Projective Parallelians and related Porisms

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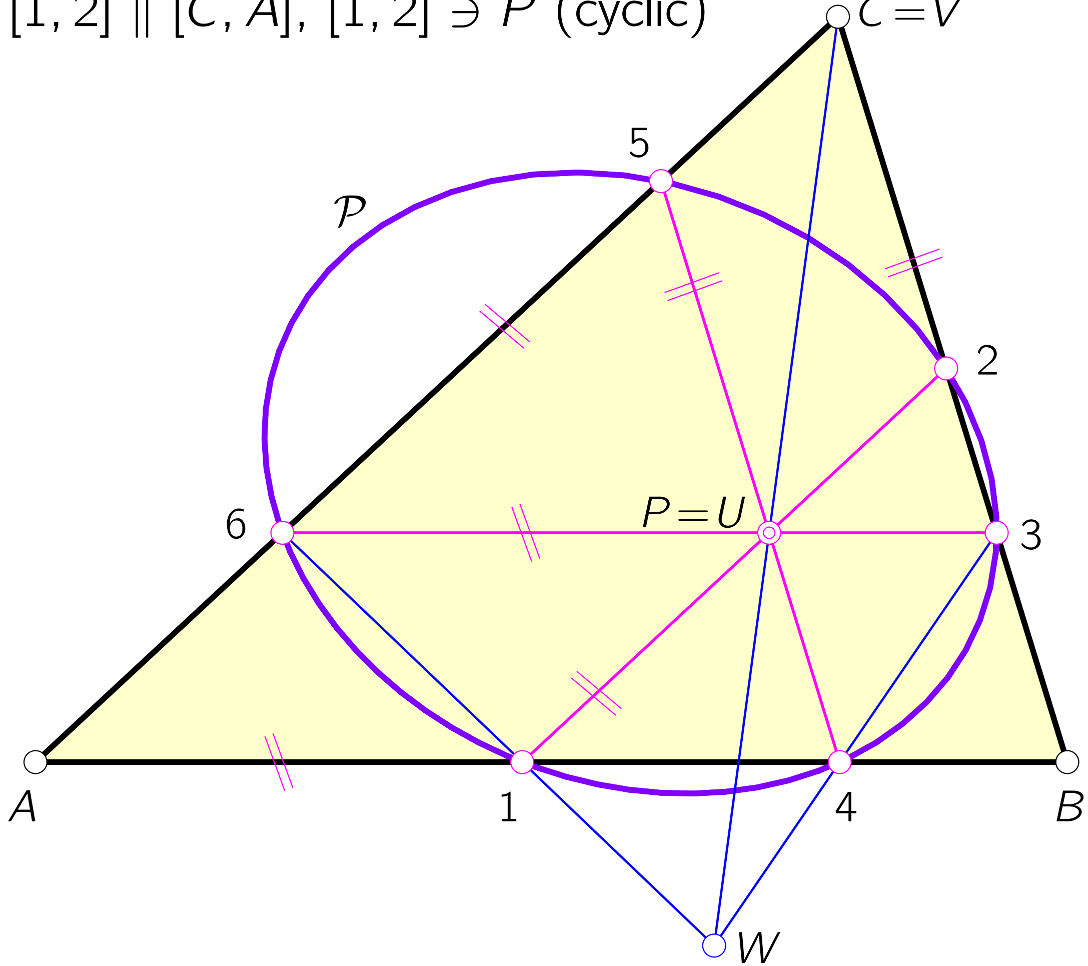
overview

Euclidean parallelians	just a special case
parallelian conics	in- and circumscribed, tangent conics
porisms	triangles, hexagons, . . .
projective view on	some Euclidean objects:
	Steiner in- & circumconic, Steiner deltoid
	elliptic curves and special pivot point
special cases	Euclidean version, Yff conics, . . .

parallelians and the parallelian conic $\mathcal{P}$

triangle $\Delta = ABC$, pivot point $P \notin \Delta^\star$
$[1,2] \parallel [C,A]$, $[1,2] \ni P$ (cyclic)



The parallelians $1, \ldots, 6$
lie on a single conic $\mathcal{P}$, parallelian conic.

proof:

$1, \ldots, 6$ fulfill the Pappos criterion

$[1,2] \cap [4,5] = P = U$,

$[2,3] \cap [5,6] = C = V$.

It remains to show that

$[3,4] \cap [6,1] = W \in [U,V]$

$63V \sim 14U$ and $[6,3] \parallel [1,4]$, ...

$\Longrightarrow$ $\exists$ central similarity with center

$[6,1] \cap [3,4] = W$

and $6 \mapsto 1$, $3 \mapsto 4$, $V \mapsto U$.

$\Longrightarrow$ U, V, W are collinear.

