Poncelet Porisms

Boris Odehnal

University of Applied Arts Vienna

overview and contents

What is a Porism?

What is a porism?

- something in between a theorem and a problem (Euclid)
- undetermined or unsolvable problem
- geometric locus
- theorems from Projective Geometry (Pappos's and Desargues's theorem)

The meaning of the word has changed (more than once)!

Nowadays: A porism is a closure theorem, or closure property,
or a geometric figure/construction that closes somehow.

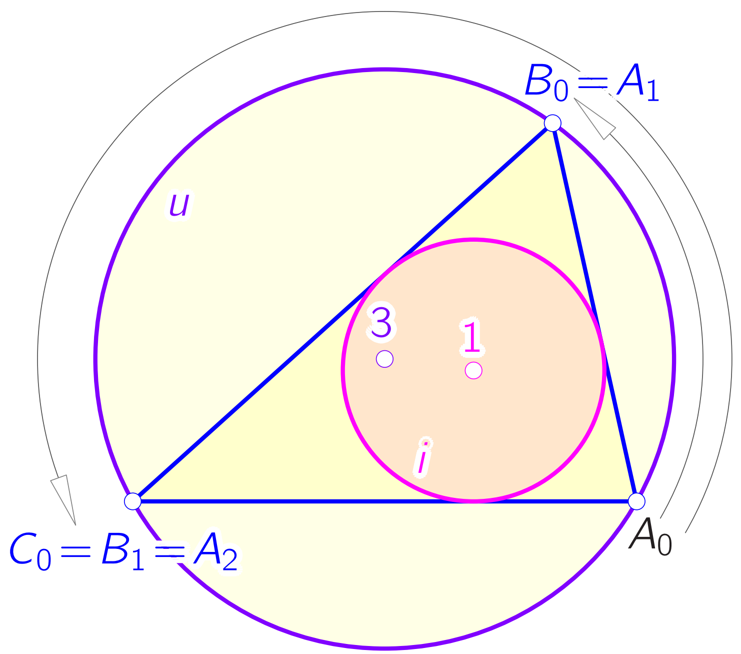
A theorem of the form:

If something closes for one particular choice, then it does for any admissible choice.

Especially Poncelet: closing polygons interscribed between two or more conics.

Chapple's porism

Chapple's porism = triangles with common circumcircle and incircle [2,3,8]



During a full turn, each point on the circumcircle plays the role of three vertices.

Orbits of points are three-fold / of multiplicity three.

True for all conic orbits and also for the limaçon.

Chapple's porism = very special Poncelet porism

The incircle i and circumcircle u span a hyperbolic pencil of circles = very special pencil of conics (pencil of the 1. kind).

where it began . . .

9.5 Poncelet porism. Suppose $\triangle ABC$ has circumcircle Γ and incircle Γ'. There are infinitely many triangles having this same circumcircle and incircle. They form a family of poristic triangles ([Gall, Chapter 3], [John, 91-95]) that "rotate" around Γ'; in Fig. 9.9, triangle $A_tB_tC_t$ is indexed by the angle t, through which the ray from center #1 (incenter) to the tangency point C'' sweeps. As you can see in the figure, center #2 (centroid) traverses a circle. Most central orbits (and noncentral orbits) remain to be explored.

