Singularities on Bézier Curves

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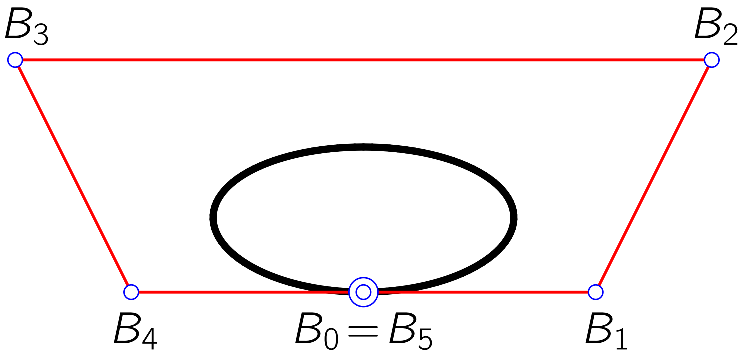
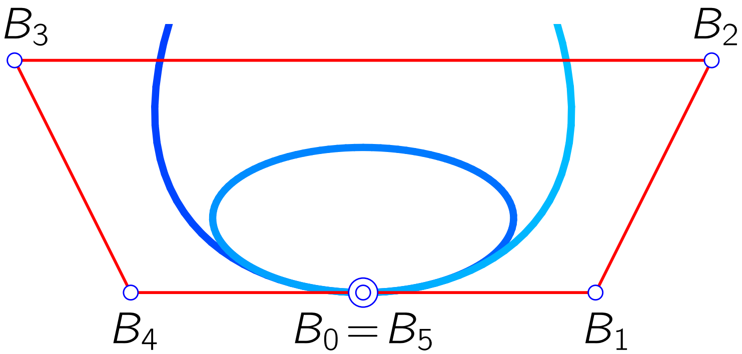
overview

rational Bézier curves of degree 3, 4, 5	and their singularities
the (im)proper choice of control points	What do we have to expect?
Bézier curves	projection of Normal Curves
singularities	results of special choices
	of the center of projection
	explains some cases, but . . .
examples	special control structures & singularities

(im)proper choice of control points





naive approach (with a quintic Bézier curve):

closed control polygon: $B_0 = B_5$

tangents at endpoint(s) coincide:

B_4, B_5, B_0, B_1 collinear

unit interval $\longrightarrow$ closed oval curve segment

even strictly convex

clear: tangent continuity at the join $B_0 = B_5$

due to symmetry: curvature continuity at the join

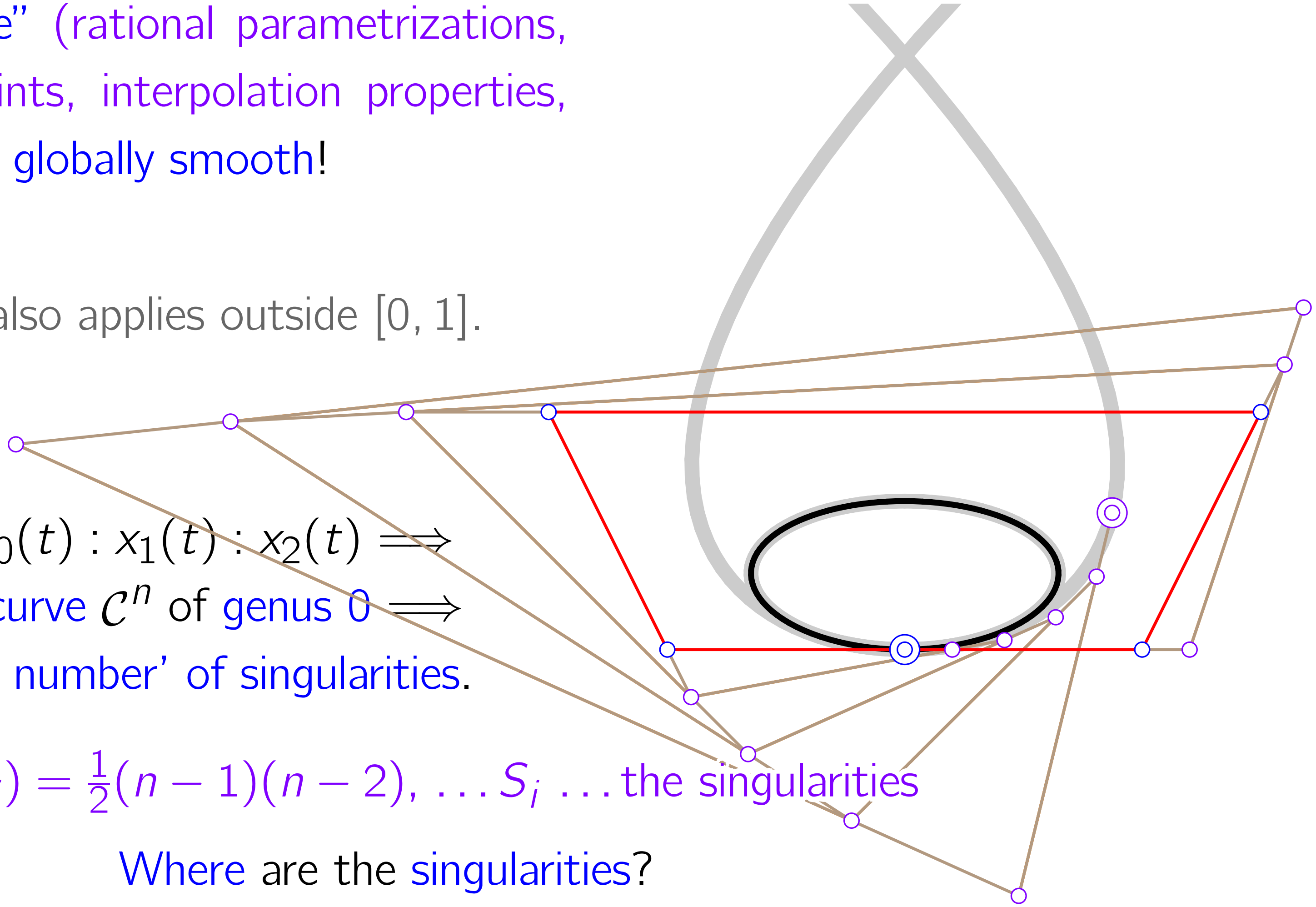
$\Longrightarrow$ B_0 is a tacnode with two osculating branches.

motivation

Bézier curves are so "nice" (rational parametrizations, linear construction of points, interpolation properties, ...) because they are not globally smooth!