projective in nature

replace ideal line ω with some line g

ideal points of $[A, B], \ldots \Longrightarrow C^\star = [B, C] \cap g \ldots$

$1, \ldots, 6 \ldots g$-parallelians of P

$1, \ldots, 6$ lie on the g-parallelian conic $\mathcal{P}$

proof:

labelling of g-parallelians already indicates that g equals the Pascal axis

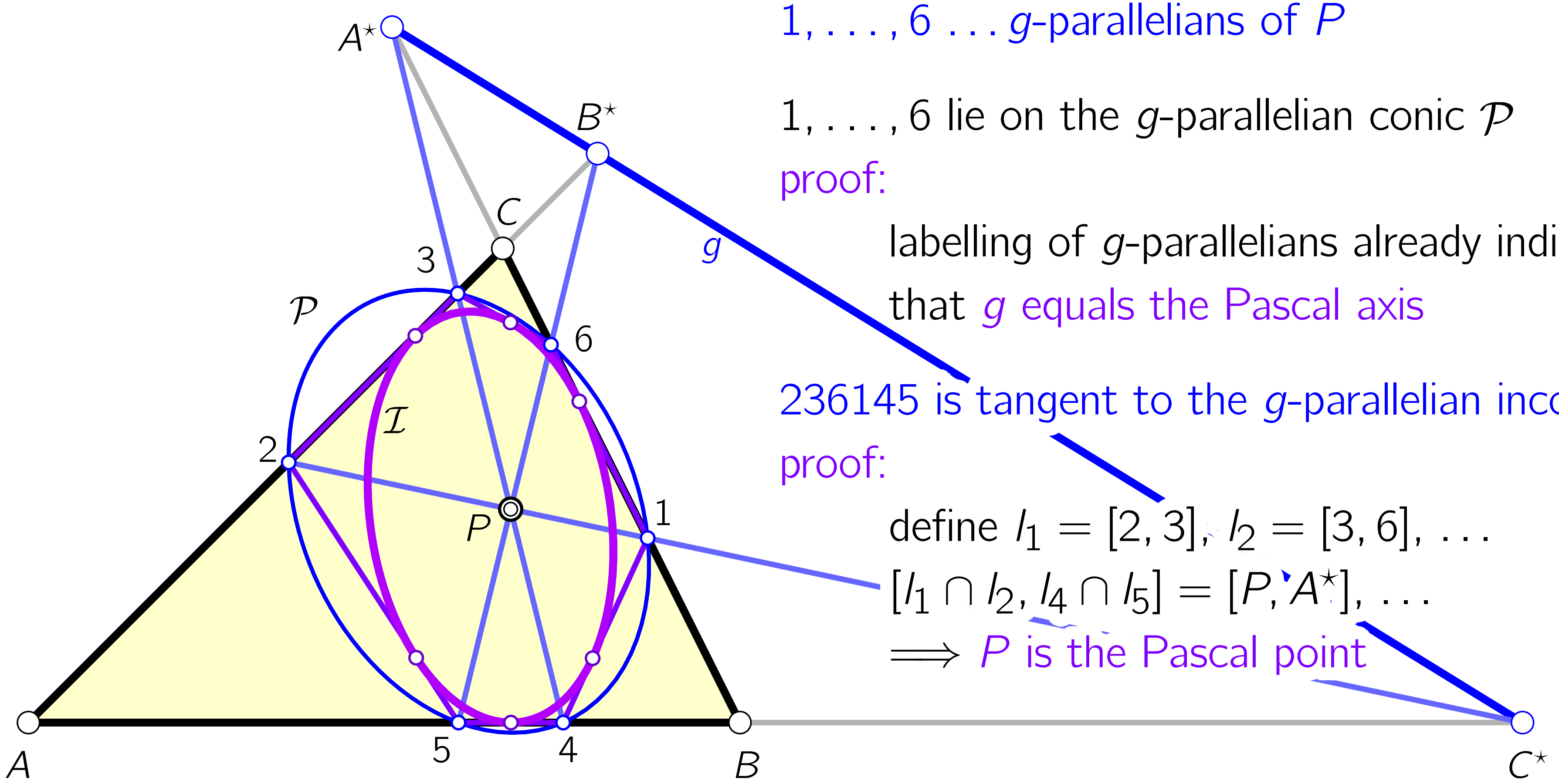
236145 is tangent to the g-parallelian inconic $\mathcal{I}$

proof:

define $l_1 = [2, 3]$, $l_2 = [3, 6]$, ...

