Bézier curves are not smooth!

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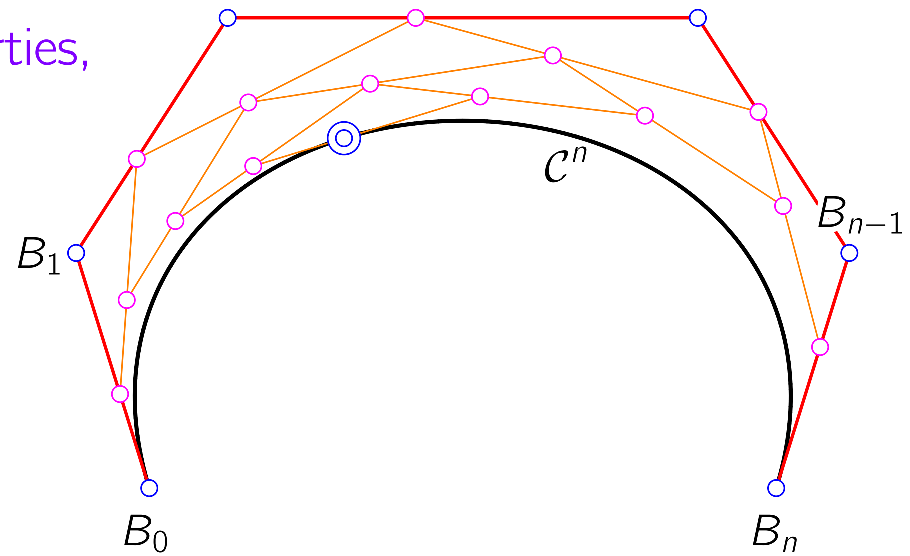
Don't worry, that's only a global result with many exceptions.

overview

improperly chosen control points	more & less obvious examples
singularities on cubics, quartics, quintics	What do we have to expect?
discussion of possible cases	classification of rational Bézier curves
	w. r. t. their singularities
	needs extension of Plücker’s formulae
Bézier curve	projection of a rational normal curve
	explains some cases, but . . .
examples	compound singularities, control structures
motivation	it’s **not** just slaughtering the holy cow

motivation

Bézier curves are so "nice" (rational parametrizations, linear construction of points, interpolation properties, ...) because they are not globally smooth!

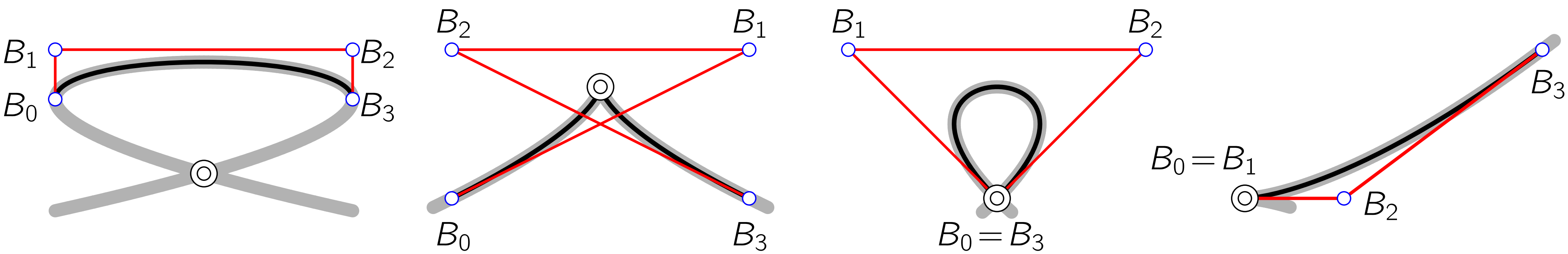


construction $\Longrightarrow$
rational parametrization $x_0(t) : x_1(t) : x_2(t) \Longrightarrow$
Bézier curve $\subset$ algebraic curve $\mathcal{C}^n$ of genus 0 $\Longrightarrow$
$\mathcal{C}^n$ exhibits the 'maximum number' of singularities.

more precise: $\sum_i \delta(S_i) = \frac{1}{2}(n-1)(n-2), \ldots S_i \ldots$ the singularities

Where are the singularities?

