a rarity in geometry: a septic curve

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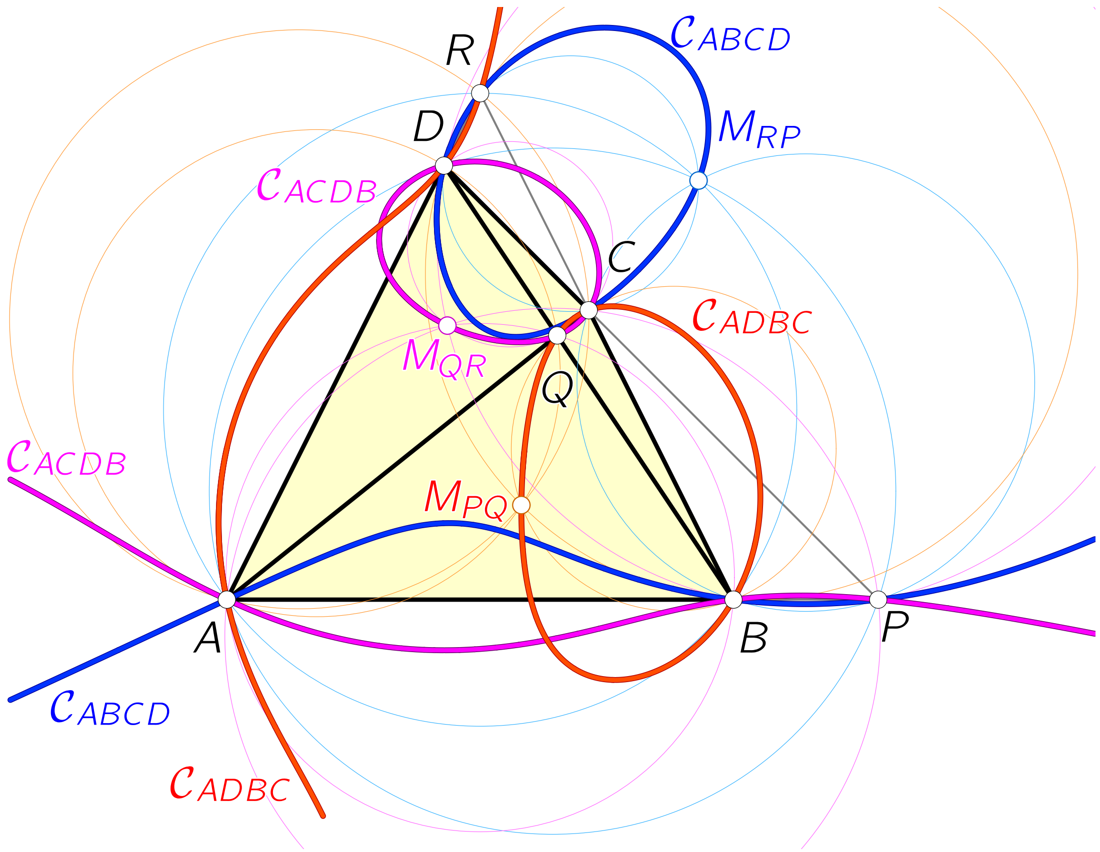
hopefully within 15 minutes

six conconic points	raising proper questions
some septic curves	examples, formulation of the problem
basic results	double points, Miquel points
equation	some algebraic properties
singular pedal conics	20 circles $\cap$ 1 septic
degeneracy, part 1	the non-quadrangle
a more general point of view	reflections in the sides, ...
degeneracy, part 2	Euclidean special cases

why? raising proper questions



⟵ The three loci $\mathcal{C}_{ABCD}$, $\mathcal{C}_{ACDB}$, $\mathcal{C}_{ADBC}$ of points with four concyclic pedal points on the sides of the three quadrilaterals on four points A, B, C, D.