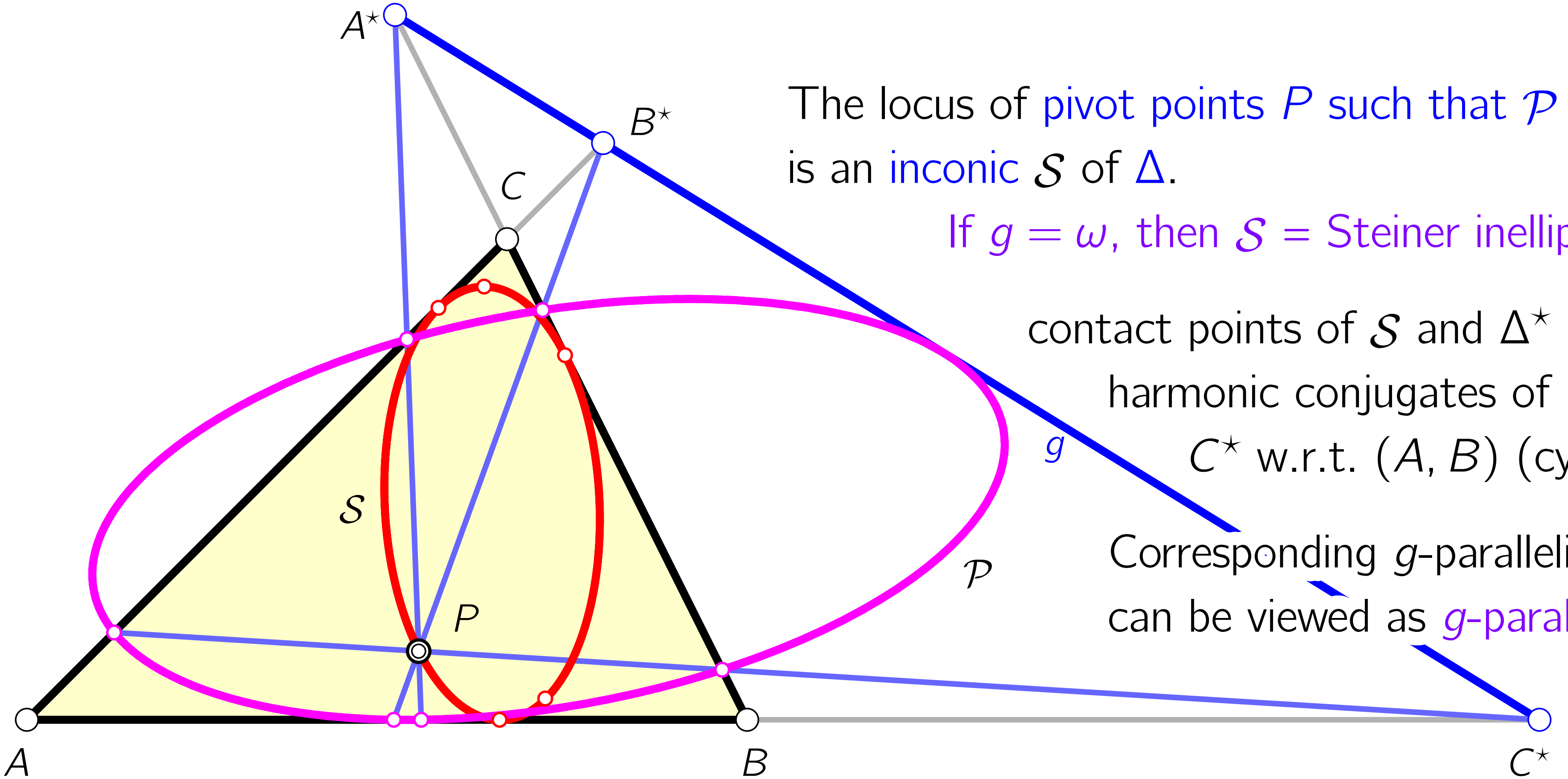
$[l_1 \cap l_2, l_4 \cap l_5] = [P, A^\star]$, ...

$\Longrightarrow$ P is the Pascal point



236145 hexagon interscribed to $\mathcal{P}$ and $\mathcal{I}$ $\Longrightarrow$ $\exists$ a hexagon porism

g-Steiner inconic



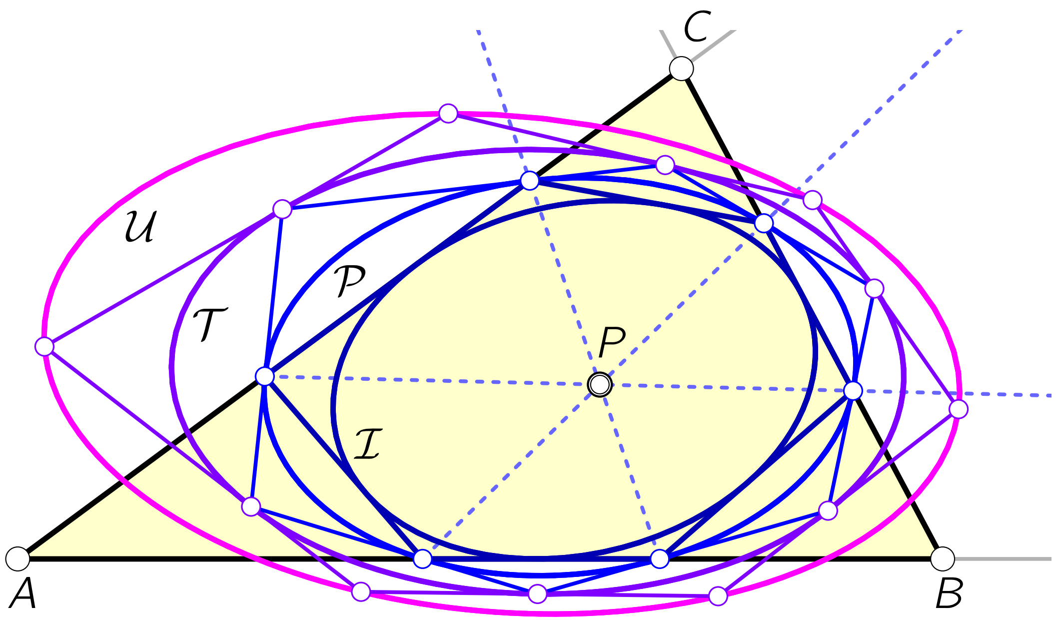
The locus of pivot points P such that $\mathcal{P}$ touches g is an inconic $\mathcal{S}$ of Δ.

If $g = \omega$, then $\mathcal{S}$ = Steiner inellipse,

contact points of $\mathcal{S}$ and $\Delta^\star$ are the harmonic conjugates of $C^\star$ w.r.t. (A, B) (cyclic).

Corresponding g-parallelian conics $\mathcal{P}$ can be viewed as g-parabolas.

parallelian conics: in-, circum-, tangent, and ...