more or less obvious examples



nothing spectacular in the unit interval (ordinary node) happens in any case

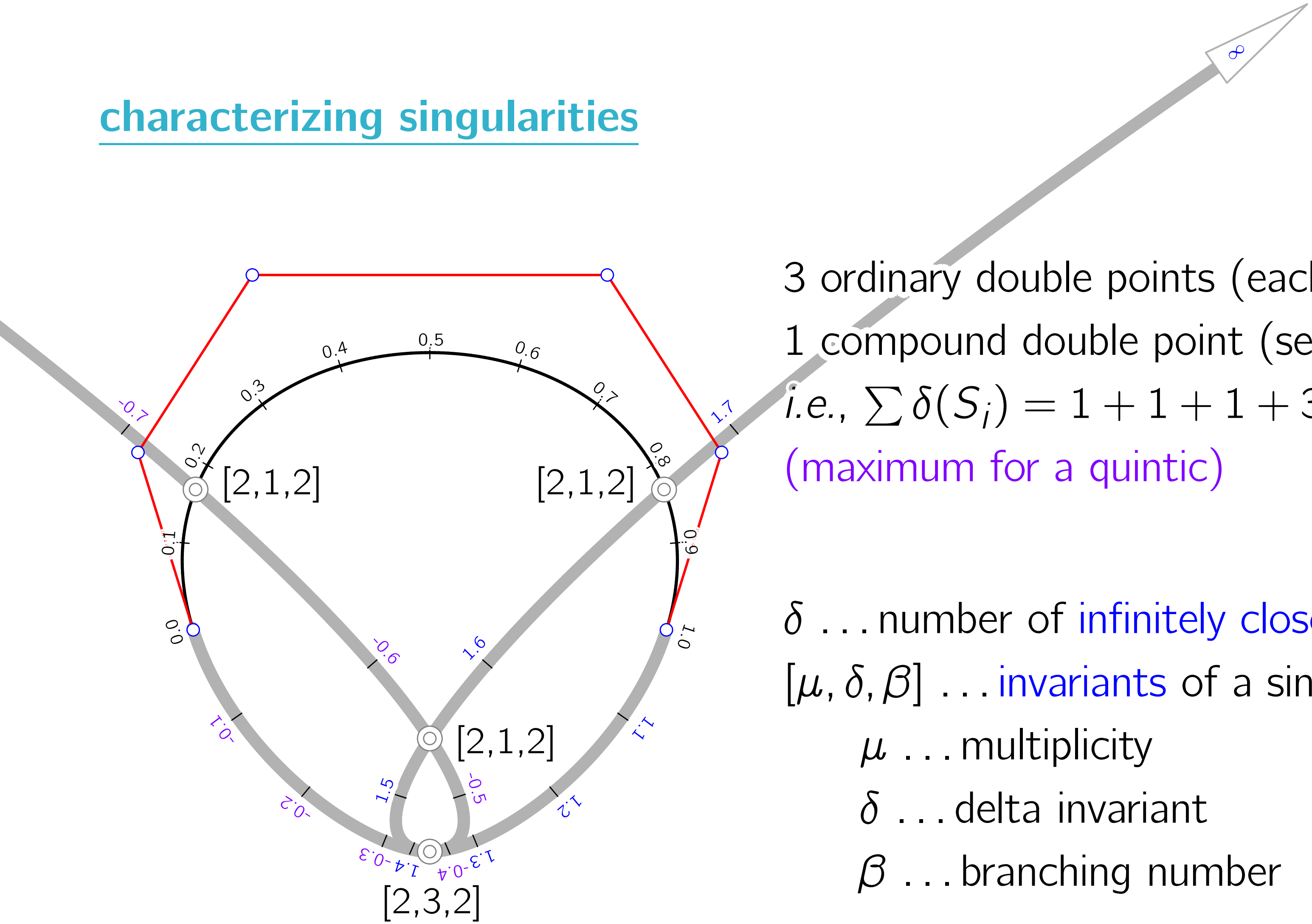
singularity in the interior of the unit interval (cusp, 1^{st} kind) sometimes

bad choice: $B_0 = B_3$, (ordinary node) obvious

bad choice: $B_0 = B_1$, (cusp, 1^{st} kind) obvious

The singularities on (Bézier) curves of degree ≥ 4 can be more spectacular.

characterizing singularities



3 ordinary double points (each with $\delta = 1$)

1 compound double point (self-osculation with $\delta = 3$)

i.e., $\sum \delta(S_i) = 1 + 1 + 1 + 3 = 6$

(maximum for a quintic)

δ . . . number of infinitely close ordinary double points

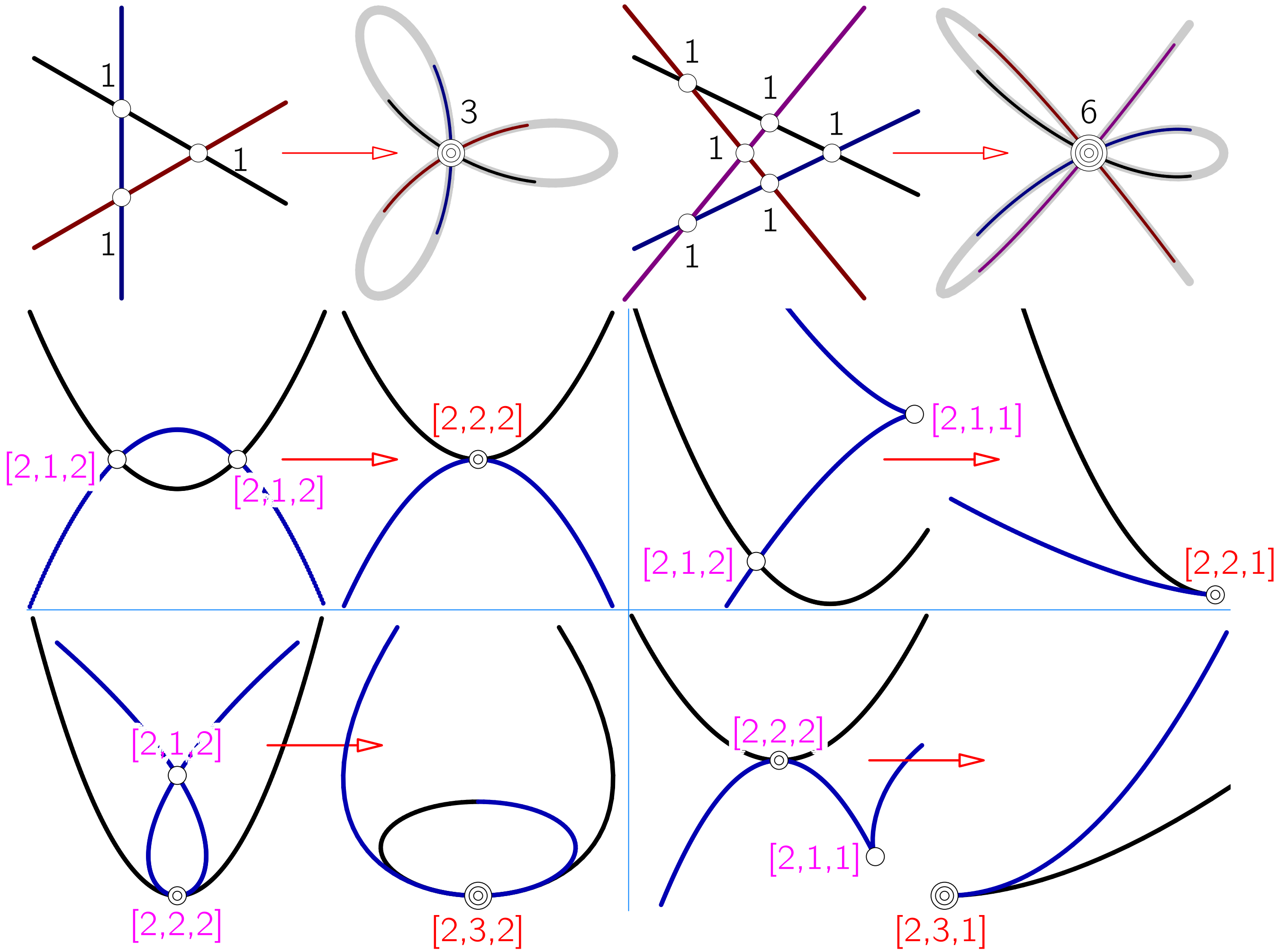
$[\mu, \delta, \beta]$ . . . invariants of a singularity w.r.t. PGL$(\mathbb{C}, 2)$