De Casteljau's algorithm also applies outside $[0, 1]$.

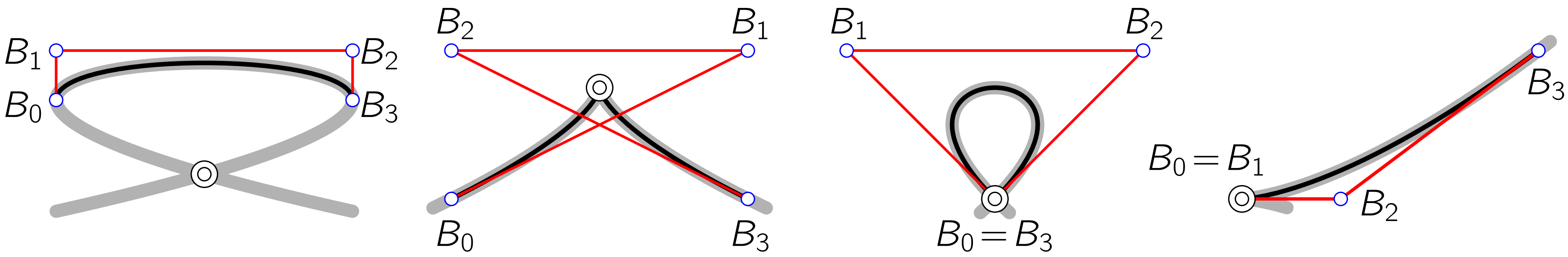
construction $\Longrightarrow$
rational parametrization $x_0(t) : x_1(t) : x_2(t) \Longrightarrow$
Bézier curve $\subset$ algebraic curve $\mathcal{C}^n$ of genus 0 $\Longrightarrow$
$\mathcal{C}^n$ exhibits the 'maximum number' of singularities.



more precise: $\sum_i \delta(S_i) = \frac{1}{2}(n-1)(n-2), \ldots S_i \ldots$ the singularities

Where are the singularities?

more or less obvious examples



nothing spectacular
in the unit interval
(ordinary node)
happens in any case

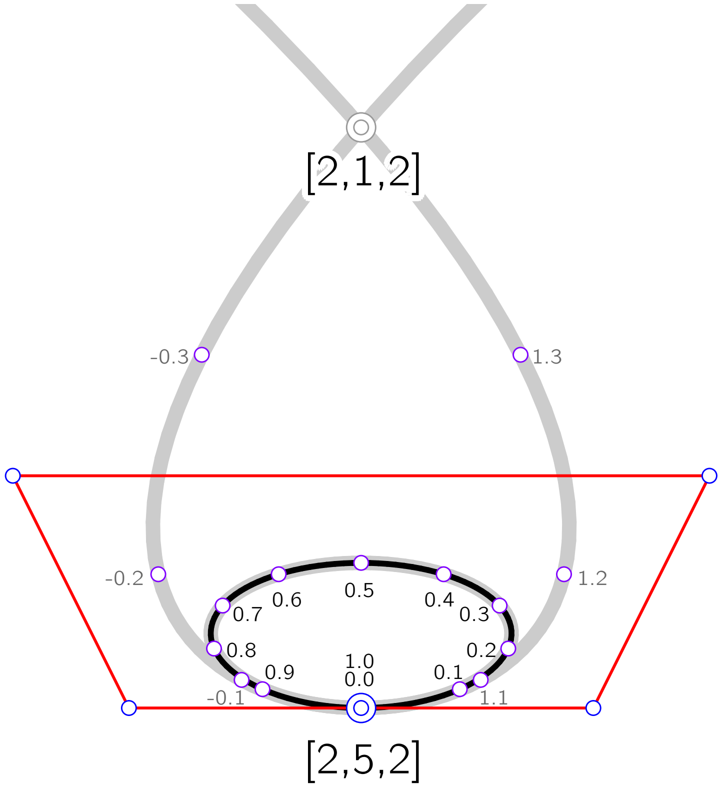
singularity in the interior of the unit interval (cusp, 1^{st} kind)
sometimes

bad choice:
$B_0 = B_3$,
(ordinary node)
obvious

bad choice:
$B_0 = B_1$,
(cusp, 1^{st} kind)
obvious

The singularities on (Bézier) curves of degree ≥ 4 can be more spectacular.

characterizing singularities



1 ordinary double point (with $\delta = 1$)

