Iterated Triangle Center Construction

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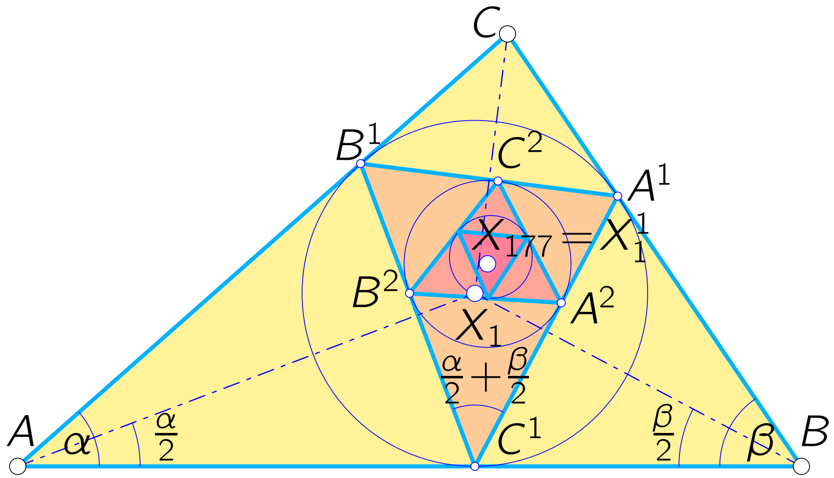
overview

triangle centers define central triangles	and further centers
constructive / analytic aspects	limits, convergence analysis
	hardly any known limits
	Some do not satisfy!
numerical approach	precision matters
aims at ideas	for algebraic proofs
	and recurring patterns
findings	different behaviour(s)

a limit that cannot escape: the incenter



interior angles of Δ_0: α, β, γ

interior angles of Δ_1: $\frac{\beta+\gamma}{2}$, $\frac{\gamma+\alpha}{2}$, $\frac{\gamma+\alpha}{2}$

$$\vdots$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \longrightarrow \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{=:\mathbf{T}} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

The incenter of the triangle $\Delta_0 = ABC$
always lies in the interior of Δ_0
and in the interior of the intouch triangle Δ_1,
which also lies completely in the interior of Δ_0.
This yields a sequence of triangle centers:

