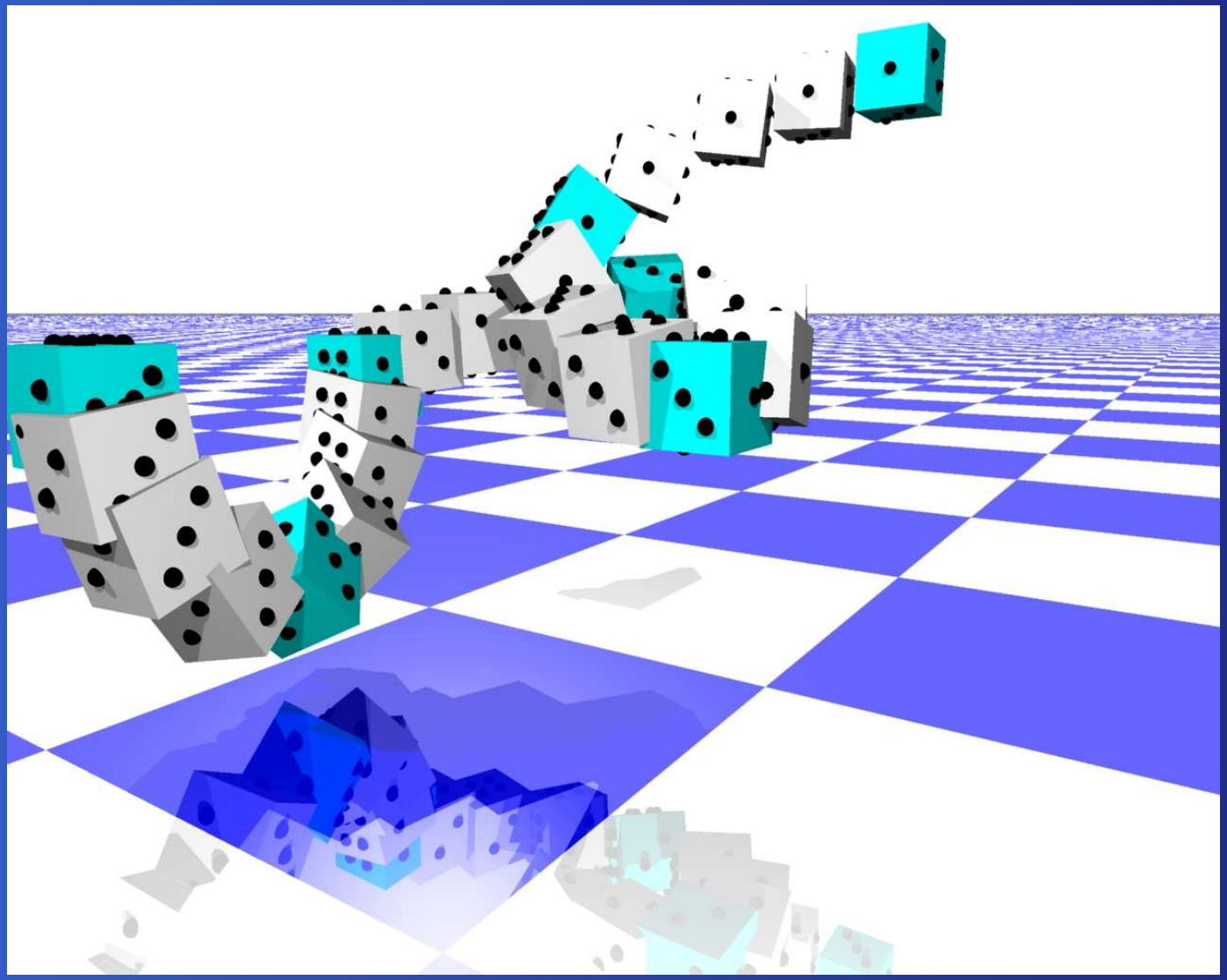


Motivation