1 compound double point (self-super-osculation with $\delta = 5$)

i.e., $\sum \delta(S_i) = 1 + 5 = 6 = \frac{1}{2}(5-1)(5-2)$

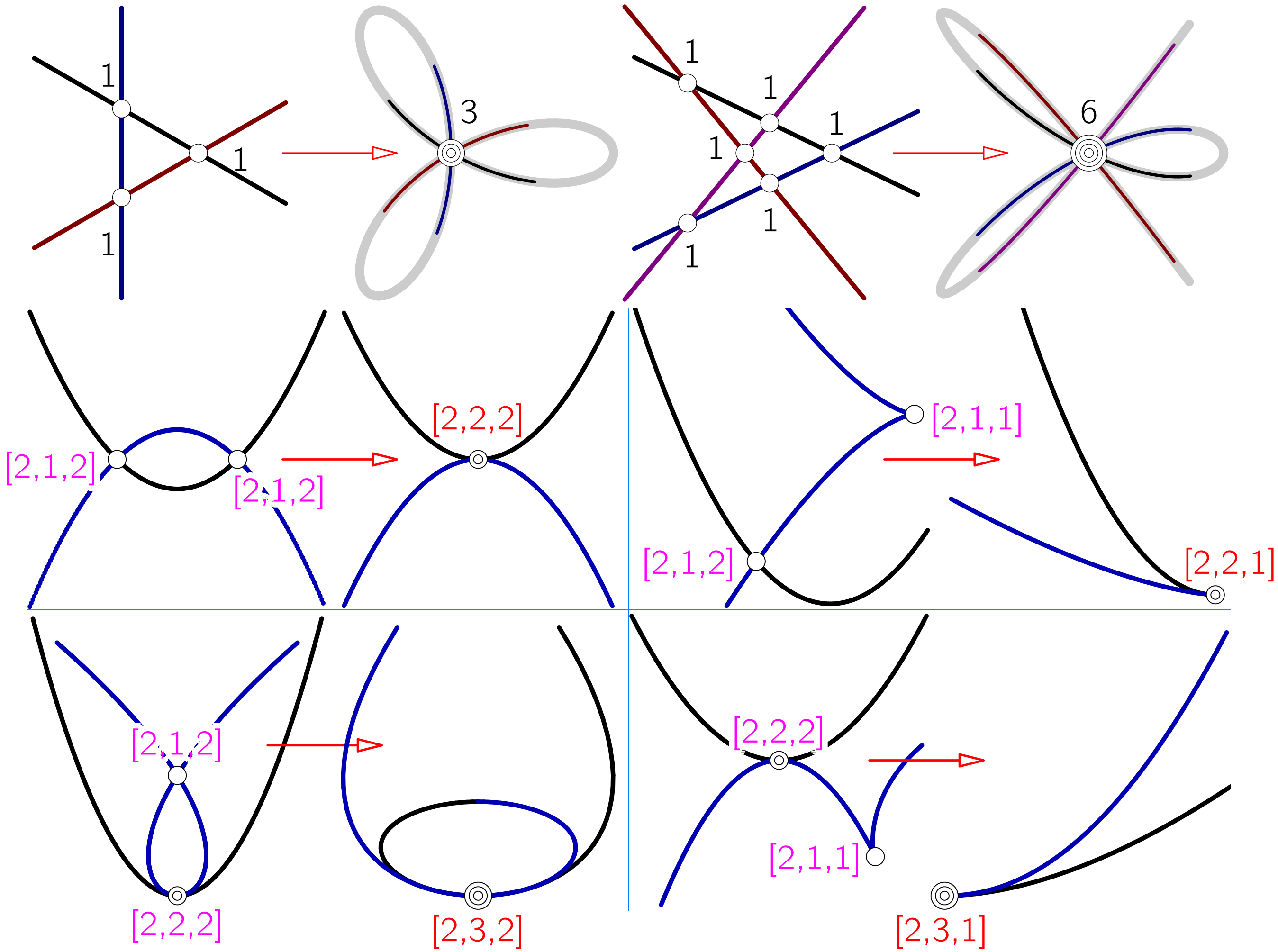
(maximum for a quintic)

δ . . . number of infinitely close ordinary double points

$[\mu, \delta, \beta]$ . . . invariants of a singularity w.r.t. $\mathrm{PGL}(\mathbb{C}, 2)$

- μ . . . multiplicity
- δ . . . delta invariant
- β . . . branching number

δ = number of ordinary double points concentrated



μ (linear) branches meet in $\frac{1}{2}\mu(\mu-1)$ ord. double pts. ($\delta = 1$) $\Longrightarrow$ 'tighten the knot' $\Longrightarrow$

$$\delta \geq \tfrac{1}{2}\mu(\mu-1)$$

compound double points on quartics obtained by merging singularities

The limit procedures towards certain compound singularities are not unique.

classification of Bézier curves - Plücker formulae . . .

$$m = n(n-1) - 2d - 3s, \qquad w = 3n(n-2) - 6d - 8s$$

are used to rule out impossible cases. Usual form not sufficient. There are singularities more complicated than d and s (node and cusp).

Therefore: $m = n(n-1) - \sum \tau_i(S_i) \geq \frac{1+\sqrt{1+4n}}{2}, \qquad w = 3n(n-2) - \sum \theta_i(S_i) \geq 0$

$n = \deg \mathcal{C}$, $m = \deg \mathcal{C}^\star$ (class), d (number of ordinary nodes, A_1, [2, 1, 2]), s (number of ordinary cusps, A_2, [2, 1, 1]), τ, θ weights for the singularities

singularity	τ	θ	singularity	τ	θ	singularity	τ	θ	singularity	τ	θ
[4, 6, 1]	15	42	[3, 5, 1]	12	34	[3, 3, 2]	7	20	[2, 4, 2]	8	24
[4, 6, 2]	14	40	[3, 5, 2]	11	32	[3, 3, 3]	6	18	[2, 3, 1]	7	21
[4, 6, 3]	13	38	[3, 5, 3]	10	30	[2, 6, 1]	13	39	[2, 3, 2]	6	18
[4, 6, 4]	12	36	[3, 4, 1]	10	29	[2, 6, 2]	12	36	[2, 2, 1]	5	15
[3, 6, 1]	14	43	[3, 4, 2]	9	26	[2, 5, 1]	11	33	[2, 2, 2]	4	12
[3, 6, 2]	13	40	[3, 4, 3]	8	25	[2, 5, 2]	10	30	[2, 1, 1]	3	8
[3, 6, 3]	12	37	[3, 3, 1]	8	22	[2, 4, 1]	9	27	[2, 1, 2]	2	6