μ . . . multiplicity

δ . . . delta invariant

β . . . branching number

δ = number of ordinary double points concentrated



μ (linear) branches meet in
$\frac{1}{2}\mu(\mu-1)$ ord. double pts. ($\delta = 1$)
$\Longrightarrow$ 'tighten the knot' $\Longrightarrow$

$$\delta \geq \frac{1}{2}\mu(\mu-1)$$

compound double points
on quartics obtained by
merging singularities

The limit procedures towards
certain compound singularities
are not unique.

classification of Bézier curves - Plücker formulae

$$m = n(n-1) - 2d - 3s, \qquad w = 3n(n-2) - 6d - 8s$$

are used to rule out impossible cases. Usual form not sufficient. There are singularities more complicated than d and s (node and cusp).

Therefore: $m = n(n-1) - \sum \tau_i(S_i) \geq \frac{1+\sqrt{1+4n}}{2}, \qquad w = 3n(n-2) - \sum \theta_i(S_i) \geq 0$

$n = \deg \mathcal{C}$, $m = \deg \mathcal{C}^\star$ (class), d (number of ordinary nodes, A_1, [2, 1, 2]), s (number of ordinary cusps, A_2, [2, 1, 1]), τ, θ weights for the singularities

singularity	τ	θ	singularity	τ	θ	singularity	τ	θ	singularity	τ	θ
[4, 6, 1]	15	42	[3, 5, 1]	12	34	[3, 3, 2]	7	20	[2, 4, 2]	8	24
[4, 6, 2]	14	40	[3, 5, 2]	11	32	[3, 3, 3]	6	18	[2, 3, 1]	7	21
[4, 6, 3]	13	38	[3, 5, 3]	10	30	[2, 6, 1]	13	39	[2, 3, 2]	6	18
[4, 6, 4]	12	36	[3, 4, 1]	10	29	[2, 6, 2]	12	36	[2, 2, 1]	5	15
[3, 6, 1]	14	43	[3, 4, 2]	9	26	[2, 5, 1]	11	33	[2, 2, 2]	4	12
[3, 6, 2]	13	40	[3, 4, 3]	8	25	[2, 5, 2]	10	30	[2, 1, 1]	3	8
[3, 6, 3]	12	37	[3, 3, 1]	8	22	[2, 4, 1]	9	27	[2, 1, 2]	2	6

classification of genus 0 cubics, quartics, quintics by means of singularities

degree	μ_{max}	$\delta_{max} = g_{max}$	β_{max}	# cases ($\mathbb{C}$)
3	2	1	2	2
4	3	3	3	13
5	4	6	4	118

bounds for δ: $\frac{1}{2}\mu(\mu-1) \leq \delta \leq \frac{1}{2}(n-1)(n-2)$

rationality / genus = 0: $\sum \delta(S_i) = \frac{1}{2}(n-1)(n-2)$

branching number: $\beta \leq \mu$

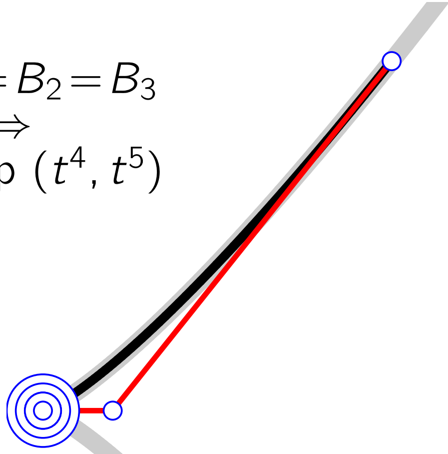
multiplicity: $\mu \leq n-1$, sum of any two $\leq n$

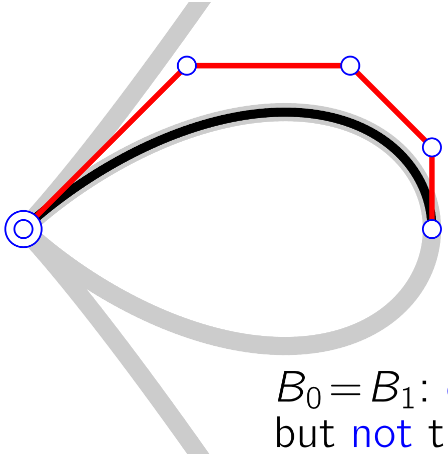
cardinal singularity	further singularities			class &, # flexes	
$[\mu, \delta, \beta]$		$\#d$	$\#s$	$\#m$	$\#w$
$[3, 3, 1]$	0	0	3	3	-1
$[2, 2, 1]$	$[2, 2, 1]$	0	2	4	-1
$[2, 2, 1]$	0	0	4	3	-2
$[2, 1, 1]$	0	1	4	3	-1
$[2, 1, 1]$	0	0	5	2	-3

some examples I

polynomial quintic Bézier curves with quadruple points / $\delta = 6$

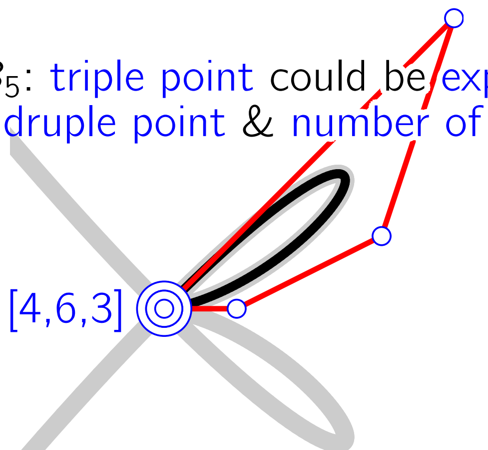
$B_0 = B_1 = B_2 = B_3$
$\Longrightarrow$
B_0 = cusp (t^4, t^5)