$$X_1 \to X_{177} \to ?$$

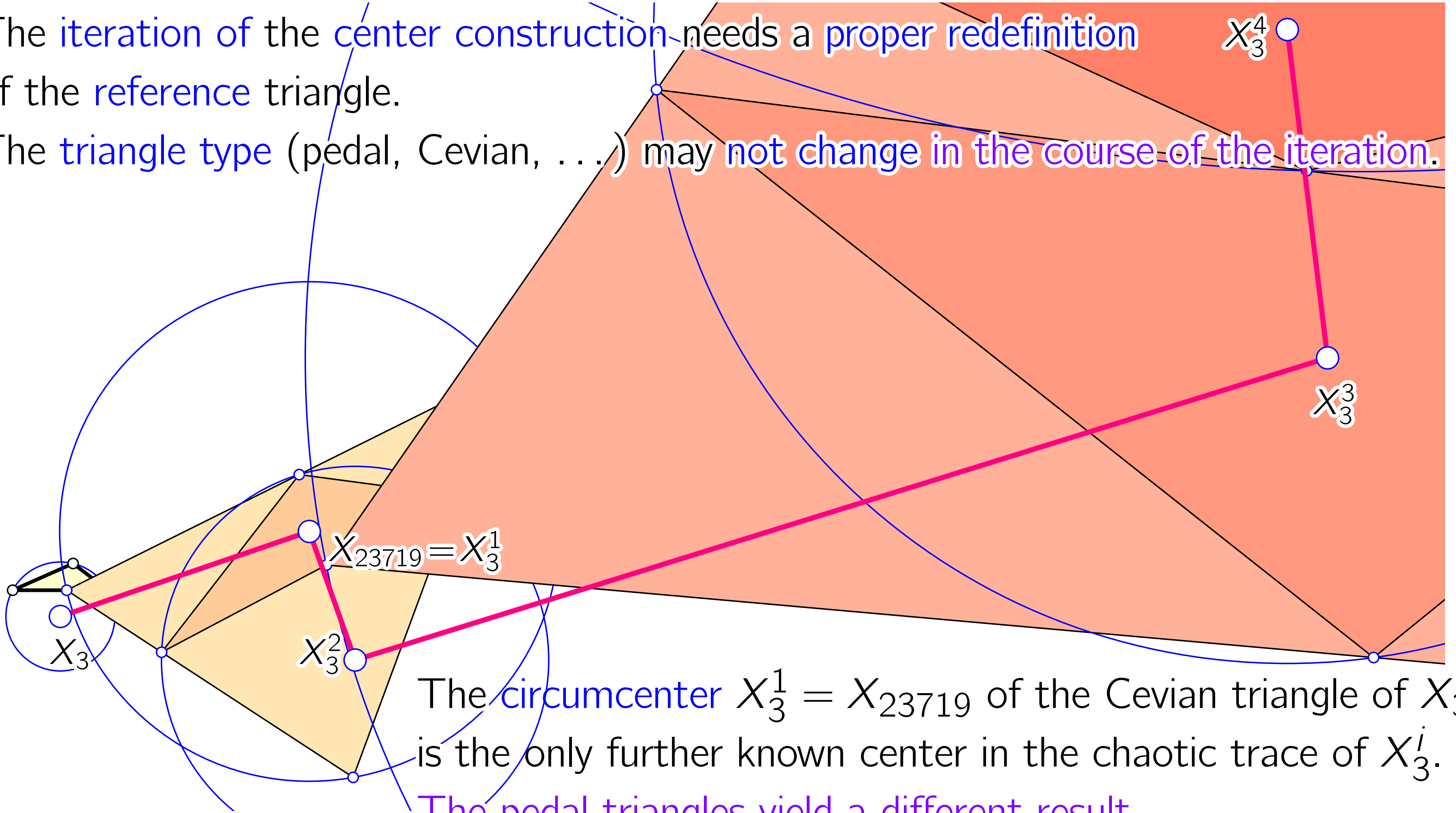
$\mathbf{T}$ has one stable eigenvector: $(1, 1, 1)$
corresponding to the eigenvalue $\frac{1}{3}$.
$\Longrightarrow$ All interior angles converge to $\frac{1}{3}(\alpha + \beta + \gamma)$.
The limit Δ_∞ of the intouch triangle is
equilateral (allthough ∞ small).
This gives no clue about X_1^∞
(except $X_1^\infty = \Delta_\infty$)!

The 1st Dilation Center X_{3513} as incenter limit does not satisfy.
It is the singular limit in the exponential pencil spanned by in- & circumcircle.

circumcenter X_3 and its Cevian triangle

The iteration of the center construction needs a proper redefinition of the reference triangle.

The triangle type (pedal, Cevian, ...) may not change in the course of the iteration.



The circumcenter $X_3^1 = X_{23719}$ of the Cevian triangle of X_3 is the only further known center in the chaotic trace of X_3^i.