classification of genus 0 cubics, quartics, quintics by means of singularities

degree	$\mu_{\max}$	$\delta_{\max} = g_{\max}$	$\beta_{\max}$	# cases ($\mathbb{C}$)
3	2	1	2	2
4	3	3	3	13
5	4	6	4	118

bounds for δ: $\frac{1}{2}\mu(\mu-1) \leq \delta \leq \frac{1}{2}(n-1)(n-2)$

rationality / genus = 0: $\sum \delta(S_i) = \frac{1}{2}(n-1)(n-2)$

branching number: $\beta \leq \mu$

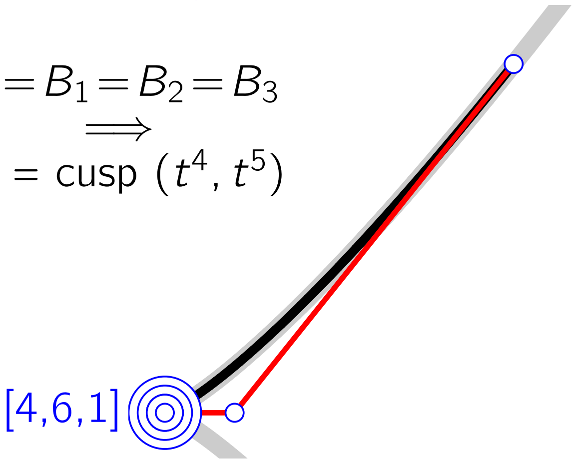
multiplicity: $\mu \leq n-1$, sum of any two $\leq n$

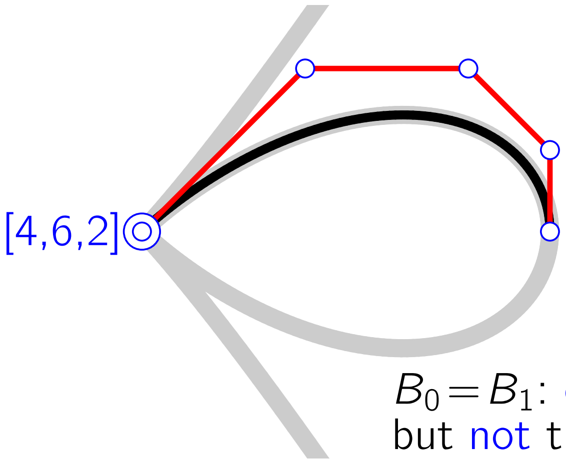
cardinal singularity	further singularities			class &, # flexes	
$[\mu, \delta, \beta]$		$\#d$	$\#s$	$\#m$	$\#w$
$[3, 3, 1]$	0	0	0	3	-1
$[2, 2, 1]$	$[2, 2, 1]$	0	2	4	-1
$[2, 2, 1]$	0	0	4	3	-2
$[2, 1, 1]$	0	1	4	3	-1
$[2, 1, 1]$	0	0	5	2	-3

some examples I

polynomial quintic Bézier curves with quadruple points / $\delta = 6$

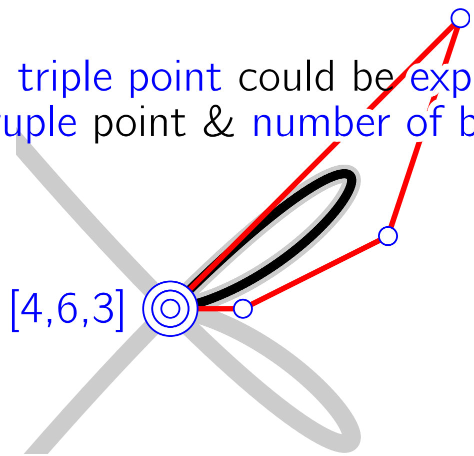
$B_0 = B_1 = B_2 = B_3 \Longrightarrow B_0 = \text{cusp } (t^4, t^5)$

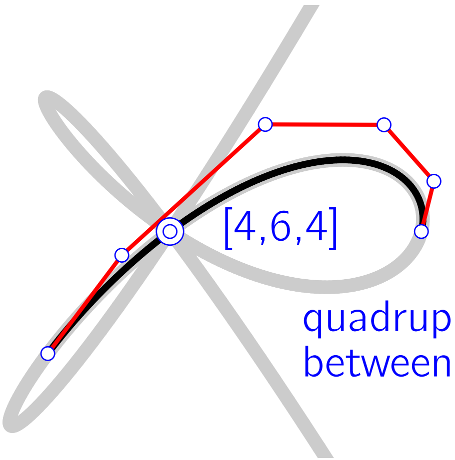




$B_0 = B_1$: double point could be expected, but not the quadruple point

$B_0 = B_1 = B_5$: triple point could be expected, but not the quadruple point & number of branches





quadruple point on the segment between B_0 and B_5

Bézier curves – projections of Normal Curves N^n

$N^n = (1, t, t^2, \ldots, t^n) \subset \mathbb{P}^n$ …rational normal curve of degree n, $[N^n] = \mathbb{P}^n$