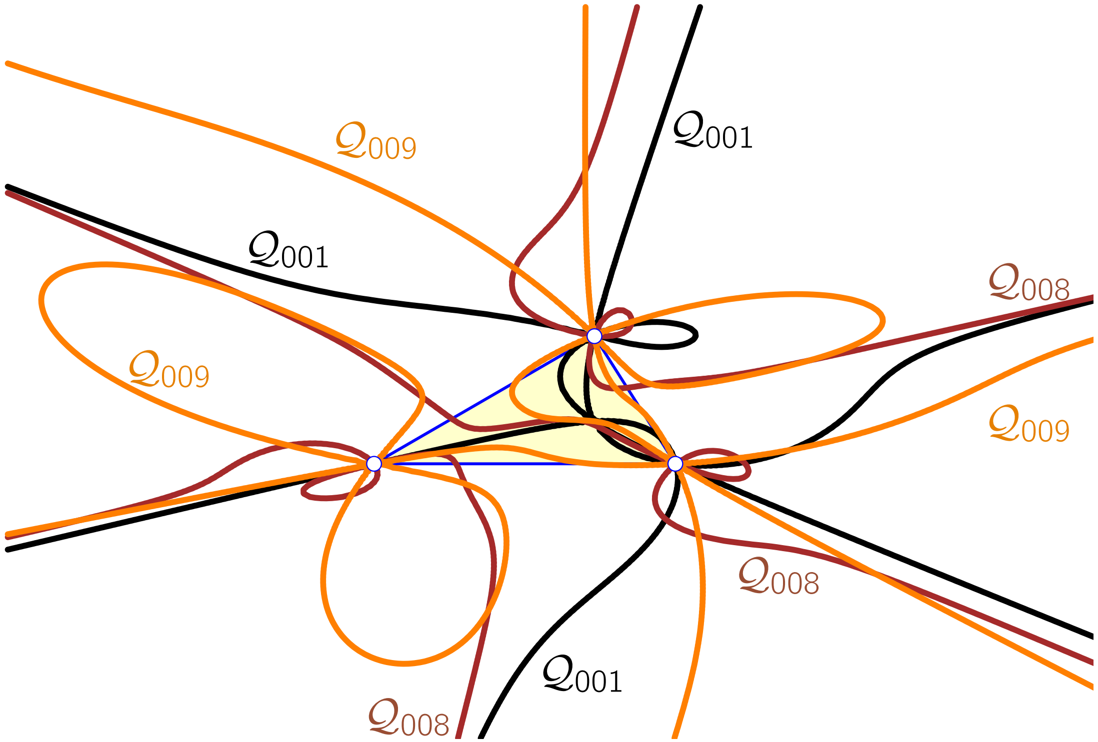
a single ordering of points [BLA] ⟶ incomplete picture

four points ⟶ six lines ⟶ six pedal points . . .

. . . six conconic points?

seems to be a proper question

some septic curves



Triangle related septics:

$\mathcal{Q}_{001}$ ... Darboux septic

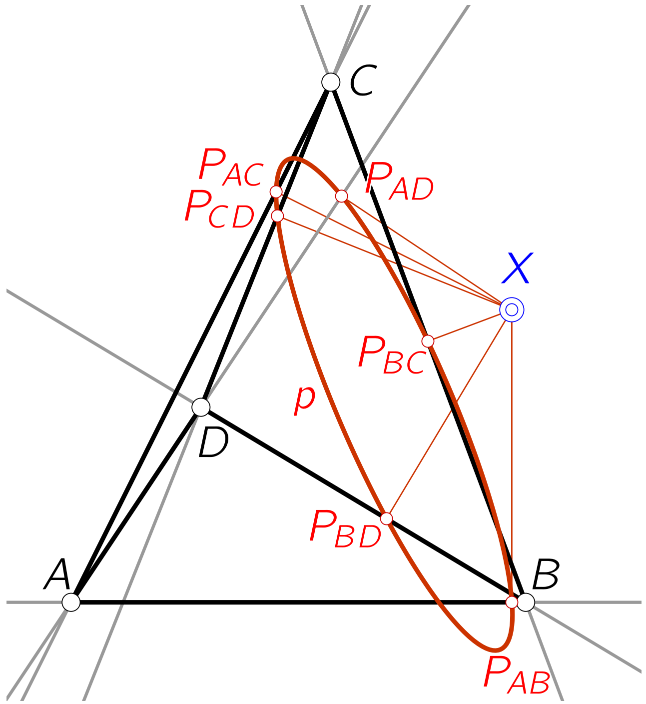
$\mathcal{Q}_{008}$ ... isogonal transform of a bicircular octic