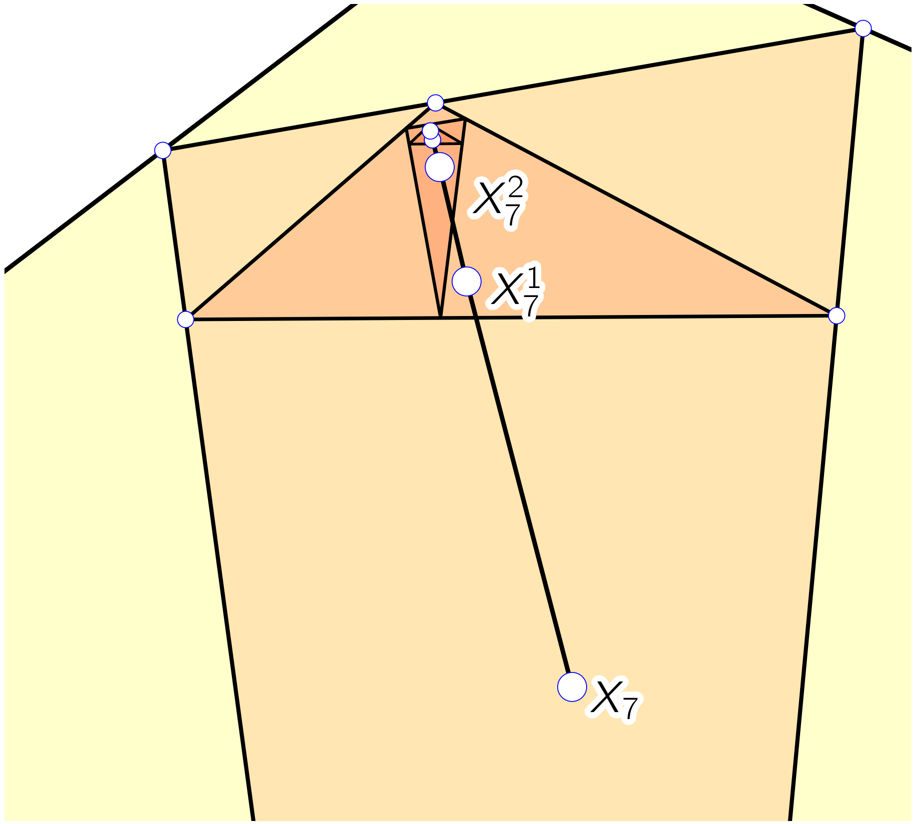
The pedal triangles yield a different result.

Gergonne point X_7 and its Cevian triangle

The Gergonne points seems to converge towards a point that is not contained in Kimberling's encyclopedia.

The (Cevian) reference triangle seems to undergo a regularization (at least numerically).

The intermediate centers are also unknown.

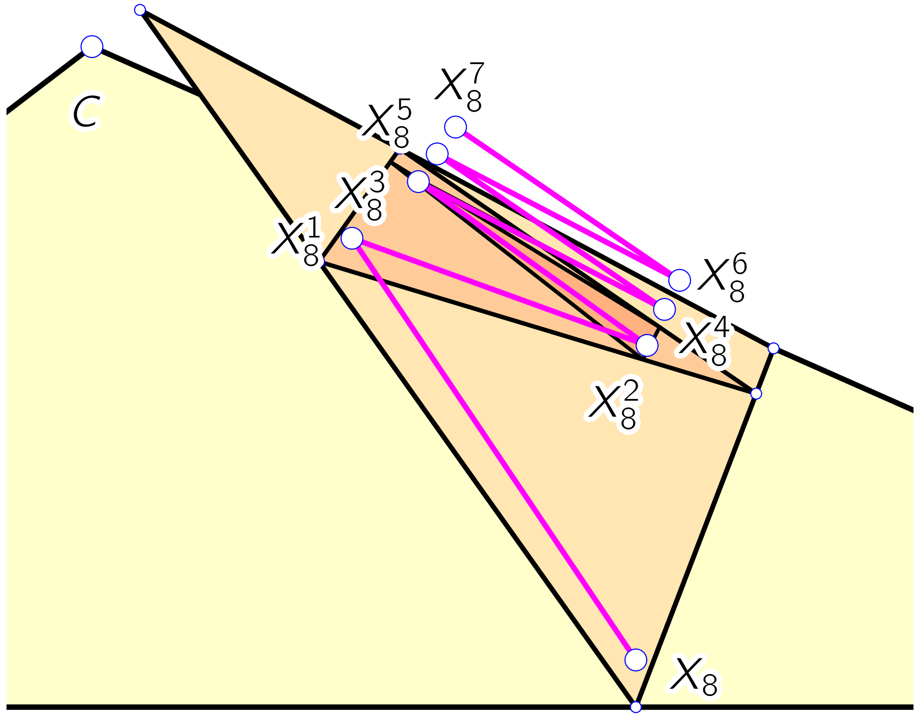


Nagel point X_8 and its pedal triangle

The Nagel point also seems to converge, but very slowly!