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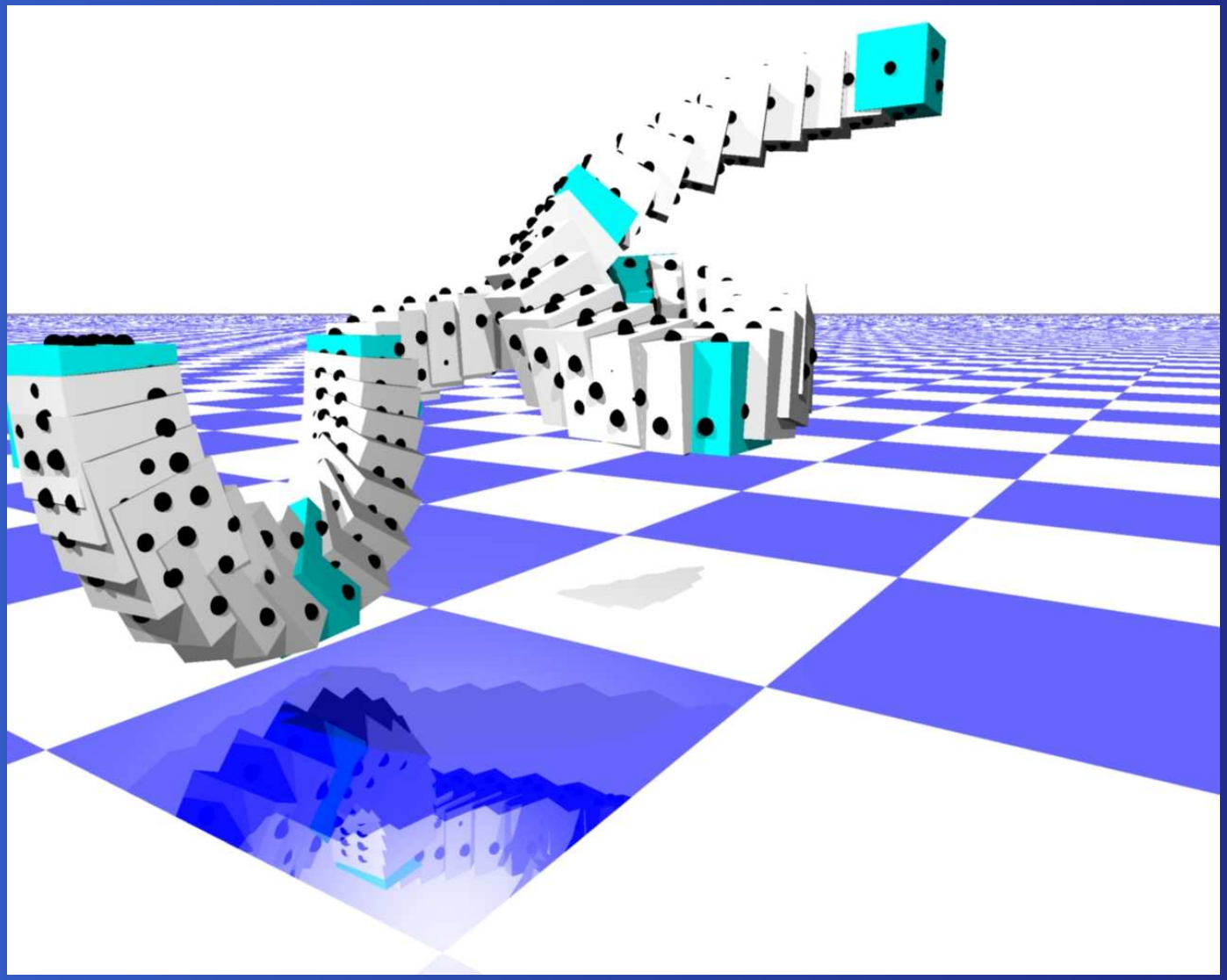
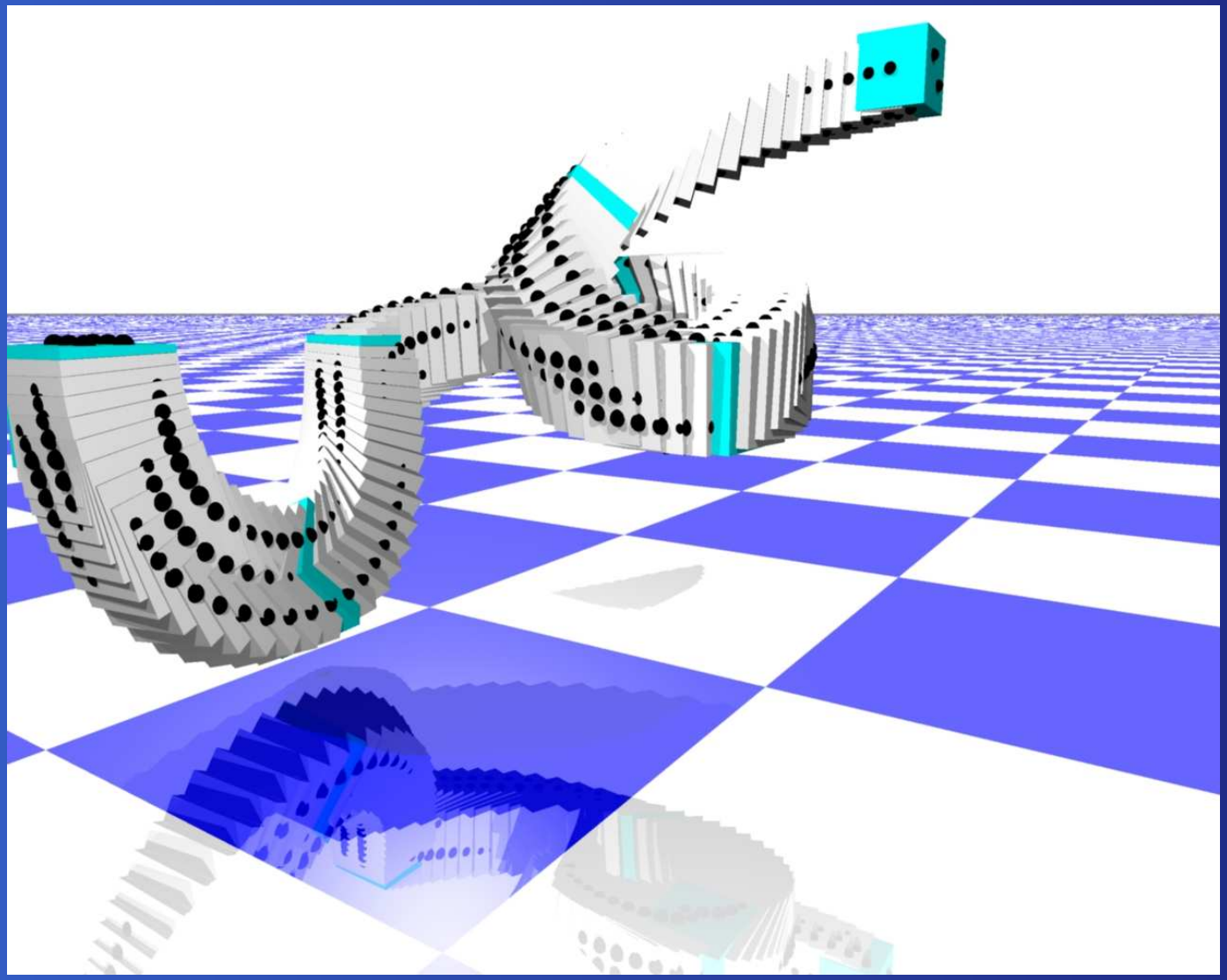