$\mathcal{Q}_{009}$ ... bicircular septic related to orthologic triangles through X_1, X_4, X_{84} (labelling according to Gibert's list [GIB])

There are only 10 more remarkable septics (9 in [GIB] plus 1 in [LEM]).

K. Fladt claims that "there could hardly be curves of degree 7 that could be of geometric relevance", see [FLA].

definition



playground:

Euclidean plane (complex + proj. extension, if necessary)

given:

the vertices A, B, C, D of a quadrilateral $\mathcal{Q}$

arbitrary point X:

6 pedal points on the 6 lines of the complete quadrangle

question:

What is the set $\mathcal{C}$ of all X with conconic pedal points?

properly raised question / natural symmetry

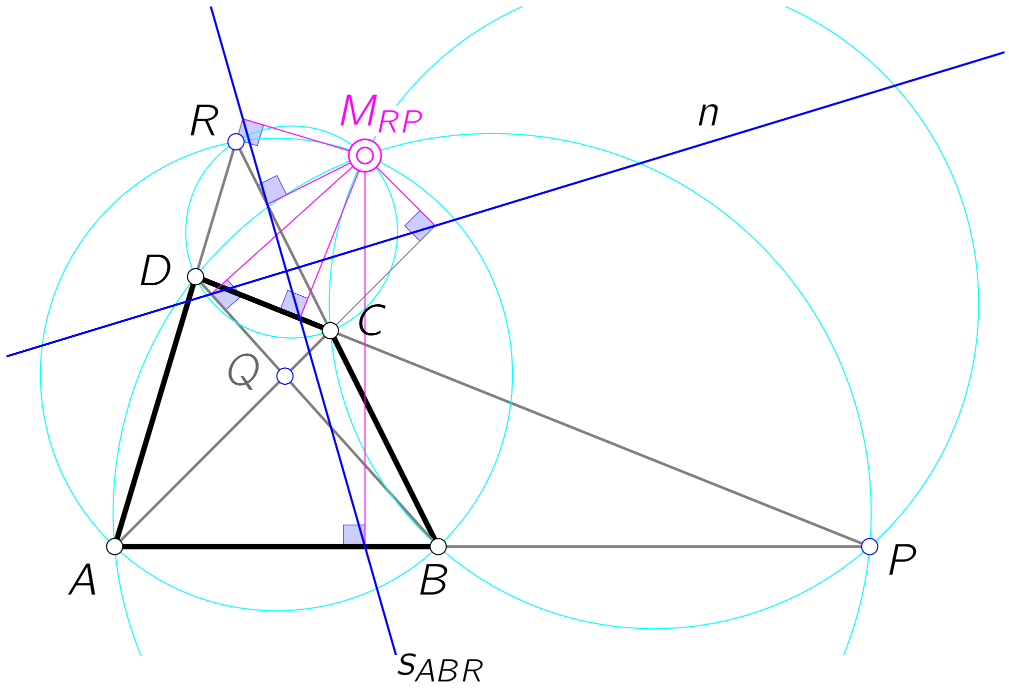
initial guess:

construction of pedal points $\longrightarrow$ linear in X's coordinates

conconicity test $\longrightarrow$ Pappos-Pascal (synthetic) or determinant of 6×6 matrix of quadratic Veronese images (analytic) [both equivalent]

$\Longrightarrow$ indicate a degree 12 locus $\mathcal{C}$

singular pedal conics



whatever $\mathcal{C}$ may look like:

Theorem: $\mathcal{C}$ passes through the vertices and the diagonal points of $\mathcal{Q}$.

vertices are singular (on $\mathcal{C}$),

3 pedal points coincide

diagonal points are regular (on $\mathcal{C}$),

2 pedal points coincide

Theorem: The three Miquel points lie on $\mathcal{C}$ and their pedal conics are degenerate.