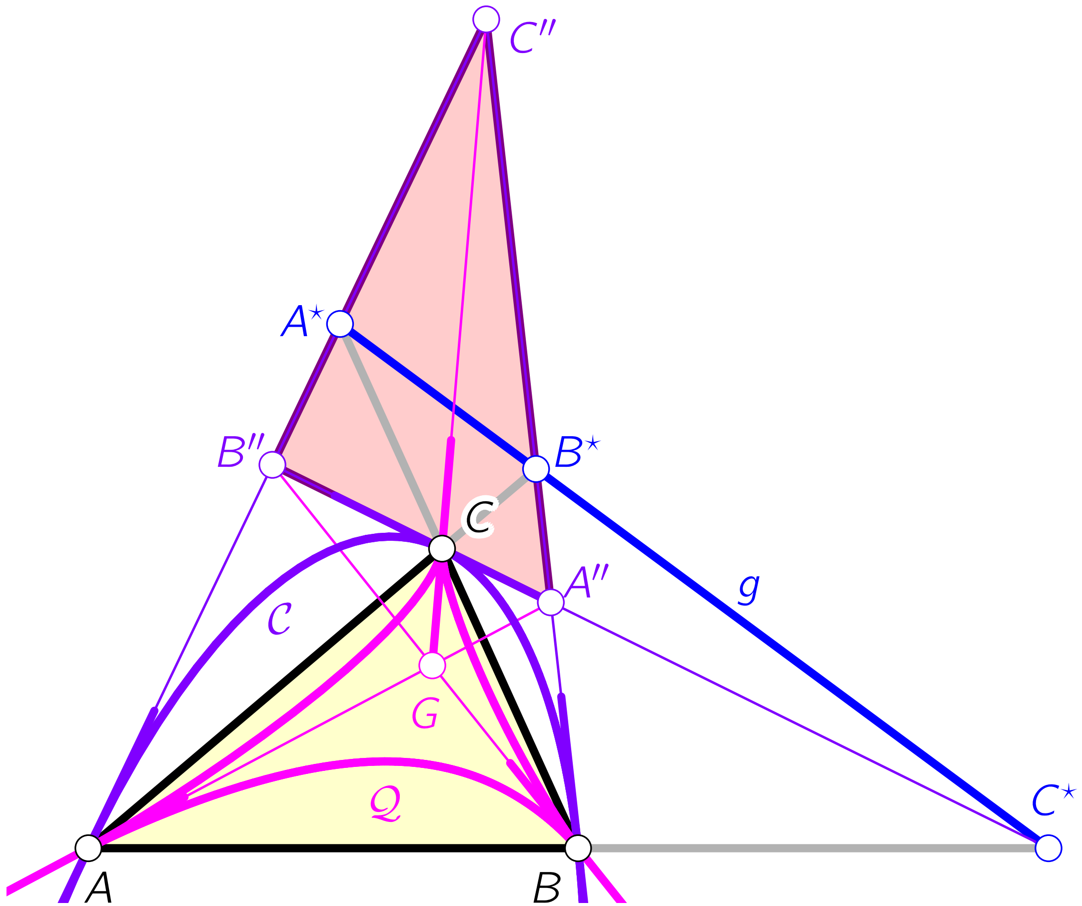
The points $T_{ij} := t_i \cap t_j$ with $(ij) \in \{(2,3),(3,6),(6,1),(1,4),(4,5),(5,2)\}$ lie on a single conic $\mathcal{T}$, the g-parallelian tangent conic.

$\mathcal{T}$ is the polar image of $\mathcal{I}$ w.r.t. $\mathcal{P}$.

$\mathcal{U}$ is the polar image of $\mathcal{P}$ w.r.t. $\mathcal{T}$.

The existence of a single interscribed hexagon between pairs of conics $(\mathcal{I}, \mathcal{P})$, $(\mathcal{P}, \mathcal{T})$, $(\mathcal{T}, \mathcal{U})$, ... guarantees the existence of poristic families of hexagons between any pair of subsequent conics.

regularity of $\mathcal{P}$, g-Steiner circumconic and g-anticomplentary triangle