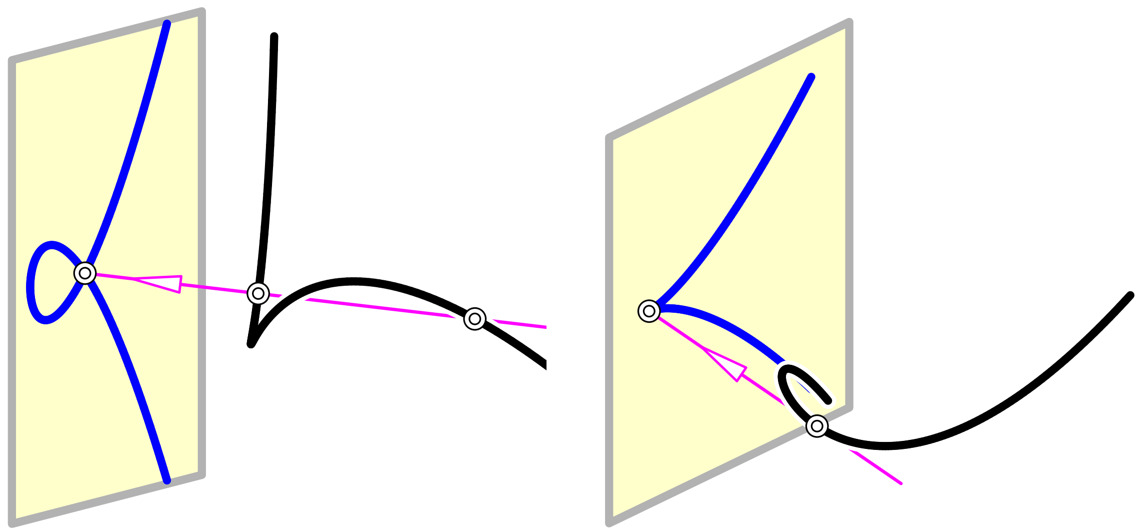
projection $\pi: \ \mathbb{P}^n \to \mathbb{P}^2$

$\mathbf{P} \in \mathbb{R}^{3 \times n+1}$ …coordinate matrix of π

$\mathbf{P} \cdot N^n$ …homogeneous parametrization of a Bézier curve

$\ker \mathbf{P}$ …center of π

The choice of the center is responsible for singularities on the image (Bézier curve).



the only relevant cases in $\mathbb{P}^3$:

fibre = secant	fibre = tangent
$[2, 1, 2]$	$[2, 1, 1]$
ordinary double point	cusp of the 1st kind

Normal Curves

Normal Curve $N^n \subset \mathbb{P}^n$

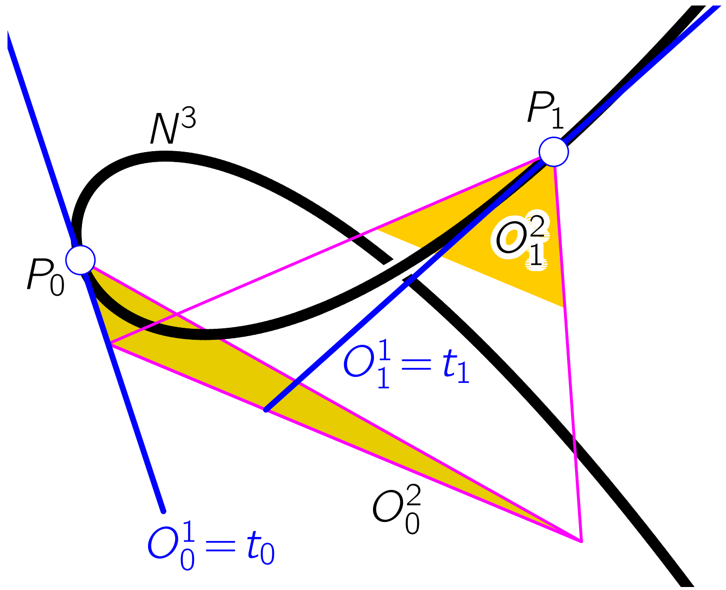
$N^n : \mathbb{P}^1 \to \mathbb{P}^n$, $(1,t) \mapsto (1, t, t^2, \dots, t^n)$, $t \in \mathbb{R} \cup \{\infty\}$

$[\mathbb{N}^n] = \mathbb{P}^n$,

$P_0 \neq P_1$, $P_0, P_1 \in N^n$: tangents t_0, t_1

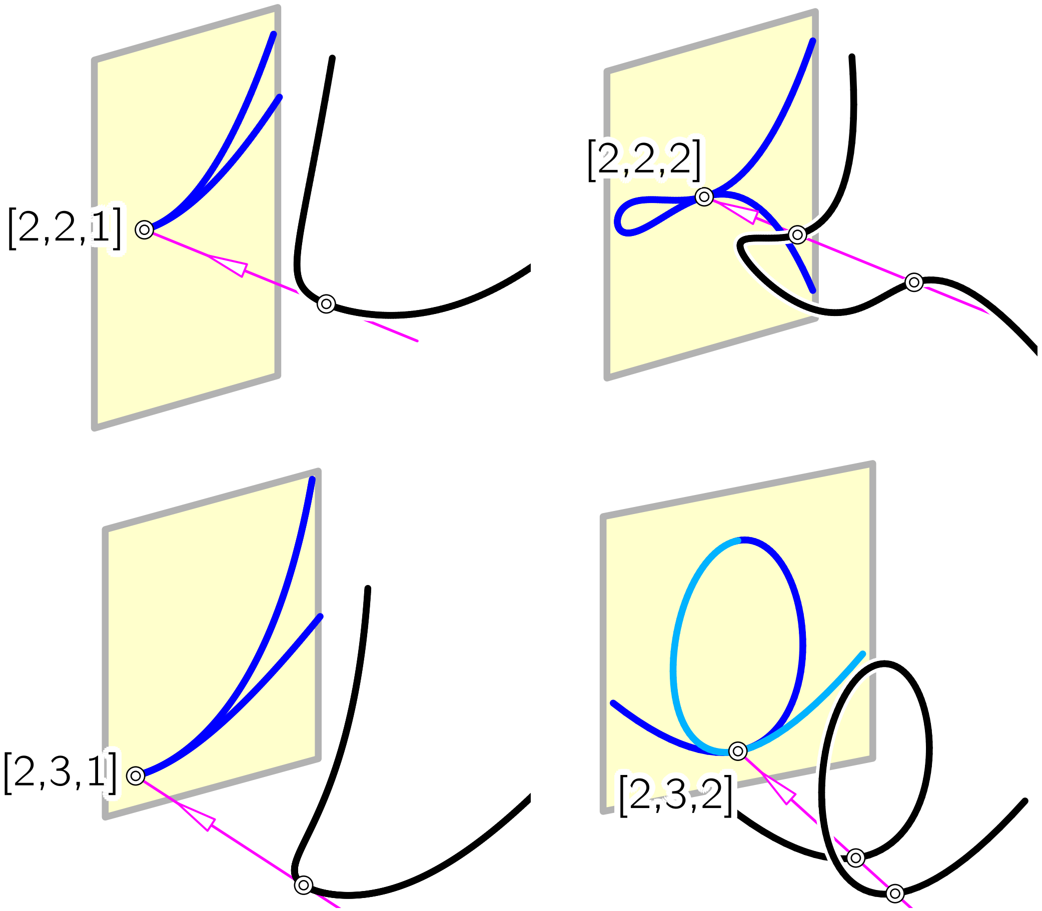
osculating subspaces O_0^k, O_1^k are maximally independent:

- $\mathbb{P}^3$: $t_0 \cap t_1 = \emptyset$
- $\mathbb{P}^4$: $O_0^2 \cap O_1^2 =$ single point
- $\mathbb{P}^5$: $O_0^2 \cap O_1^2 = \emptyset$, $O_0^3 \cap O_1^3 =$ straight line
- n odd: $P \in N^n \mapsto O^{n-1}(P)$ = (regular) null polarity



projections: compound double points on quartics

projection $\pi: \mathbb{P}^4 \to \mathbb{P}^2$



↑ they only look like handle points

center . . . straight line Z

fibres . . . planes

Z tangent to $N^3 \Longrightarrow$ contact point mapped to a $[2,1,1]$

Z secant of $N^3 \Longrightarrow$ intersection points mapped to a $[2,1,2]$

Fibre can contain up to 3 distinct points of N^3. $\Longrightarrow$ Triple points can occur first on quartics.

(That's clear from the algebraic point of view.)

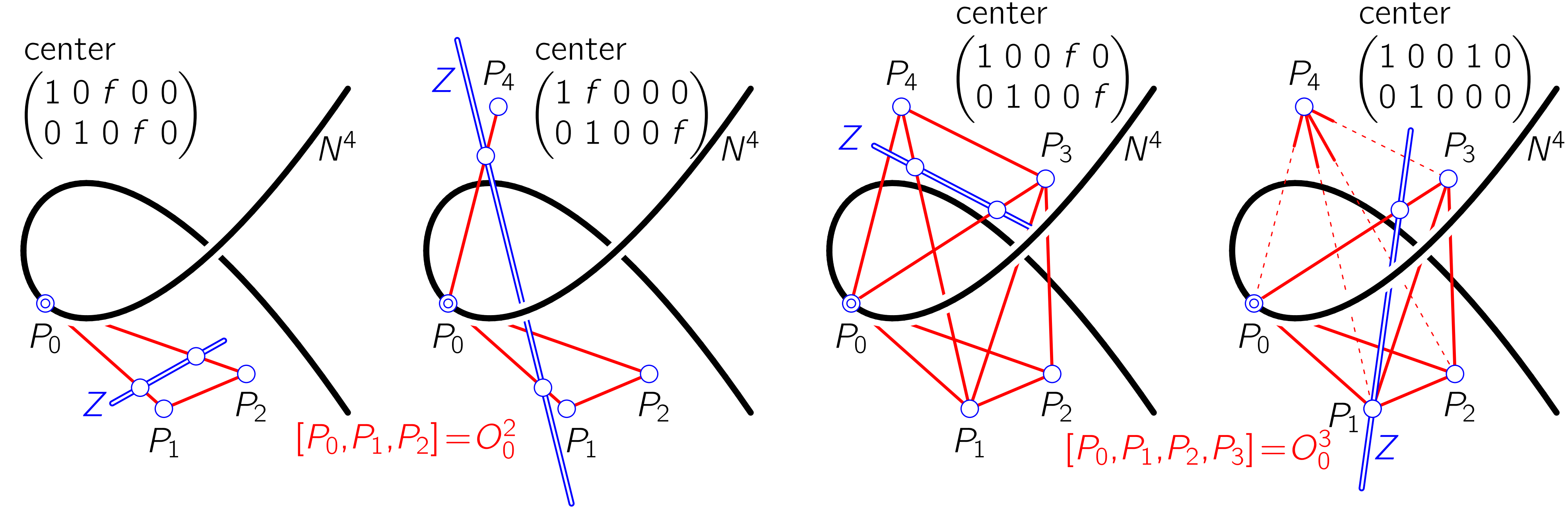
projections from $\mathbb{P}^4$

centers of the projection $\pi : \mathbb{P}^4 \to \mathbb{P}^2$ (straight line) and the corresponding singularities
eventually: projective displacement in the image space (Verlagerungskollineation)