Miquel point on 4 circumcircles $\longrightarrow$

4 Simson lines coincide $\longrightarrow$

4 collinear pedal points $\longrightarrow$

degenerate pedal conic (3 times)

equation

Cartesian frame such that $A=(0,0)$, $B=(a,0)$, $C=(b,c)$, $D=(d,e)$

$(a,b,c,d,e \in \mathbb{R}$, and some side relations between them for a non-degenerate $\mathcal{Q})$

$X=(\xi,\eta)$ arbitrary point

compute pedal points $P_{AB}, \ldots$ of P to $[A,B], \ldots$

homogenize by $\xi \to x_1 x_0^{-1}$, $\eta \to x_2 x_0^{-1}$ / ideal line $\omega:\ x_0=0$

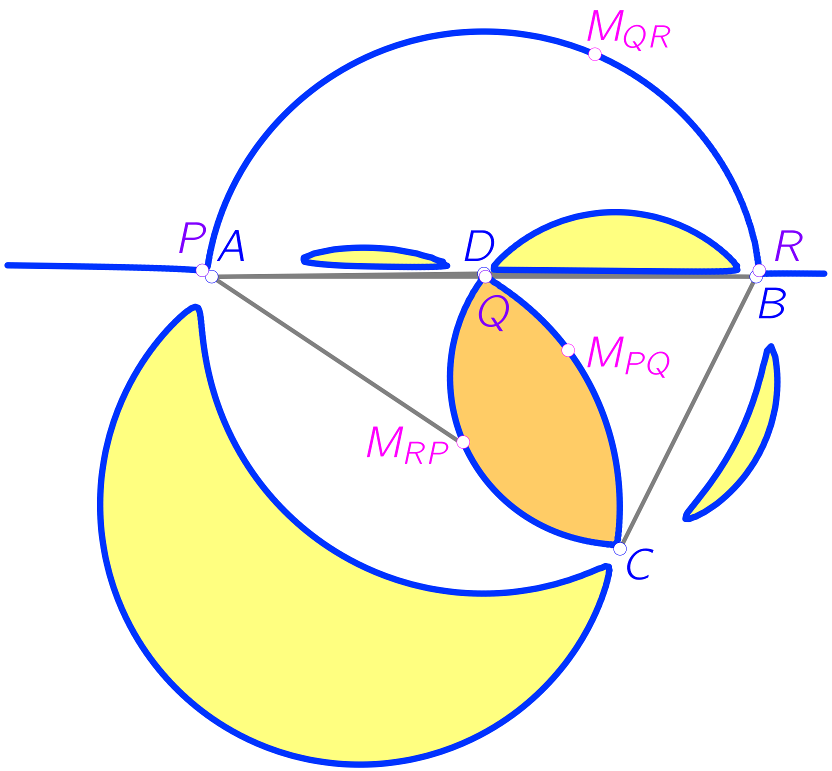
determine quadratic Veronese images v of all six $P_{..}$

linear dependency of $v(P_{AB}), \ldots, v(P_{CD}) \iff V=\det(v(P_{..}))=0$

$$V=\underbrace{-2^8 I_1^{-1} F_A^2 F_B^2 F_C^2 F_D^2}_{\text{constant}} \cdot \underbrace{x_0^5}_{\omega,\,\mu=5} \cdot \underbrace{\mathbf{P_7}}_{\deg=7} \Longrightarrow \mathcal{C}:\ \underbrace{\mathbf{P_7(x_0,x_1,x_2)=0}}_{\text{septic curve}}$$