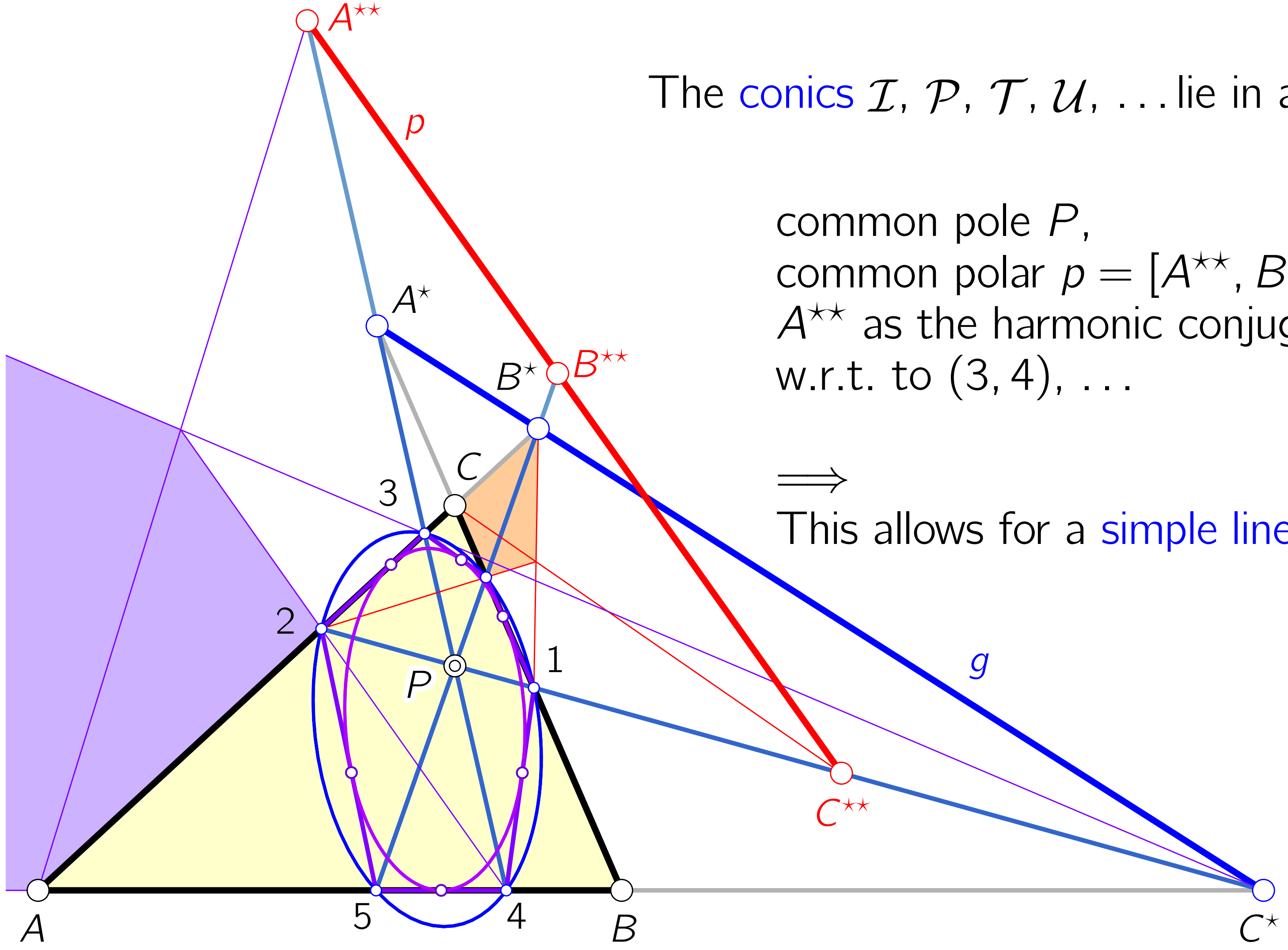


The g-parallelian conic $\mathcal{P}$ is singular $\Longleftrightarrow$ pivot P on g-Steiner circumconic $\mathcal{C}$.

The "vertices" of singular g-parallelian conics $\mathcal{P}$ lie on the three-cusp-quartic $\mathcal{Q}$.
$\mathcal{Q}$ = polar image of $\mathcal{C}^\star$ w.r.t. Δ
$\Longrightarrow$ $\mathcal{Q}$ = g-Steiner deltoid
cusp tangents concur in g's Δ-pole G

The pivots on the sides of the g-anticomplentary triangle $\Delta_a = A''B''C''$ produce singular g-parallelian tangent conics $\mathcal{T}$.

a pencil of the thrid kind



The conics $\mathcal{I}$, $\mathcal{P}$, $\mathcal{T}$, $\mathcal{U}$, ... lie in a pencil of the 3[rd] kind.

common pole P,
common polar $p = [A^{\star\star}, B^{\star\star}, C^{\star\star}]$ with
$A^{\star\star}$ as the harmonic conjugate of P
w.r.t. to $(3, 4)$, ...

$\Longrightarrow$
This allows for a simple linear construction of p.

making $\mathcal{T}$ a g-parabola

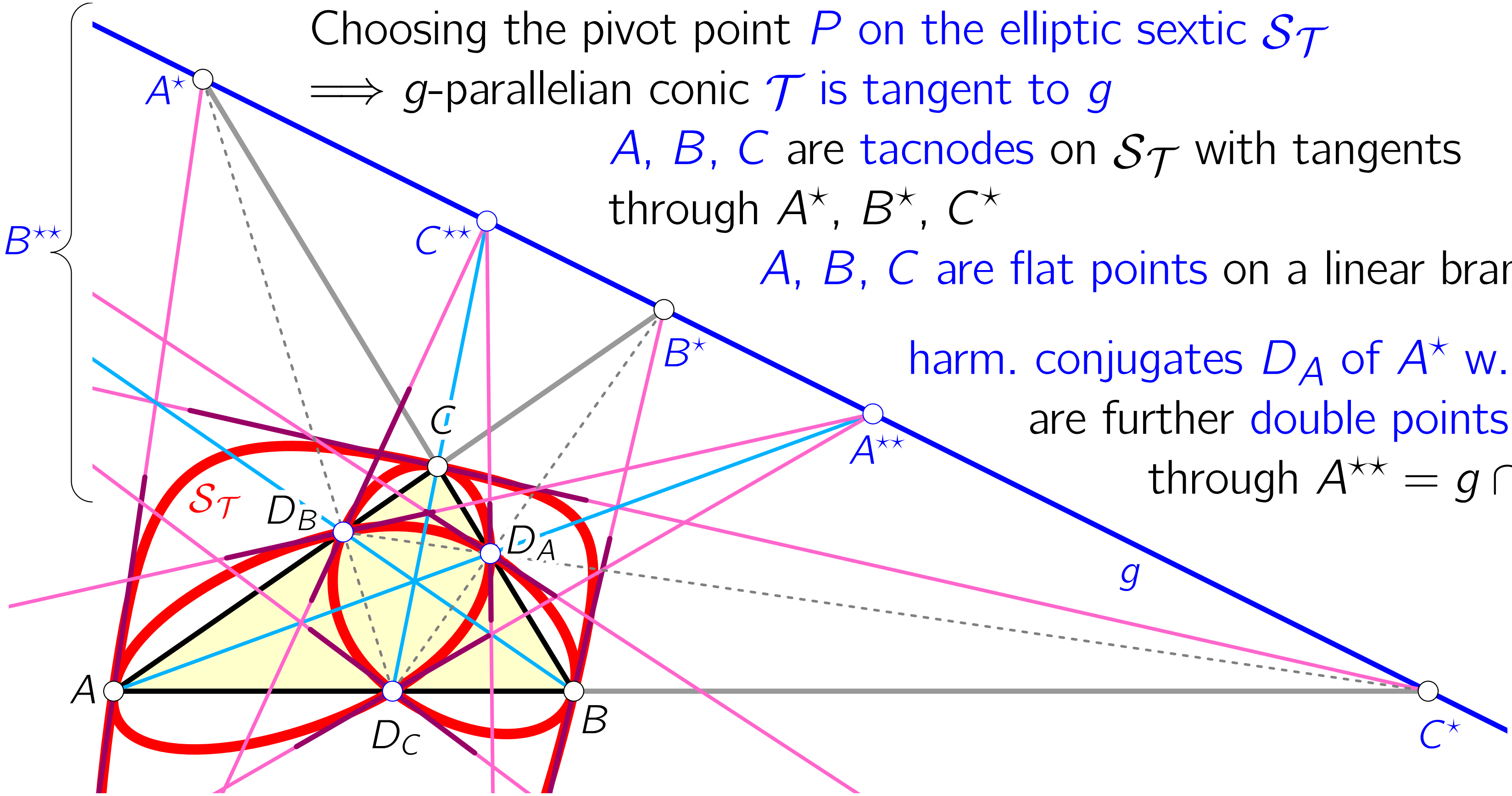
Choosing the pivot point P on the elliptic sextic $\mathcal{S}_\mathcal{T}$