Motivation



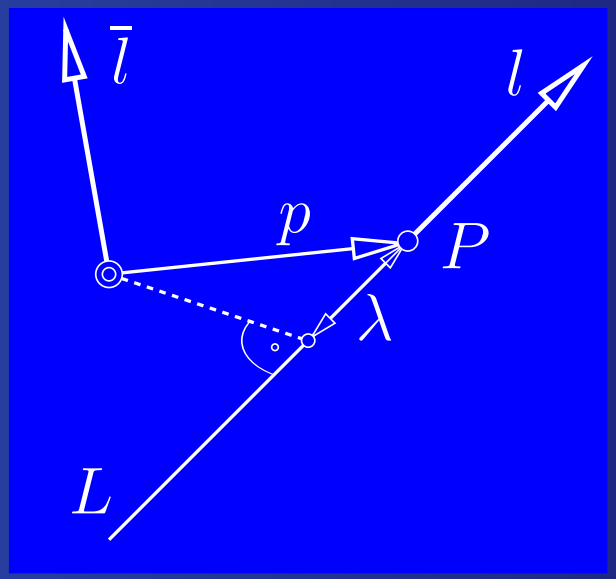
Coordinates of flags

line $L \to (l, \bar{l}) \in \mathbb{R}^6$ ... Plücker coordinates

$\bar{l} := p \times l \qquad \langle l, \bar{l} \rangle = 0$ ... $G_{3,1}$

line element (P, L) = line L + point P on it

$(P, L) \to (l, \bar{l}, \lambda) \in \mathbb{R}^7$ ... coordinates of line elements



$\lambda := \langle p, l \rangle \in \mathbb{R}$

$\|l\| = 1 \Longrightarrow$

λ = P's coordinate on L

Coordinates of flags

plane $E: \ \langle \widehat{l}, x \rangle = c$, $E \supset L \Longrightarrow \widehat{l} \perp l \Longrightarrow \langle l, \widehat{l} \, \rangle = 0$

flag $\mathcal{F} = (P, L, E)$

= line element + plane (through it)

$\mathcal{F} \to (l, \bar{l}, \widehat{l}, \lambda) \in \mathbb{R}^{10}$ ... coordinates of flags

assumption: $\|l\| = \|\widehat{l}\|$ $(= 1)$

restrictions: $l, \ \widehat{l} \neq 0$, $\langle l, \bar{l} \rangle = 0$, $\langle l, \widehat{l} \rangle = 0$

Coordinates of flags

coordinates $(l, \bar{l}, \widehat{l}, \lambda)$ determine $\mathcal{F}$ uniquely
(up to orientations)

components can be recovered:

point $P = l \times \bar{l} + \lambda l$

line $L = (l, \bar{l})$

plane $E: \ \langle \widehat{l}, x \rangle = \det(l, \bar{l}, \widehat{l}\,)$

Equation of the manifold M_8^6

$$
\begin{array}{rcl}
(l, \bar{l}, \widehat{l}, \lambda) & = & (l_1, l_2, l_3; l_4, l_5, l_6; l_7, l_8, l_9; l_{10}) \\
\langle l, \bar{l} \rangle & = & l_1 l_4 + l_2 l_5 + l_3 l_6 \quad = \quad 0 \\
\langle l, \widehat{l} \rangle & = & l_1 l_7 + l_2 l_8 + l_3 l_9 \quad = \quad 0 \\
\langle l, l \rangle - \langle \widehat{l}, \widehat{l} \rangle & = & l_1^2 + l_2^2 + l_3^2 - l_7^2 - l_8^2 - l_9^2 \quad = \quad 0
\end{array}
$$