$$\mathbf{P_7}=\sum_{i=0}^{7} q_i(x_1,x_2)x_0^i, \quad \text{with } q_7=q_6=0$$

algebraic properties of $\mathcal{C}$



A, B, C, D are acnodes and focal points

$q_7 = q_6 = 0$, $q_5 = 2^4 l_1 l_2 F_A F_B F_C F_D (x_1^2 + x_2^2)$

$\Longrightarrow$ A = acnode, isotropic tangents

$\Longrightarrow$ A = isolated focal point of/on $\mathcal{C}$

analogously for B, C, D (tanslate coord. frame)

$\mathcal{C}$ tricyclic

$q_0 = (x_1^2 + x_2^2)^3(\alpha_1 x_1 + \alpha_2 x_2)$, $(\alpha_1\alpha_2 \neq 0)$

$\Longrightarrow$ ordinary triple points at $I, J = 0 : 1 : \pm \mathrm{i}$

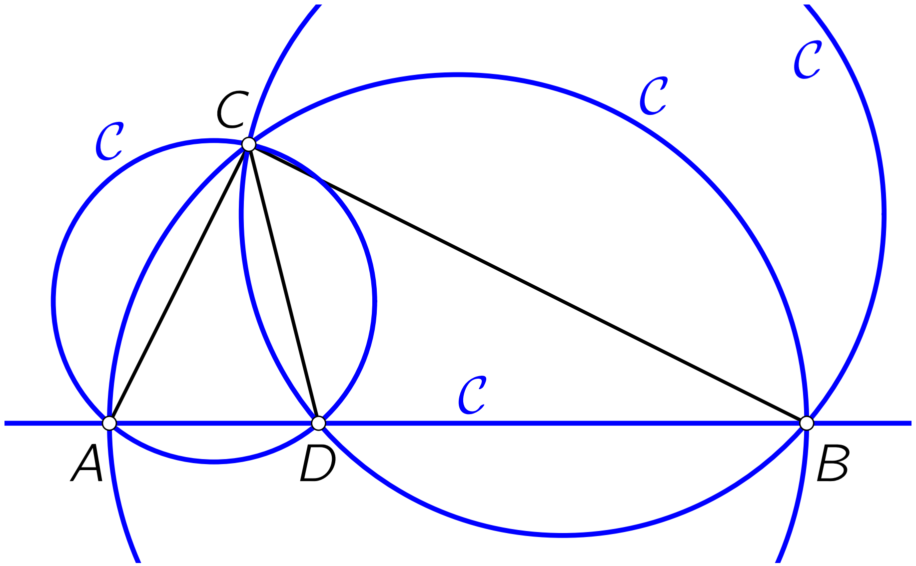
no further singularities $\Longrightarrow$

class($\mathcal{C}$) $= 7(7-1) - 2\cdot 4 - 6 \cdot 2 = 22$,

genus($\mathcal{C}$) $= \frac{1}{2}(7-1)(6-1) - 4 \cdot 1 - 2 \cdot 3 = 5$

genus = 5: $\mathcal{C}$ may have up to six separated components (over $\mathbb{R}$)

degeneracy, part 1



Theorem:

$\mathcal{C}$ splits into a line and three circumcircles if exactly 3 vertices of $\mathcal{Q}$ are collinear.

shown by means of computation

The degree of $\mathcal{C}$ does not drop.