$\Longrightarrow$ g-parallelian conic $\mathcal{T}$ is tangent to g

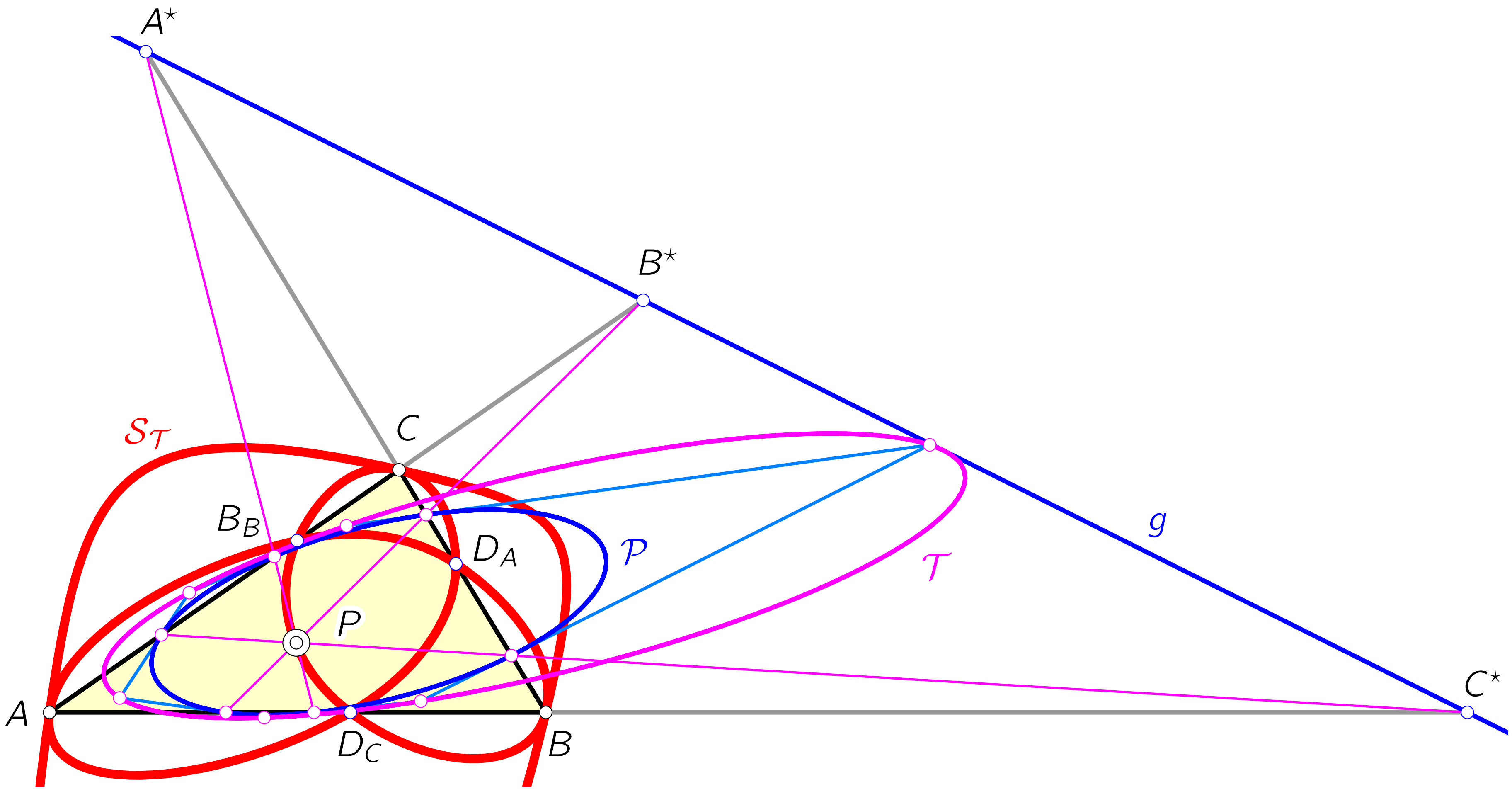
A, B, C are tacnodes on $\mathcal{S}_\mathcal{T}$ with tangents through $A^\star$, $B^\star$, $C^\star$