degree = 8, dimension = 6

l_i's . . . homogeneous coordinates of points in $\mathbb{P}^9$

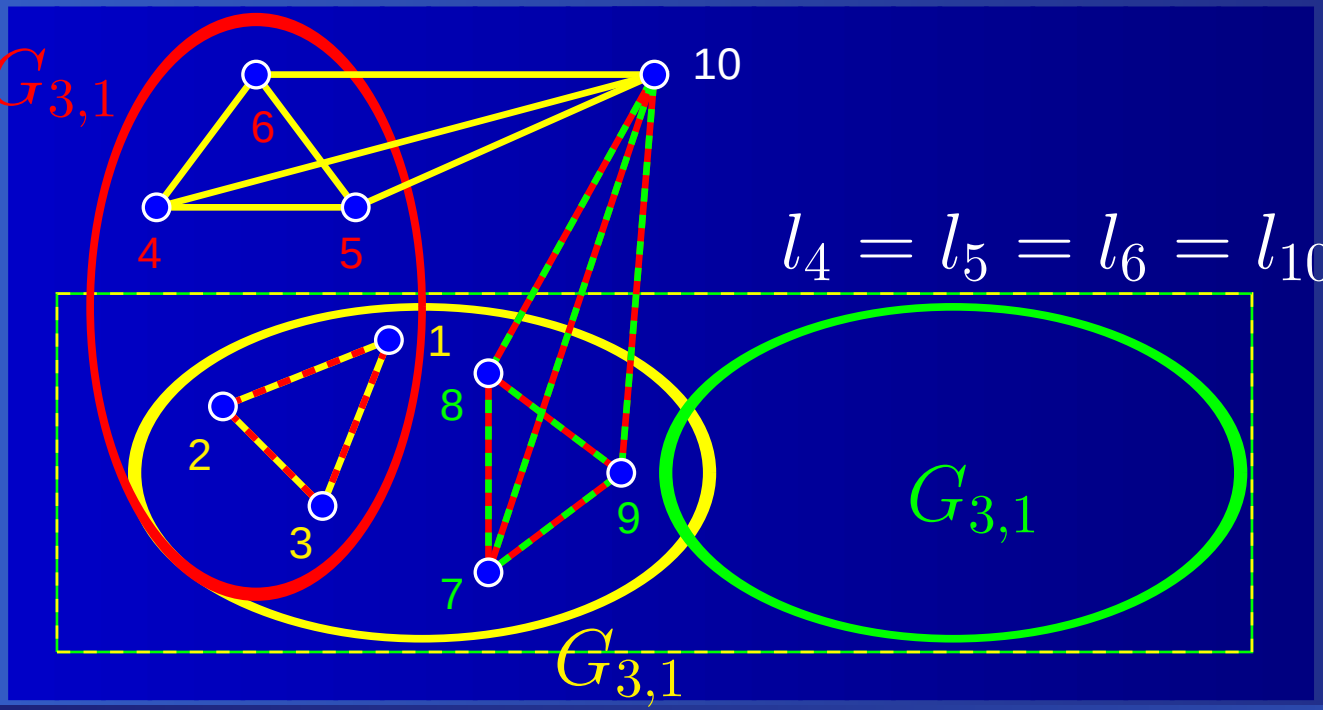
l_{10} does not show up in any equation

$\Longrightarrow M_8^6$ is a cone

Properties

$M_8^6 \subset \Delta_1 \cap \Delta_2 \cap \Delta_3$ . . . intersection of three quadratic cones carrying each others vertices

Δ_i . . . 3-dim. vertices, . . . 6-dim. generators



Rational parametrization

Theorem: There exists one.

Proof:

$l = (\frac{2u_1}{N}, \frac{2u_2}{N}, \frac{1-u_1^2-u_2^2}{N})$, $N = 1 + u_1^2 + u_2^2$

l ... isothermal param. of $S^2 \Longrightarrow$

$\langle l, l_{,i} \rangle = 0$ and $\langle l_{,i}, l_{,j} \rangle = 4\delta_{ij} N^{-2} \Longrightarrow$