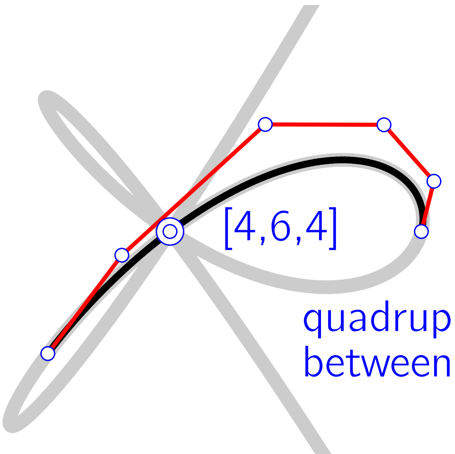




$B_0 = B_1$: double point could be expected,
but not the quadruple point

$B_0 = B_1 = B_5$: triple point could be expected, but
not the quadruple point & number of branches

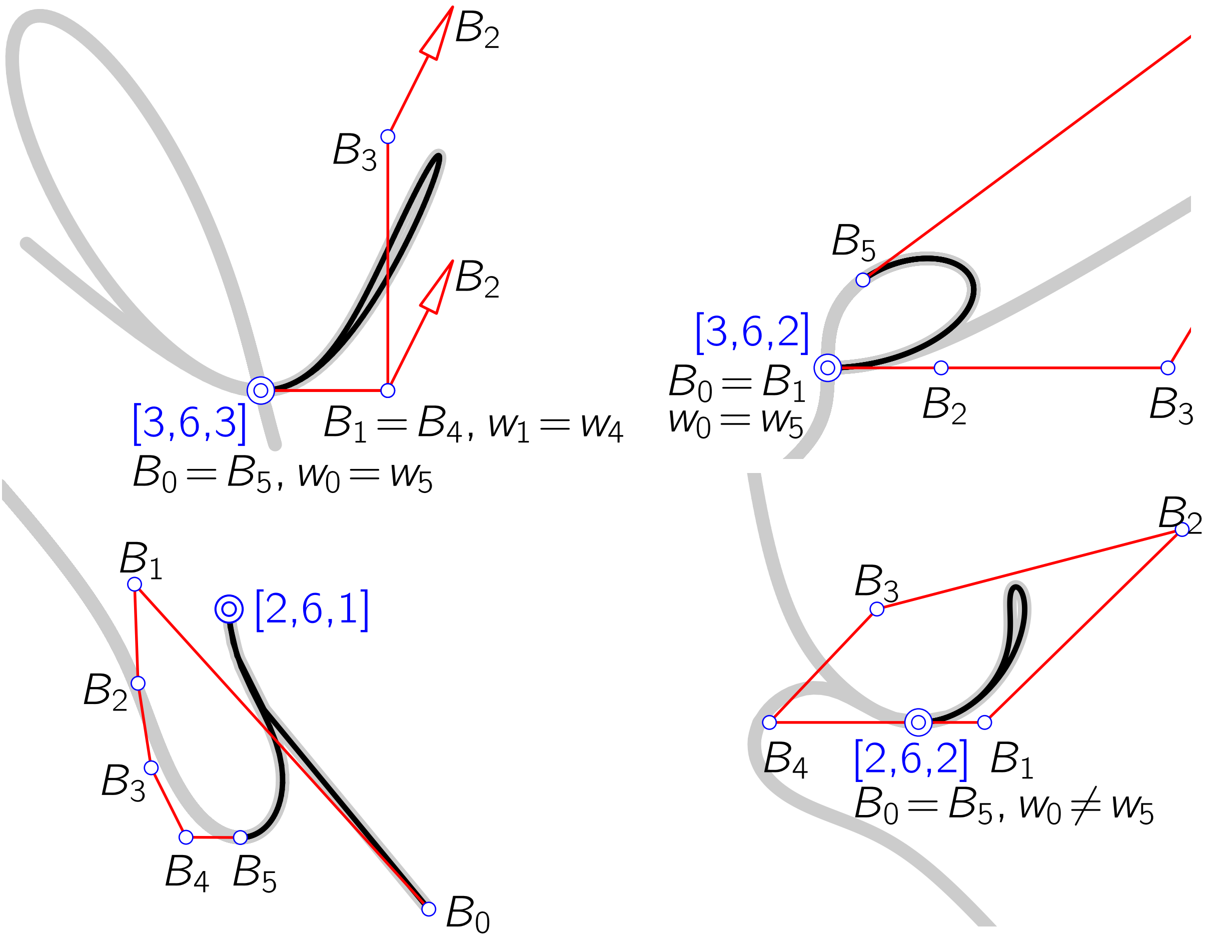




quadruple point on the segment
between B_0 and B_5

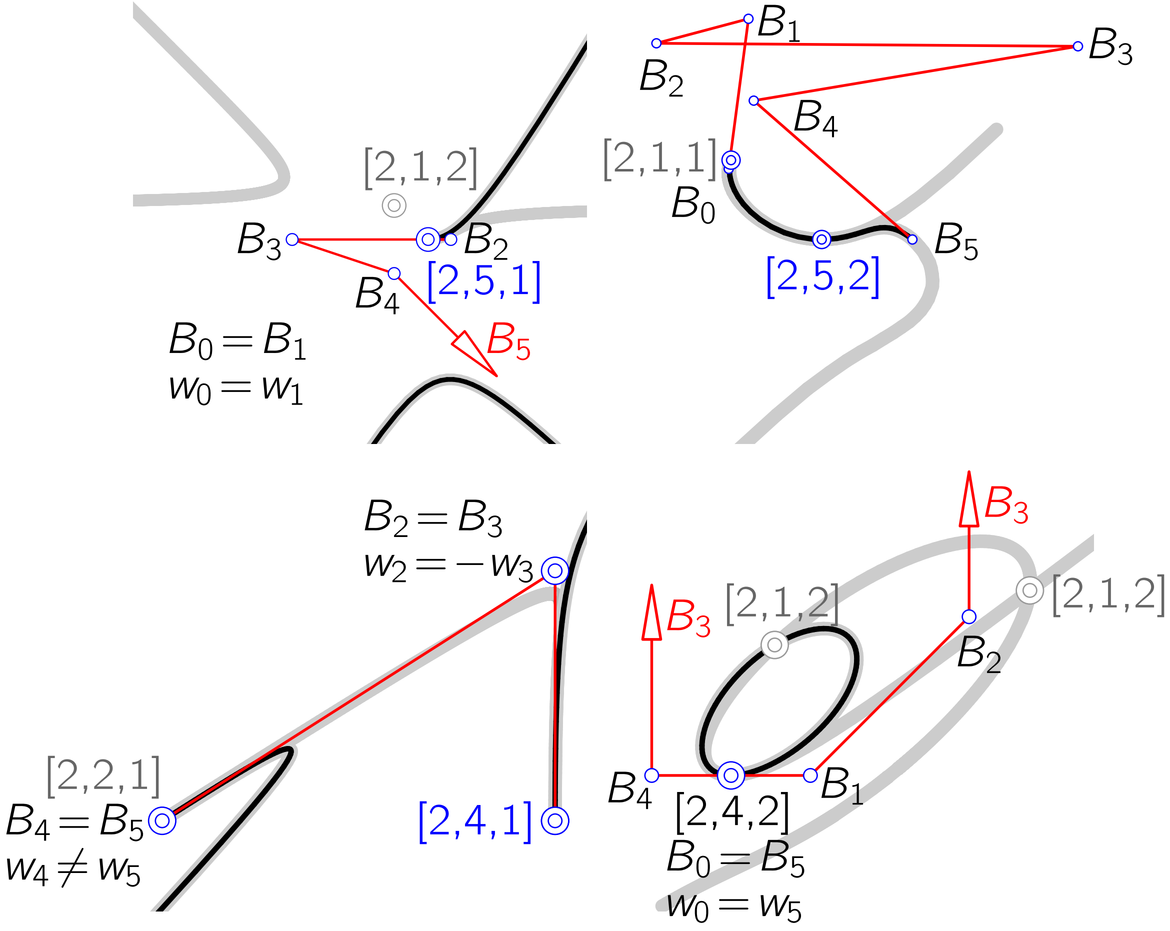
examples II

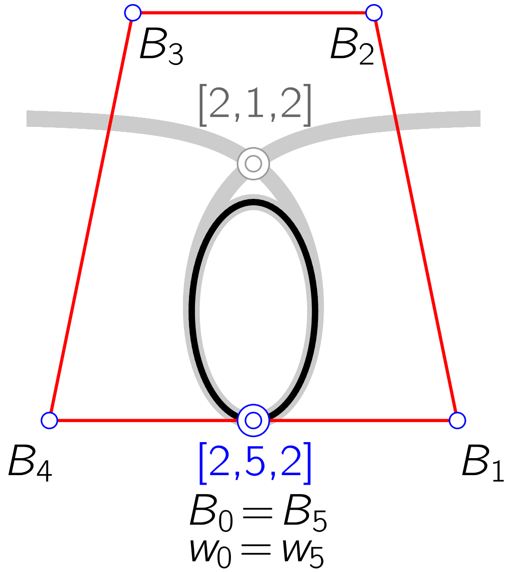
rational quintic Bézier curves with double and triple points / $\delta = 6$