X_8 is running on a zig-zag path.
Even with highest numerical accuracy, the (pedal) reference triangles collaps after seven steps.

The reference triangles collapse even earlier in the case of Cevian triangles.



numerical results to be compared with the $6-9-13$ search table

	Cevian triangle	pedal triangle
X_1	1.95770029904487735665	0.80433539504925636000
X_2	2.62936879248871824114	1.76867523171775377550
X_3	-3891699654776.25808763607293483169	2.62936879248116398887
X_4	-1.02844023546083472296	-1.028440235460834722964
X_5	0.76566640603164320837	1.32186957169792197941
X_6	1.09217049764661208244	0.10296691685647652550
X_7	0.80433539504925636000	0.15823641292070149571
X_8	2.53702599581424750311	1.56932209282972051791
X_9	3.12652311458376376898	2.15889133044926341090
X_{11}	2.57245781384282079537	2.17029063766605551907

The Spieker center X_{10} needs a separate treatment.

	Intouch triangle of medial triangle
X_{10}	3.19337592231807969123

$$\Delta_P(X_4) = \Delta_C(X_4) \Longrightarrow \text{same (unknown) limit.}$$

X_3^∞, X_6^∞ using pedal triangles show up in Kimberling's search table as X_2 and X_{1285}, respectively.

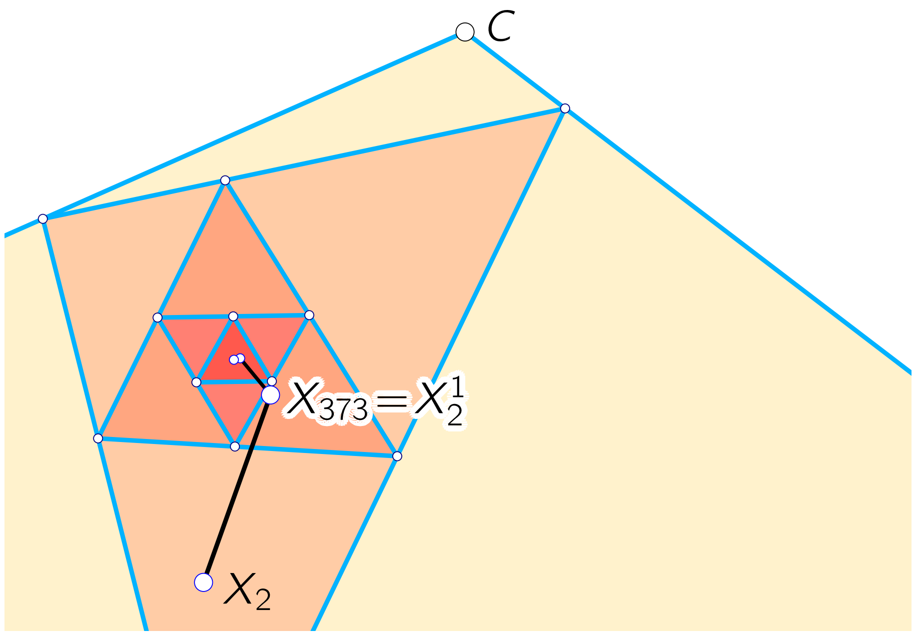
How to achieve sufficient numerical precision?

Arbitrary Precision Floating Point Numbers are obtained, *e.g.*, with *BigDecimal*.

80 digits are sufficient in most cases.

Critical operations are:
length calculation, normalization of vectors, halving angles, evaluating sine and cosine

sometimes ...



... convergence is observed, but a groth rule is not detected.