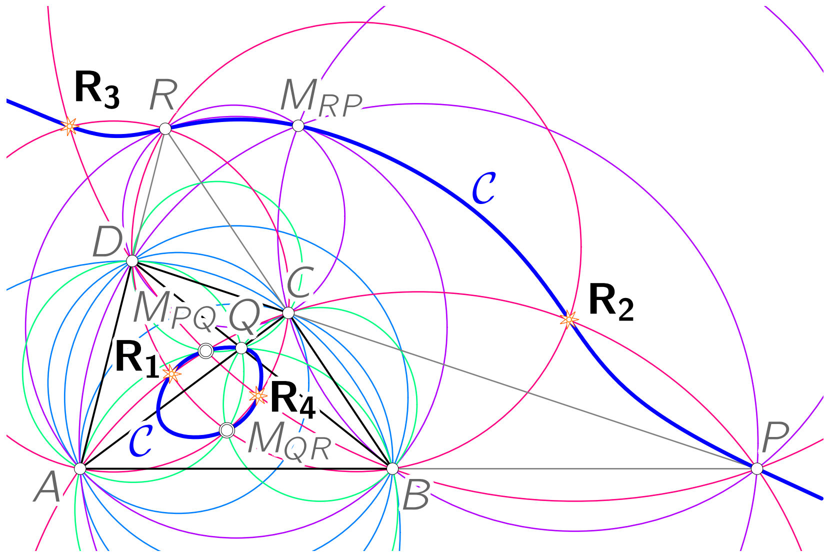
only further condition on $\mathcal{C}$ to degenerate:

$x_1^2 + x_2^2$ is a common divisor of q_3 and q_4.

$\Longrightarrow$ yields only degenerate quadrangles

$\Longrightarrow$ $6 \leq \deg \mathcal{C} \leq 7$ in any case

singular pedal conics



Theorem:

There exist 4 real points R_i ($\neq A, B, C, D,$ M_{PQ}, M_{QR}, M_{RP}) with **singular** pedal conics.

$\mathbf{x}^T \mathbf{C}_{AB}\mathbf{x} = 0$ ... conic on 5 pedals, except P_{AB}

$\det(\mathbf{C}_{AB})$ factors, do this for all $\binom{6}{5}$ pedals

L ... least comon multiple of all 6 $\det(\mathbf{C}..)$

$\mathcal{L}$: $L = 0$ ... cycle of degree 40 (20 "circles")

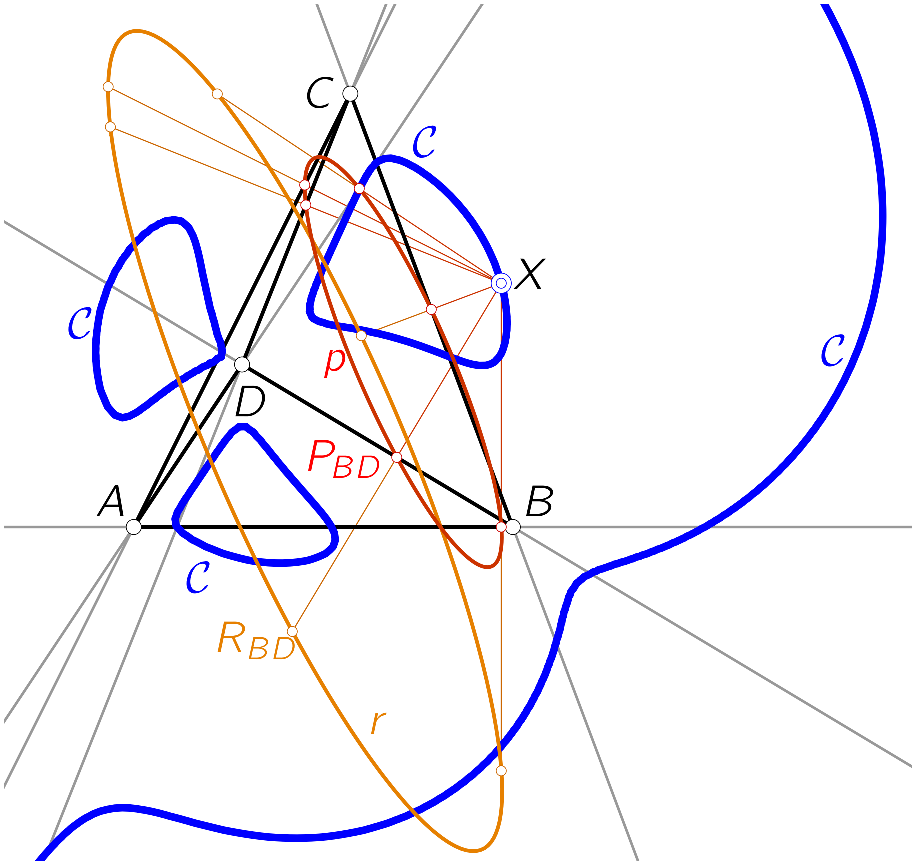
$\mathcal{C} \cap \mathcal{L}$ ... candidates of points with deg. pedal conics.