some examples III

rational quintic Bézier curves with double and triple points / $\delta = 4, 5$



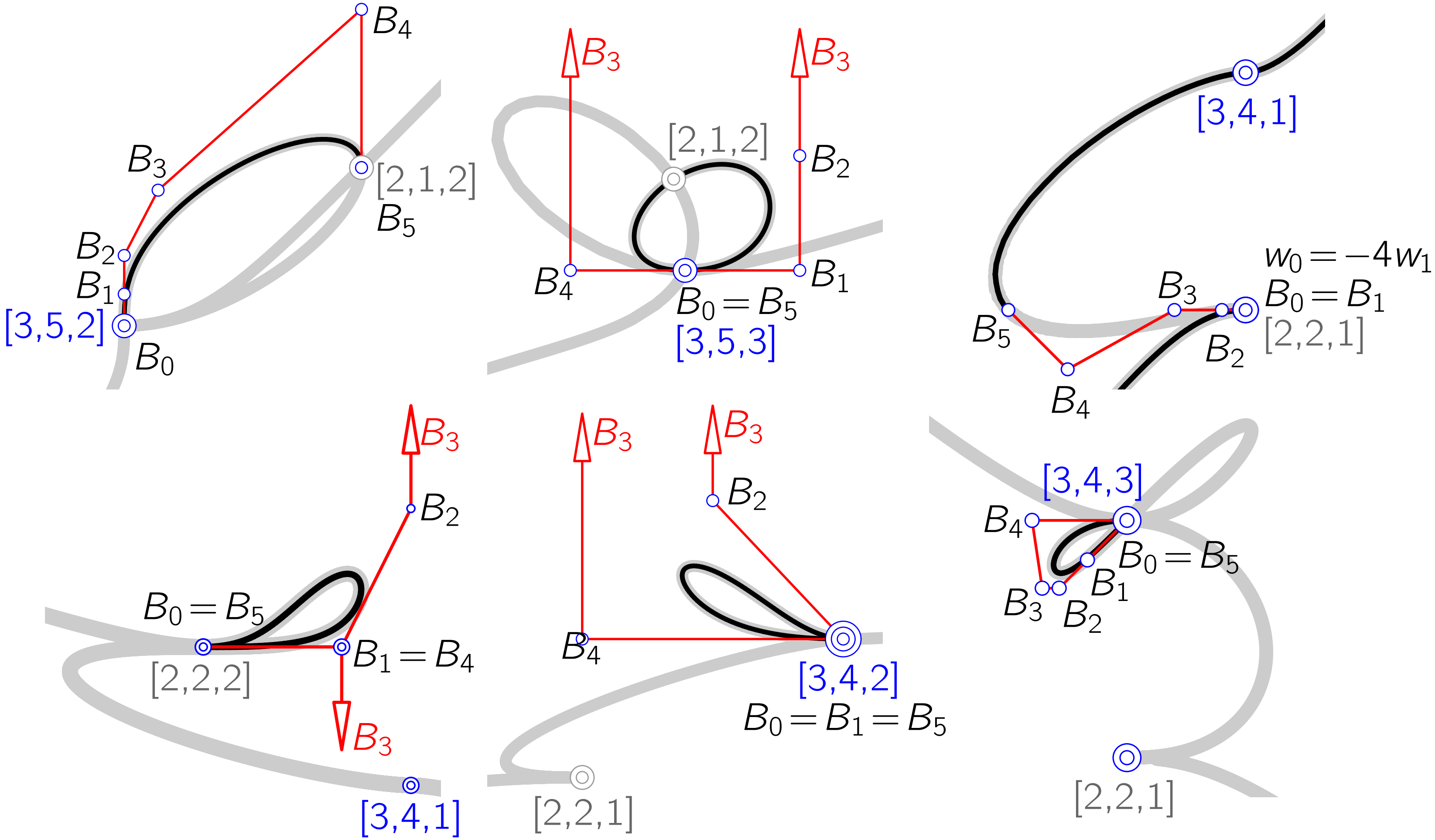


$\delta < 6$:

More singularities have to show up in order to make the curve rational.