Analytical approaches fail due to complexity.

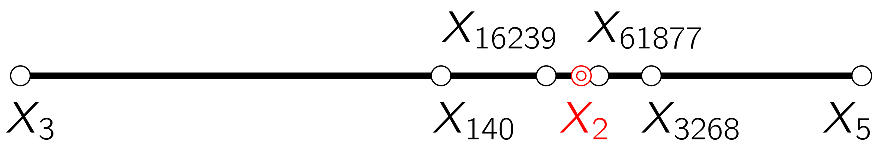
The limit of $X_2^{i+1}(\Delta_p(X_2^i))$ is not yet known and the growth rules of the polygon $X_2X_2^1X_2^2\ldots$ cannot easily be read off from the figure.

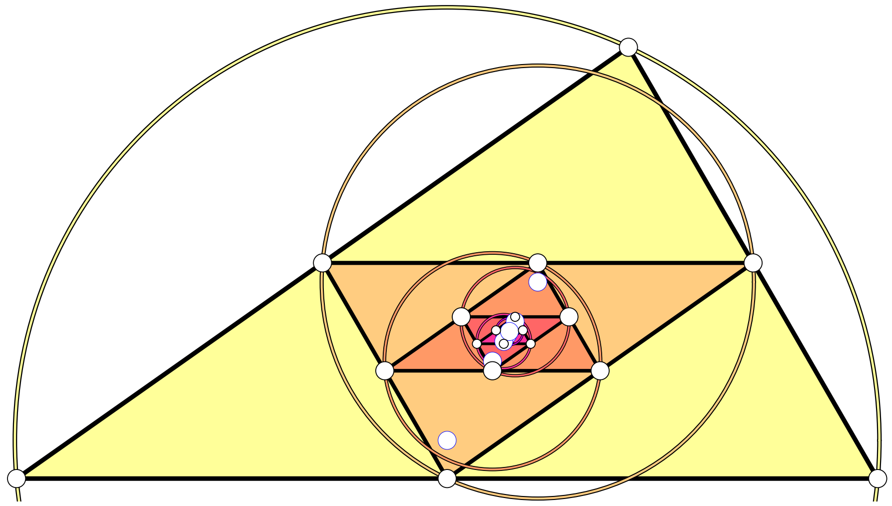
circumcenter limit with pedal triangles

The $\Delta_P(X_3)$ equals the medial triangle Δ_m, *i.e.*, the $\Delta_C(X_2)$.

The centroid is the circumcenter limit.

All intermediate centers are midpoints of some centers on the Euler line.





circumcenter limit with pedal triangles = centroid

$$\begin{array}{ll}
X_3^0 = X_3, & \text{circumcenter} \\
X_3^1 = X_5, & \text{nine-point center} \\
X_3^2 = \frac{1}{2}(X_3^1 + X_3^0) = X_{140}, & \text{Midpoint of } X_3 \text{ and } X_5 \\
X_3^3 = \frac{1}{4}(X_3^2 + X_3^1) = X_{3268}, & \text{isotomic conjugate of } X_{476} \\
X_3^4 = \frac{1}{8}(X_3^3 + X_3^2) = X_{16239}, & \text{complement of } X_{3628} \\
X_3^5 = \frac{1}{16}(X_3^4 + X_3^3) = X_{61877}, & [X_2, X_5] \cap [X_{10}, X_{61280}] \\
X_3^6 = \ldots &
\end{array}$$