singularity	center of the projection
$[3,3,1]$, $[3,3,2]$, $[3,3,3]$	$\begin{pmatrix} 1 & f & 0 & 0 & 0 \\ 0 & 1 & f & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & f & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & f \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & f & 0 & 0 \\ 0 & 1 & 0 & f & 0 \end{pmatrix}$
$[2,3,1]$, $[2,3,2]$	$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}$
$[2,2,1] + [2,1,1]$ / $+ [2,1,2]$	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & f & 0 \\ 0 & 1 & 0 & f & 0 \end{pmatrix}$
$[2,2,2] + [2,1,1]$ / $+ [2,1,2]$	$\begin{pmatrix} 1 & 1 & 3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 & f \\ 0 & 1 & 0 & f & 0 \end{pmatrix}$
$3 \times [2,1,1]$	$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$
$2 \times [2,1,1] + 1 \times [2,1,2]$	$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$
$1 \times [2,1,1] + 2 \times [2,1,2]$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$
$3 \times [2,1,2]$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{pmatrix}$, in any case: $f \in \mathbb{R}$

projections from $\mathbb{P}^4$: some examples

$P_0, P_1, \ldots, P_k$ basis of the osculating k frame O^k at P_0



center $Z \subset O_0^2$
[3, 3, 1]

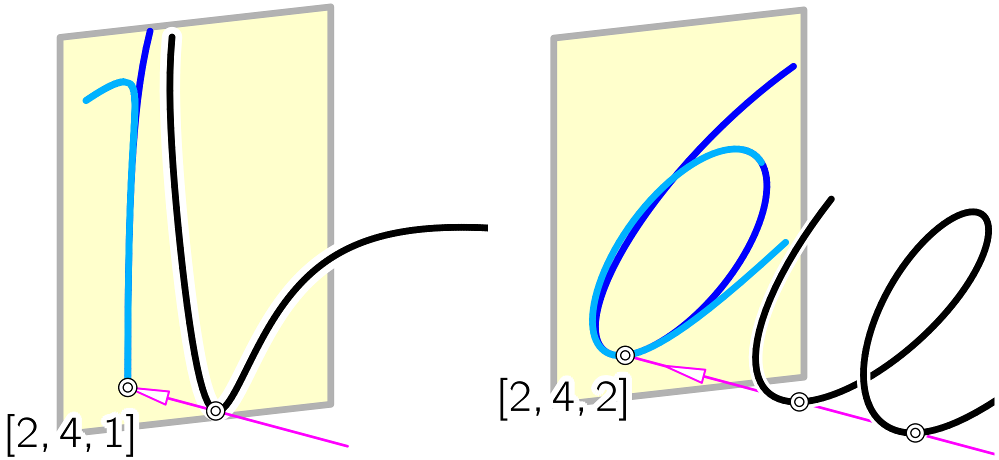
$Z \cap [P_0, P_1] \neq \emptyset$
[3, 3, 2]

$Z \subset [P_0, P_3, P_4]$
[3, 3, 3]

$Z \subset [P_0, P_1, P_3]$
[2, 3, 1]