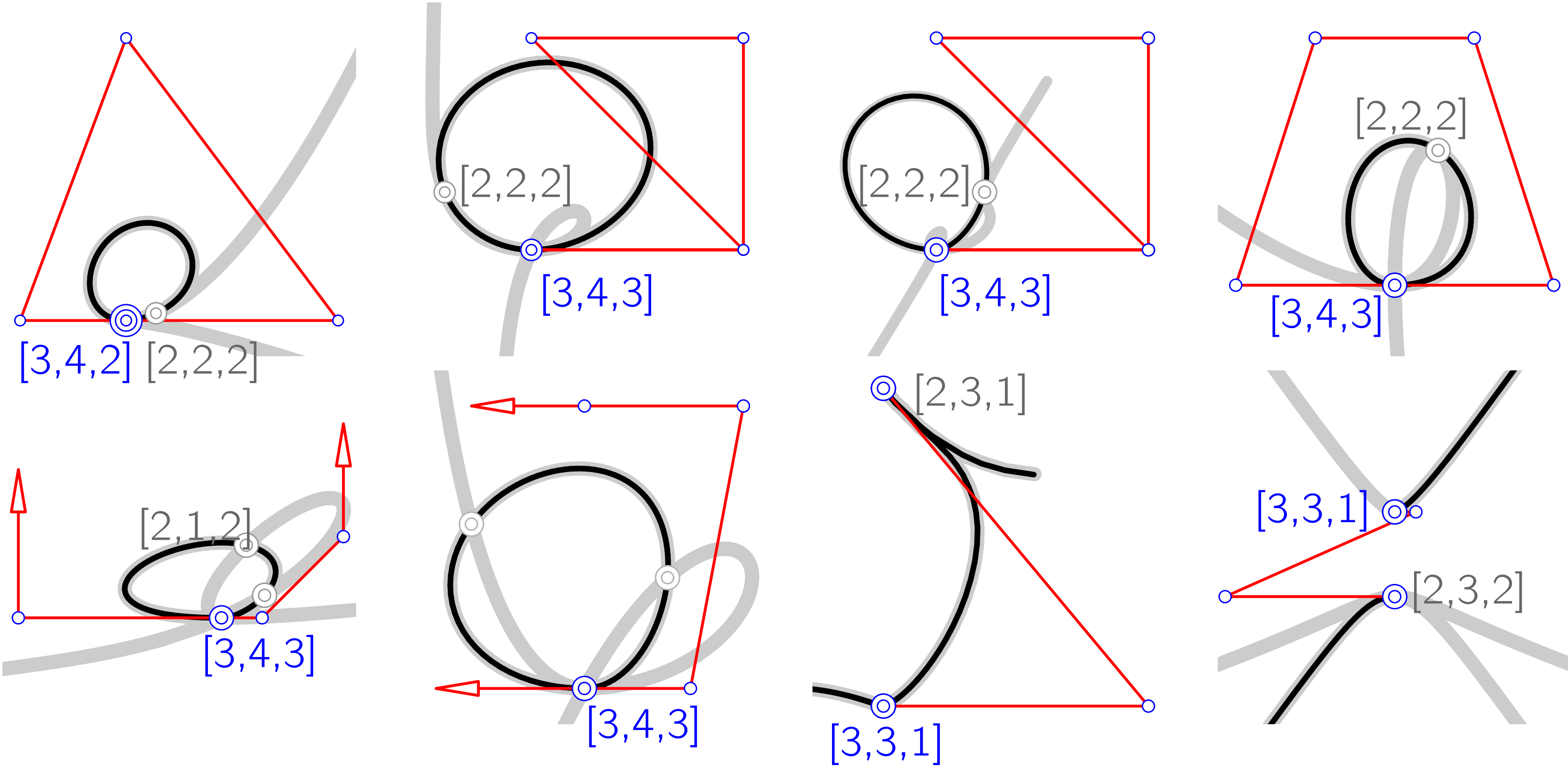
some examples IV

rational quintic Bézier curves with triple points / $\delta = 4, 5$

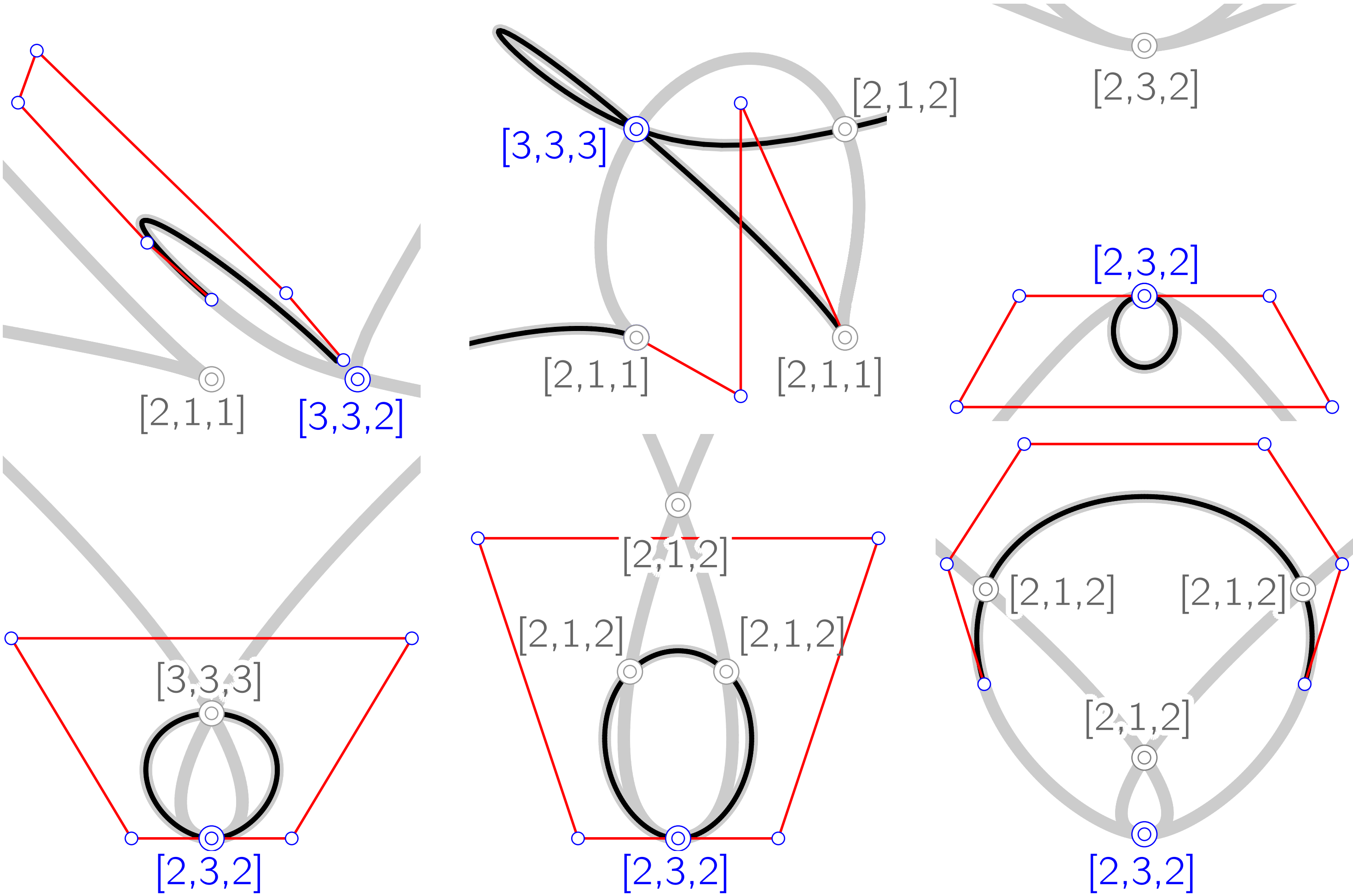