A, B, C are flat points on a linear branch of $\mathcal{S}_\mathcal{T}$

harm. conjugates D_A of $A^\star$ w.r.t. (B, C) (cyclic) are further double points with tangents through $A^{\star\star} = g \cap [A, D_A]$ (cyclic)



$\mathcal{T}$ tangent to g



tangent triangles – triangle and hexagon porisms

tangents of $\mathcal{P}$ at 1, 3, 5 $\longrightarrow$ triangle $\Delta_U = U_1U_2U_3$

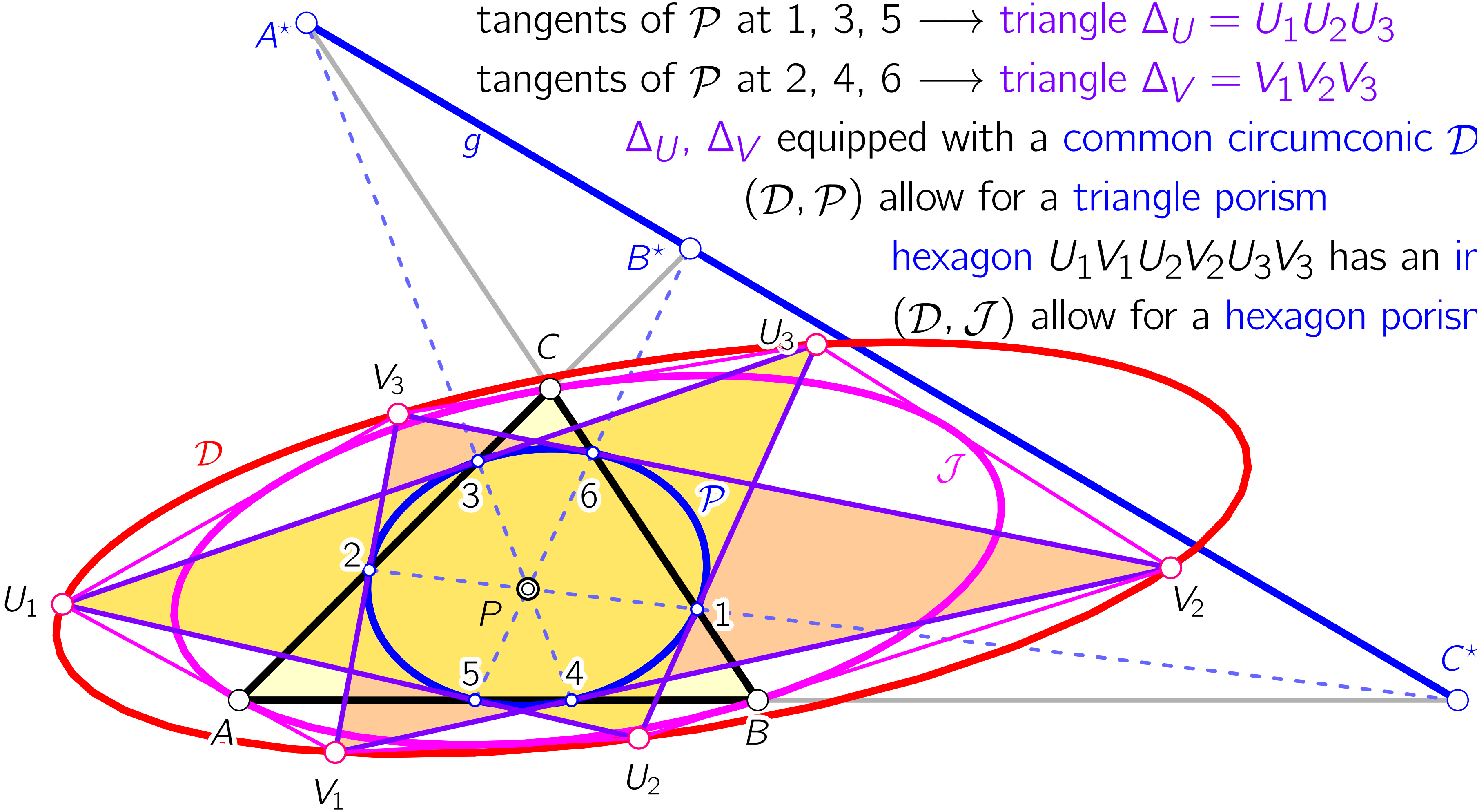
tangents of $\mathcal{P}$ at 2, 4, 6 $\longrightarrow$ triangle $\Delta_V = V_1V_2V_3$