projections from $\mathbb{P}^5$

projection $\pi : \mathbb{P}^5 \to \mathbb{P}^2$

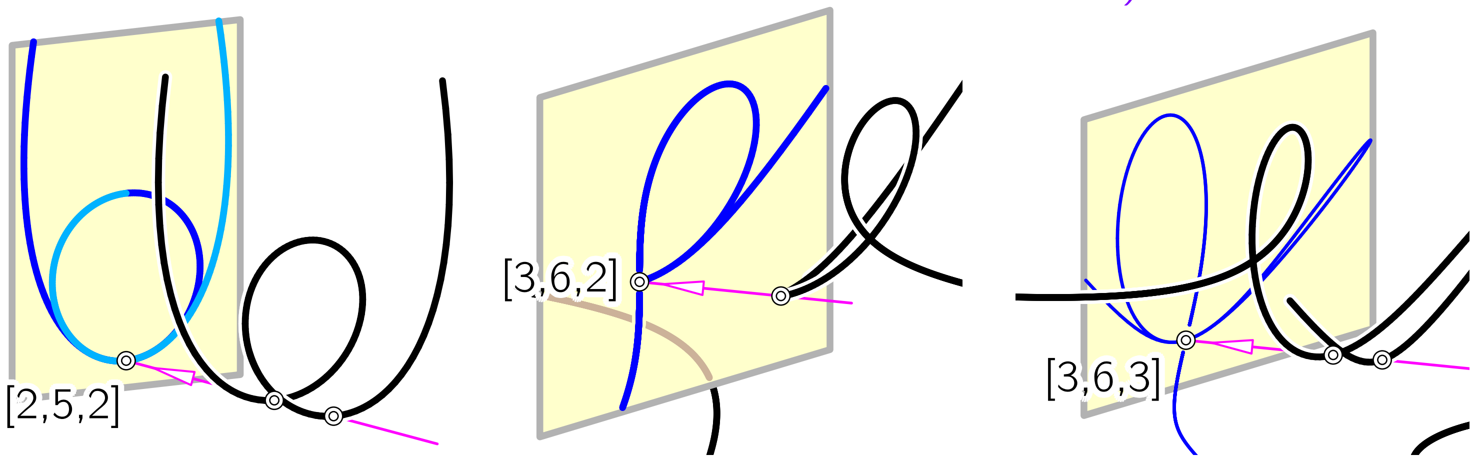


center ... plane Z

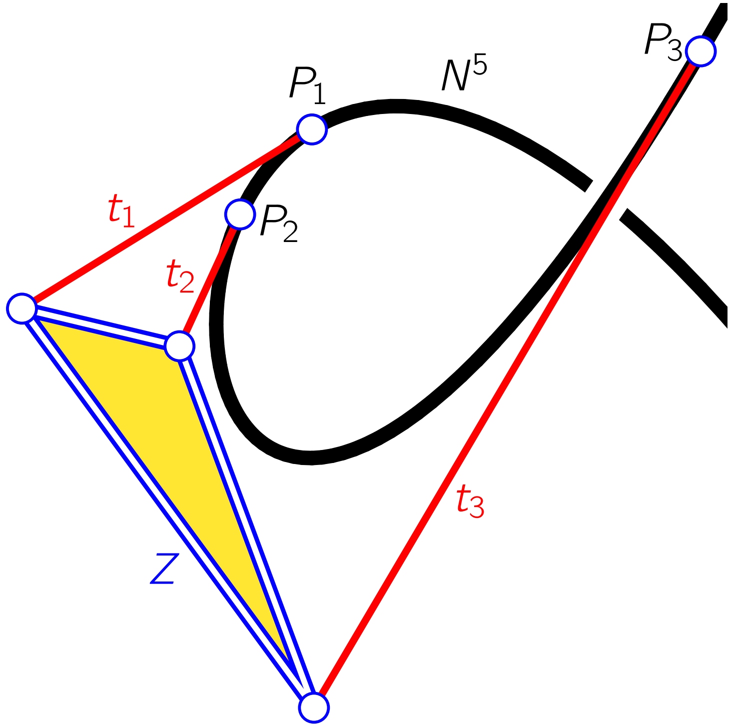
fibres ... 3-spaces

Fibre can contain up to 4 distinct points of N^5. $\Longrightarrow$ Quadruple points can occur first on quintics.

(That's clear from the algebraic point of view.)



projections from $\mathbb{P}^5$: simple example



2-dimensional center =
transversal of 3 tangents t_i at P_i
image curve equipped with 3 cusps at P_i'
(images of P_i), since
tangents t_i are contained in the fibres of P_i

The images of $N^5 = (1, t, \ldots, t^5)$ in the
2-dimensional faces of the canonical frame
are Bézier curves exhibiting one of the following
singularities:
$[2, 1, 1]$, $[2, 2, 1]$, $[3, 3, 1]$, $[3, 4, 1]$, $[4, 6, 1]$.

What did we learn from this?

1. It is possible to choose control points such that singularities of any type occur.
2. Nevertheless, singularities are present in any case.
3. Some singularities occur in the interior of the convex hull of the control polygon.
4. Unexpected and unwanted singularities may show up on the standard segment.
5. Projections of Normal curves may explain some singularities but not all.

Thank You for Your Attention!

some references

[1] V.I. Arnold, S.M. Gusein-Zade, A.N. Varachenko: *Singularities of differentiable maps.* Vol. II. Birkhäuser, Boston-Basel-Berlin, 1988.

[2] B.A. Basset: *Compound singularities of curves.* Quart. J. **37** (1905), 313–328.

[3] B.A. Basset: *Compound singularities of quintic curves.* Quart. J. **36** (1904), 359–372.

[4] E. Brieskorn, H. Knörrer: *Planar Algebraic Curves.* Birkhäuser, Basel, 1986.

[5] A. Brill: *On singularities of plane algebraic curves and a new species of curves.* Clebsch Ann. **XVI** (1880), 348–408.

[6 W.G. Bullard: *On the general classification of plane quartic curves.* PhD thesis, Worcester, Mass., 1899.

[7] W. Burau: *Algebraische Kurven und Flächen. I – Algebraische Kurven der Ebene.* De Gruyter, Berlin, 1962.

[8] I. Cheltsov: *Worst singularities on plane curves of given degree.* `https://doi.org/10.48550/arXiv.1409.6186`

[9] J.L. Coolidge: *A Treatise on Algebraic Plane Curves.* Oxford University Press, 1931.

[10] R.T. Farouki: *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable.* Springer-Verlag, Berlin Heidelberg, 2008.

[11] K. Fladt: *Analytische Geometrie spezieller ebener Kurven.* Akademische Verlagsgesellschaft, Frankfurt am Main, 1962.

[12] R. Gentry: *On the forms of plane quartic curves.* PhD thesis, Brynn Mawr, PA (1896), R. Drummond, NY, 73p.

[13] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Conics. From the ancient Greeks to 21st century developments.* 2nd ed. Springer-Spektrum, Springer-Verlag, Heidelberg, 2024.

[14] D.A. Gudkov, G.A. Utkin, M.L. Tai: *Vollständige Klassifizierung der nicht-zerfallenden Kurven vierter Ordnung.* Mat. Sb., N. Ser. **69**/111 (1966), 222–256.

[15] H. Hilton, S.D. Jervis: *On plane quartic curves with a triple point.* Proc. London Math. Soc. **2**/25 (1926), 385–400.

[16] J. Hoschek, D. Lasser: *Fundamentals of Computer Aided Geometric Design.* A.K. Peters / CRC Press, 1996.

[17] R.L. Krueger: *Plane quintic curves, a classification and projective construction.* Tôhoku math. Journ. **41** (1935), 27–51.

[18] E. Kunz: *Introduction to plane algebraic curves.* Birkhäuser, Boston, 2000.

[19] J.D. Lawrence: *A catalog of special plane curves.* Dover Publications, New York, 1972.

[20] G.B. Mathews: *On a certain class of plane quartics.* London M. S. Proc. **XXII** (1891), 173–176.

[21] J.H. Neelley: *Compound singularities of the Rational Plane Quartic Curve.* Amer. J. Math. **49**/3 (1927), 389–400.

[22] C.T.C. Wall: *Geometry of quartic curves.* Math. Proc. Camb. Phil. Soc. **117**/3 (1995), 415–423.

[23] C.T.C. Wall: *Highly singular quintic curves.* Math. Proc. Camb. Phil. Soc. **119**/2 (1996), 257–277.

[24] D.J. Walton, D.S. Meek: *Planar G^2 transition with a fair Pythagorean hodograph quintic curve.* J. Comput. Appl. Math. **138** (2002), 109–126.