some examples V

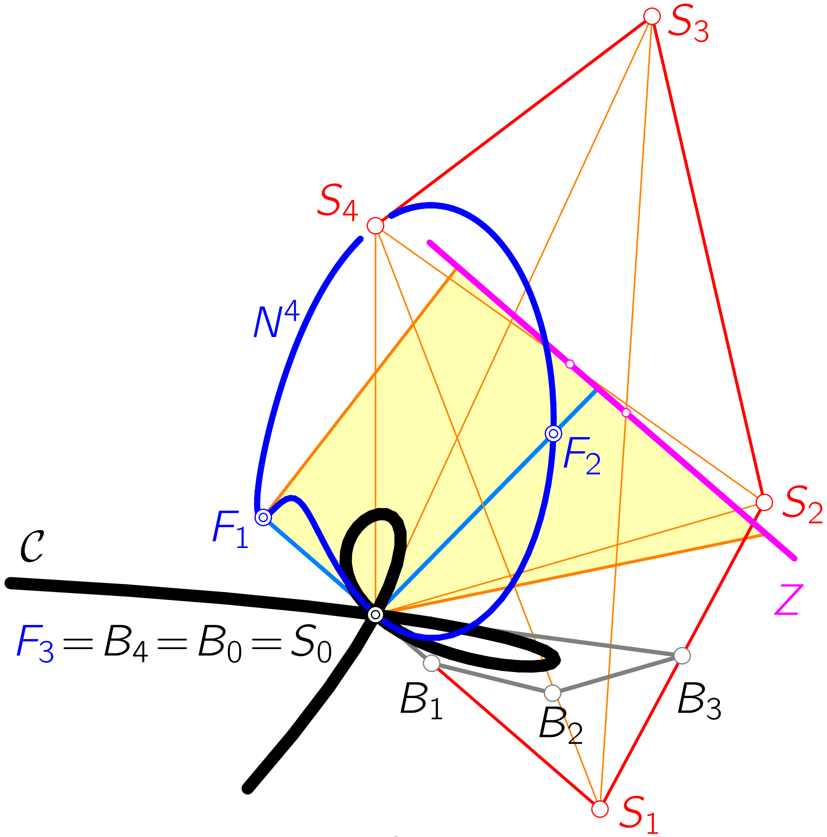
some triple points on rational quintic Bézier curves
The additional nodes complete the sum of δ invariants.