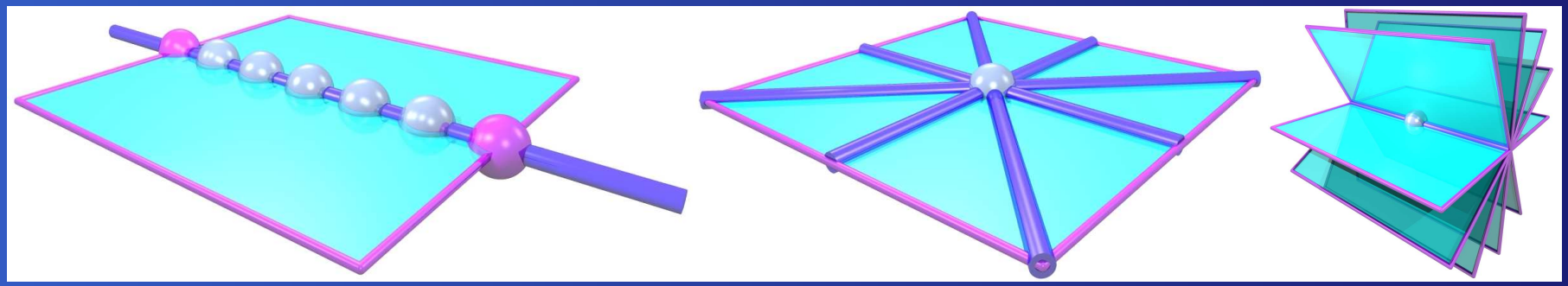
$N\widehat{l} = 2c_3 l_{,1} + 2s_3 l_{,2}$

with $c_3 = (1 - u_3^2)/(1 + u_3^2)$, $s_3 = 2u_3/(1 + u_3^2)$

$P = (u_4, u_5, u_6) \longrightarrow \bar{l}$, λ.

Properties

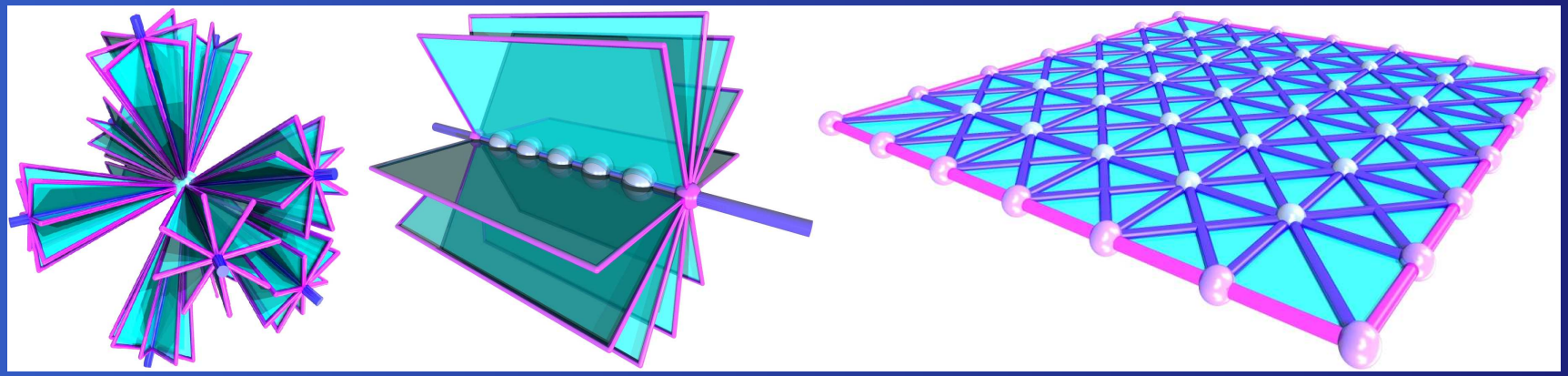
pencils of flags: two components fixed



correspond to lines in M_8^6

Properties

bundles of flags: one component fixed



correspond to oval quadrics, planes, and quadratic cones in M_8^6

Non-Euclidean geometries

bundle with fixed point component:
elliptic three-space

bundle with fixed plane component:
quasi-elliptic three-space

Study's quadric R_2^6 =
point model for the set of Euclidean motions =
point model for the set of flags in Euclidean $\mathbb{R}^3$

Thank You For Your Attention!

Overview

- Motivation
- Basics
- Coordinates
- Equations of the manifold
- Parameterization
- Properties