$$L = \underbrace{\iota_A \cdot \iota_B \cdot \iota_C \cdot \iota_D}_{\text{isotropic lines through vertices}} \cdot \underbrace{k_A \cdot k_B \cdot k_C \cdot k_D}_{\text{circumcircles of subtriangles}} \cdot \underbrace{k_{ABR} \cdot k_{CDR} \cdot k_{ADP} \cdot k_{BCP}}_{\text{circles through the Miquel point } M_{RP}} \cdot$$

$$\cdot \underbrace{k_{ACP} \cdot k_{BDP} \cdot k_{ABQ} \cdot k_{CDQ}}_{\text{circles through the Miquel point } M_{PQ}} \cdot \underbrace{k_{ADQ} \cdot k_{BCQ} \cdot k_{ACR} \cdot k_{BDR}}_{\text{circles through the Miquel point } M_{QR}}$$

intersection multiplicities					
I	J	A	B	C	D
60	60	24	24	24	24
P	Q	R	M_{PQ}	M_{QR}	M_{RP}
4	4	4	4	4	4
$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_4$	compl. pts.	
2	**2**	**2**	**2**	32	

a more general point of view



⟶ The conics p / r collect the pedal points / reflections of $X \in \mathcal{C}$.

Theorem:

$\mathcal{Q} = ABCD$ . . . quadrilateral in $\mathbb{R}^2$

κ_{kl}^{δ} . . . 6 persp. collineations, $(k, l) \in \{A, B, C, D\}$

axes = side line $[k, l]$ of $\mathcal{Q}$

center = $[k, l]^{\perp} \in \omega$

$\delta \in \mathbb{R} \setminus \{0\}$ = characteristic crossratio

The set $\mathcal{C}^{\delta}$ of all points X whose images P_{kl}^{δ} under the six perspective collineations κ_{kl}^{δ} lie on a single conic form the septic curve $\mathcal{C}$ (the one we already know) independent of $\delta \neq 0$.

degeneracy, part 2

Theorem:
$\mathcal{Q} = ABCD$ ... proper quadrilateral + D **= orthocenter of** ABC (wlog)
$\Longrightarrow \deg \mathcal{C} = 6$, $g(\mathcal{C}) = 1$, $\mathcal{C}$ has 9 (isolated) double points and no further singularities, class $m(\mathcal{C}) = 12$, $\mathcal{C}$ has no real branch.

Theorem:
$\mathcal{Q} = ABCD$ ... **parallelogram**
$\Longrightarrow \deg \mathcal{C} = 6$, $g(\mathcal{C}) = 3$, has 7 (isolated) double points and no further singularities, class $m(\mathcal{C}) = 16$, $\mathcal{C}$ has no real branch.

The latter theorem covers the case of rectangles, squares, and rhombuses.

$\mathcal{Q}$ = parallelogram $\Longrightarrow$ 2 diagonal points on $\omega \Longrightarrow x_0$ splits off from V once more
$\Longrightarrow \deg \mathcal{C} = 6$

related work

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Dover Publications, New York, 1959.

[FLA] K. Fladt: *Analytische Geometrie spezieller ebener Kurven.*
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Thank You For Your Attention!