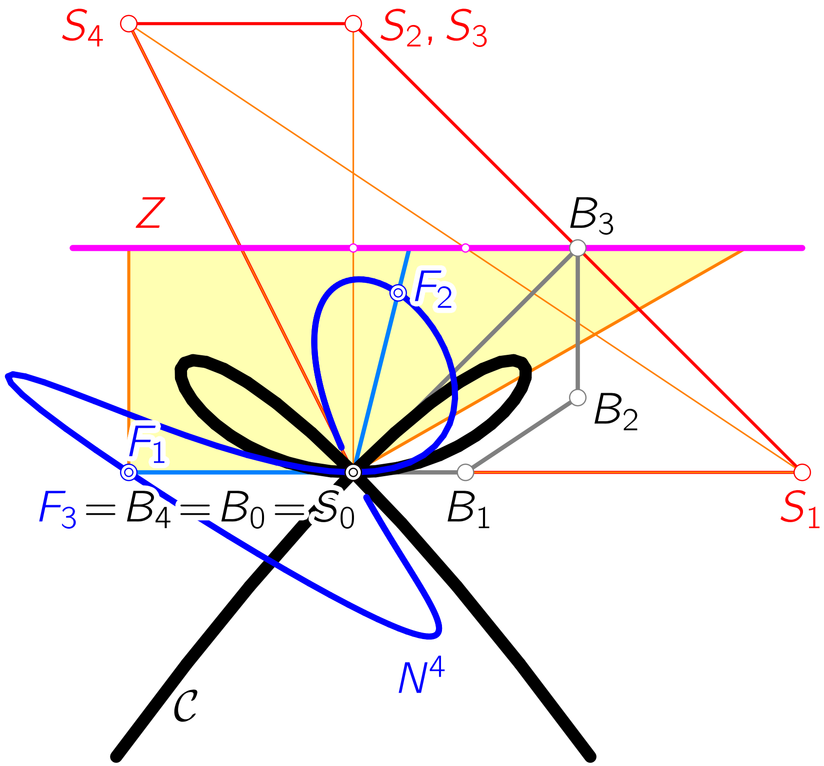


some examples VI



projections from $\mathbb{P}^3$, $\mathbb{P}^4$, $\mathbb{P}^5$ onto $\mathbb{P}^2$



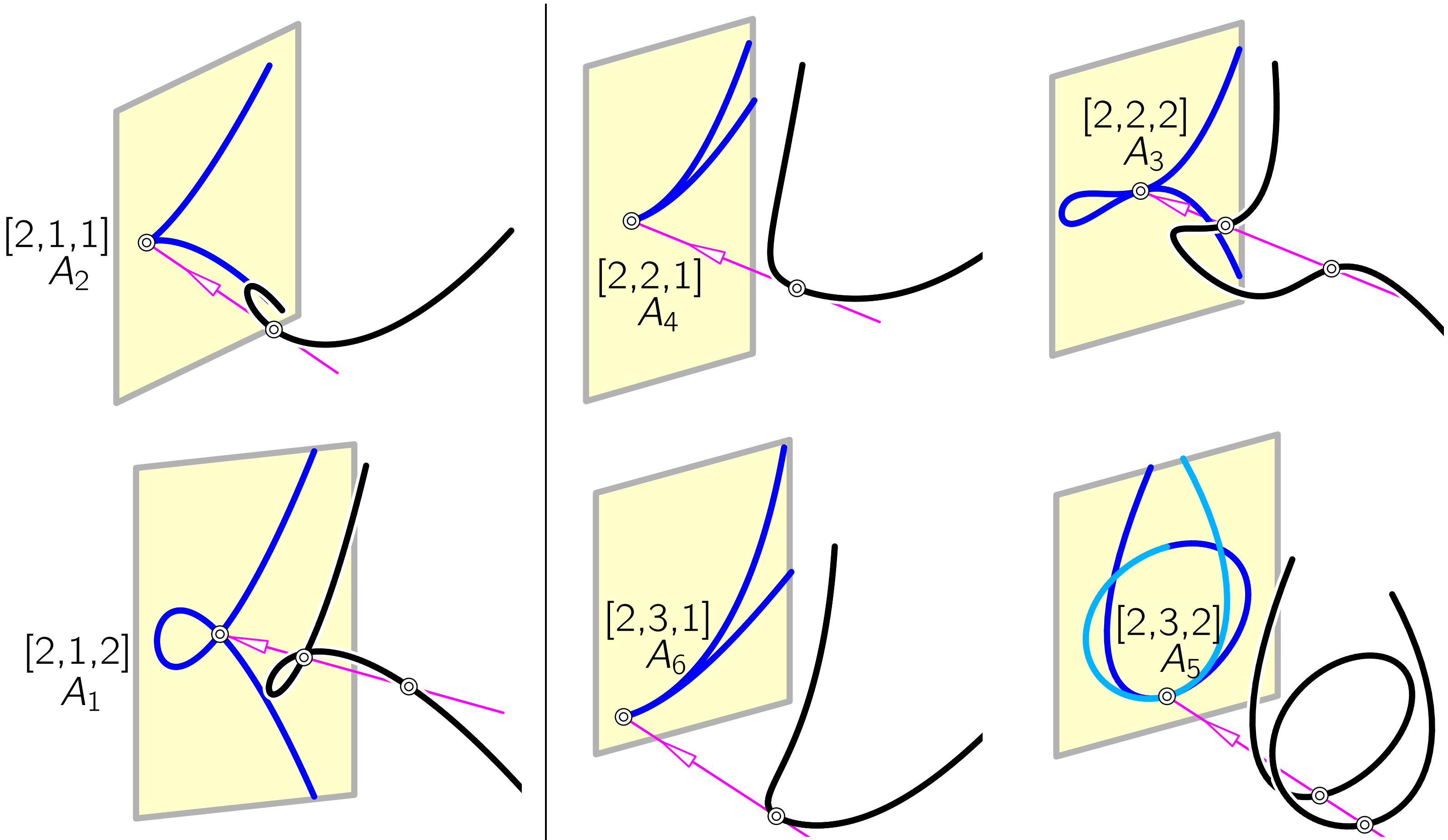


projection $\mathbb{P}^4 \to \mathbb{P}^2$ (center Z)
rational normal curve $N^4 \longrightarrow$ quartic Bézier curve $\mathcal{C}$ (triple point at S_0)
fibre plane meets N^4 in 3 distinct points
$F_1, F_2, F_3 \longrightarrow [3,3,3]$ at S_0

planar Bézier curve and its control structure
edges of osculating simplex at $S_0 \longrightarrow$ edges of the control polygon

projections from $\mathbb{P}^3$, $\mathbb{P}^4$, P^5 onto $\mathbb{P}^2$

3-space | 4-space (at least)



↑ they only look like handle points

What did we learn from this?

1. It is possible to choose control points such that singularities of any type occur.
2. Nevertheless, singularities are present in any case.
3. Some singularities occur in the interior of the convex hull of the control polygon.
4. Unexpected and unwanted singularities may show up on the standard segment.
5. Projections of Normal curves may explain some singularities but not all.

Thank You for Your Attention!

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