Δ_U, Δ_V equipped with a common circumconic $\mathcal{D}$

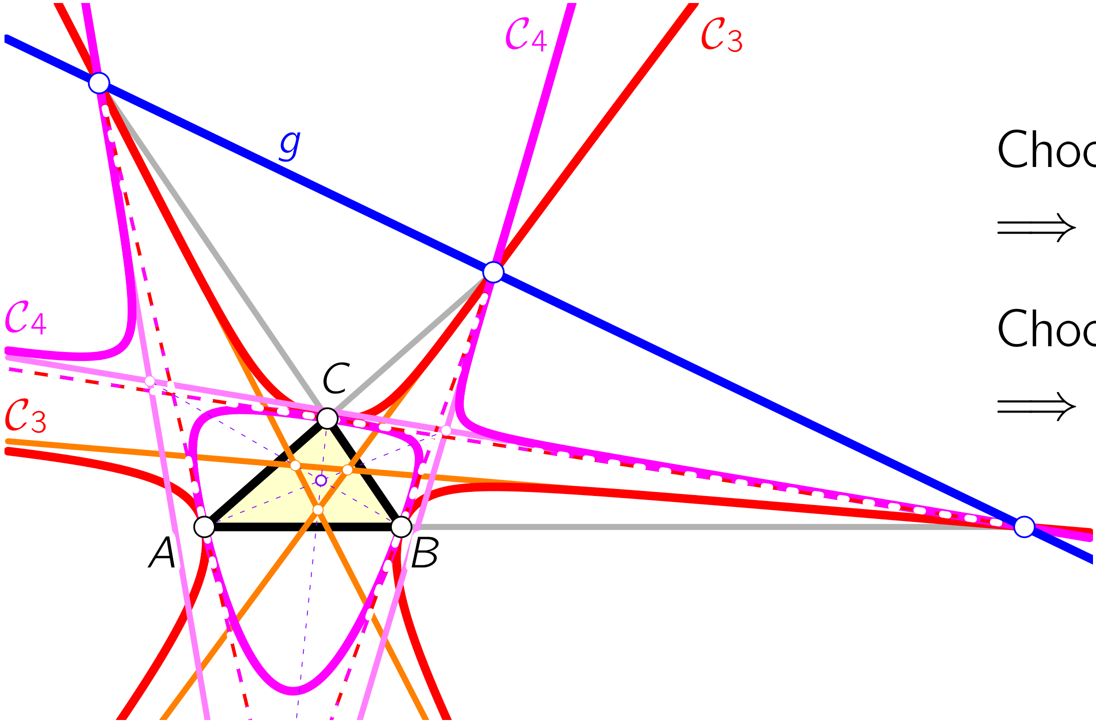
$(\mathcal{D}, \mathcal{P})$ allow for a triangle porism

hexagon $U_1V_1U_2V_2U_3V_3$ has an inconic $\mathcal{J}$

$(\mathcal{D}, \mathcal{J})$ allow for a hexagon porism



porisms of triangles, quadrilaterals, . . .



Choosing the pivot P on the elliptic cubic $\mathcal{C}_3$
$\Longrightarrow$ triangle porism between $\mathcal{J}$ and $\mathcal{P}$

Choosing the pivot P on the elliptic cubic $\mathcal{C}_4$
$\Longrightarrow$ quadrangle porism between $\mathcal{J}$ and $\mathcal{P}$

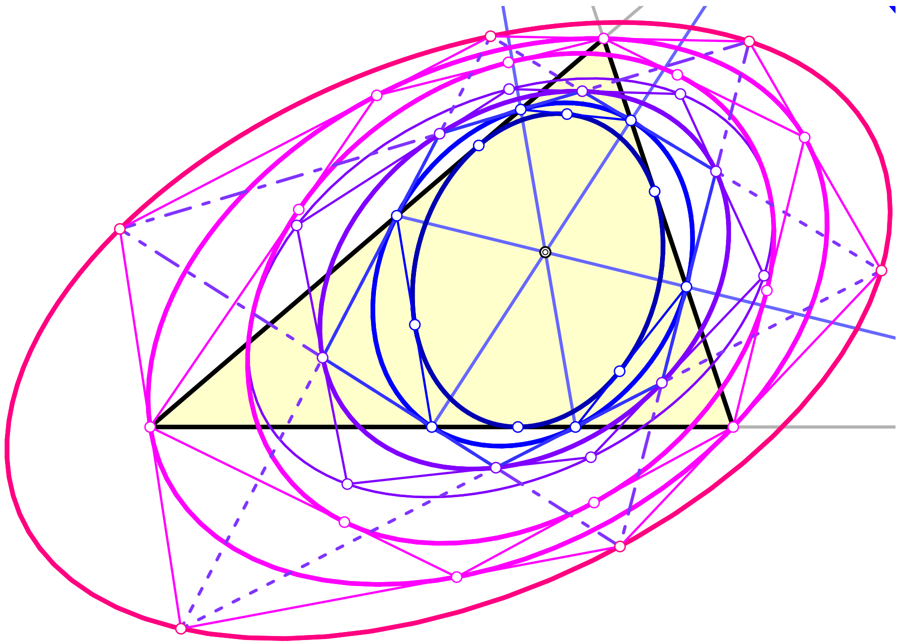
The triangles built by triangles at $A^\star$, $B^\star$, $C^\star$ are perspective to each other and Δ with common perspectrix g and its Δ-pole as common perspector.

repeated polarization . . .

. . . produces a chain of nested porisms.

1. chain generated by $(\mathcal{P}, \mathcal{T})$, family of hexagons
2. chain genearted by $(\mathcal{D}, \mathcal{P})$, family of triangles
3. chain genearted by $(\mathcal{D}, \mathcal{J})$, family of hexagons

$\Longrightarrow$ The g-parallelians give rise to three different and independent chains of porisms!



Are there more universal porisms?
except colllinear images of Yff pencil

Thank You for Your Attention!

some references

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