names and indices according to the Kimberling telephone directory

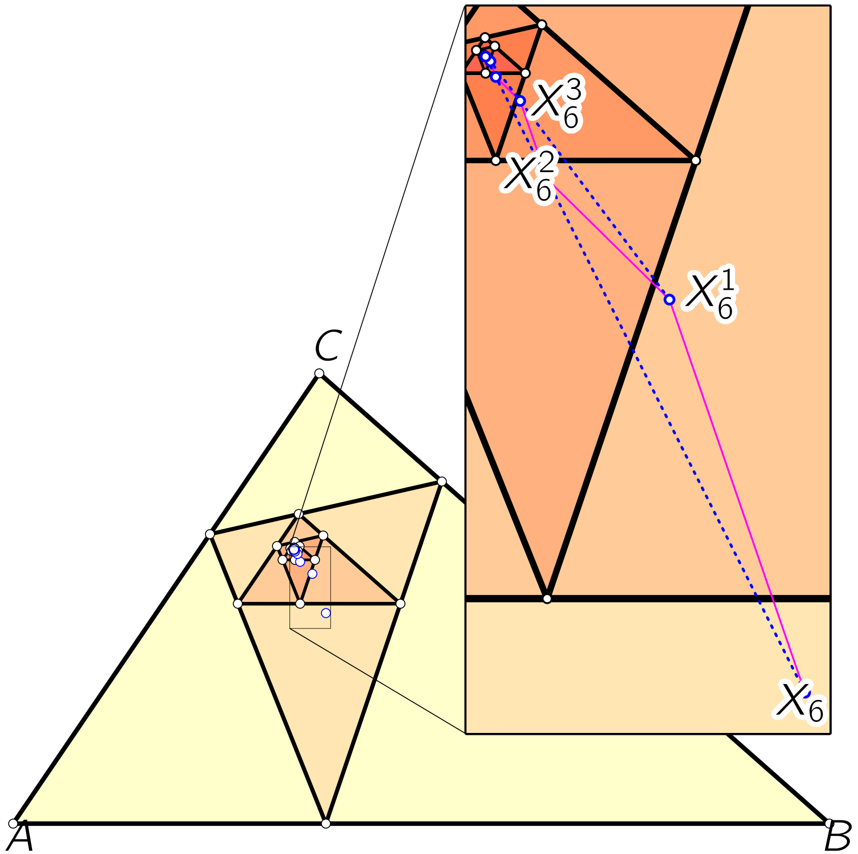
$$\Longrightarrow \quad X_3^k = \frac{1}{2^{k-2}}\left(\mathcal{J}(k-1)X_3^{k-1} + \mathcal{J}(k)X_3^{k-2}\right)$$

$\mathcal{J}(k) = \frac{1}{3}(2^k - (-1)^k)$ is the k-th Jacobsthal number, with $\mathcal{J}_k \xrightarrow{\infty} \frac{1}{3}2^k$

$$\lim_{k\to\infty} X_3^k = \frac{1}{3}\lim_{k\to\infty}\frac{1}{2^{k-1}}((2^{k-1}-(-1)^{k-1})X_3^1+(2^k-(-1)^k)X_3^0) = \frac{2}{3}X_5+\frac{1}{3}X_3 = X_2$$

symmedian (point) limit with pedal triangles = X_{1285}

X_{1285} = Lemoine homothetic center



similar triangles: $X_6 X_6^1 X_6^2 \sim X_6^3 X_6^4 X_6^5, \ldots$
$\Longrightarrow$ The dotted lines joining the "even" and "odd" points intersect in the similarity center L. X_{1285}.

$$L = [X_6, X_6^2] \cap [X_6^1, X_6^3] \overset{\star}{=}$$

$$\frac{1}{2c}\left(\frac{-4F(a^2-b^2-3c^2)(a^2-b^2+3c^2)}{\sum_{\text{cyclic}} 7a^4+2b^2c^2}.\right)$$

The second coordinate of L (after one cyclic permuation $a \to b \to c \to a$ & cutting out all cyclic symmetric functions) yields the trilinear center function $bc(9a^4-(b^2-c^2)^2)$ of X_{1285}.

What is left?

Will increasing numerical accuracy lead to results in the case of X_8?

Where does X_1 really move to?

What about infinite chains of quadratic Cremona transformations?

It is left as homework to determine limits of further centers.

Does the isogonal/isotomic conjugate of a center X_i converge if X_i does?

Thank You for Your Attention!

some references

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