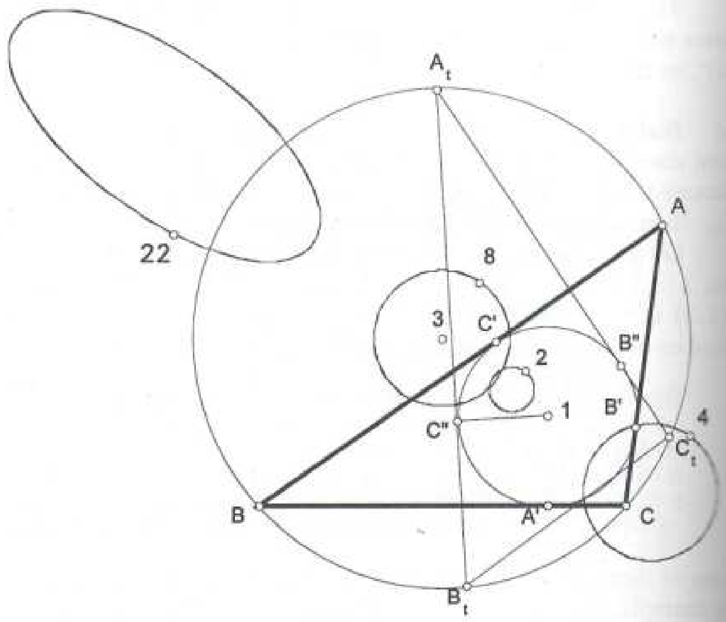


Fig. 9.9 Circular orbits of centers 2, 4 and 8; noncircular orbit of center 22

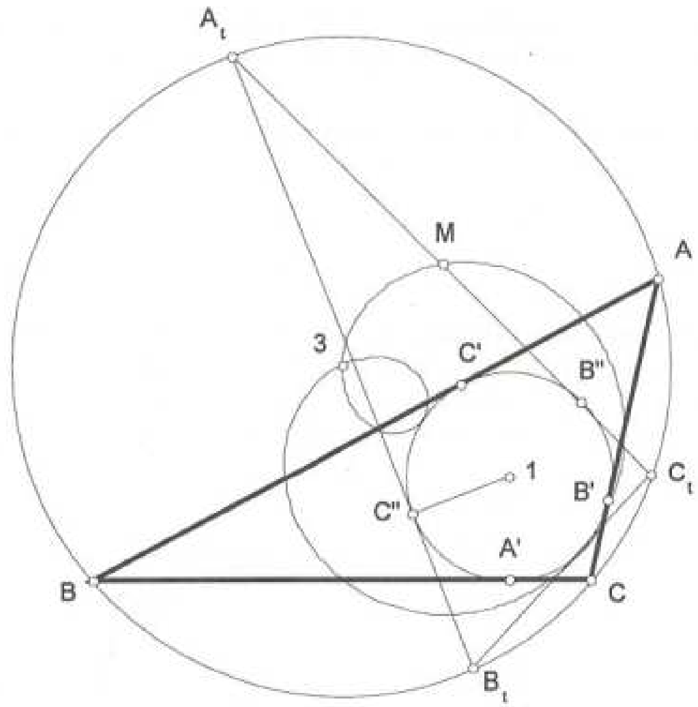


Fig. 9.10 Noncircular orbit of the midpoint of a side of the dynamic triangle $A_tB_tC_t$. Conjectures, anyone?

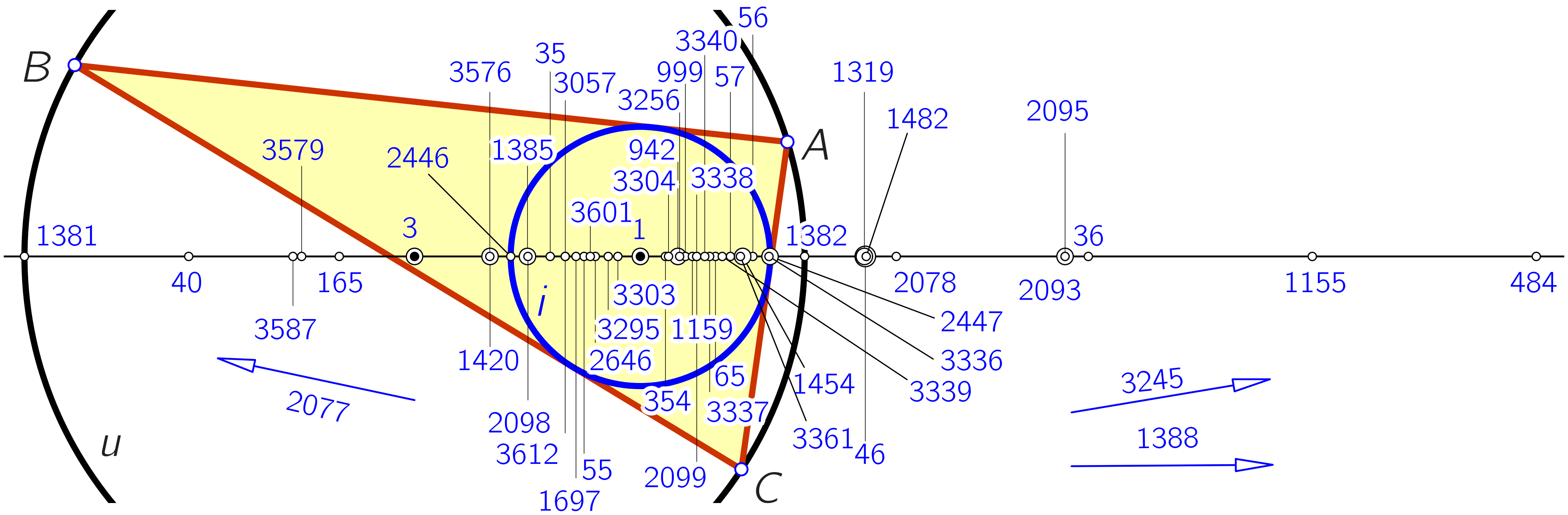
from C. Kimberling's book:

Triangle Centers and Central Triangles $\left[10,\ \text{p. } 1+2^{2^3}\right]$

circles, ellipses, and a limaçon

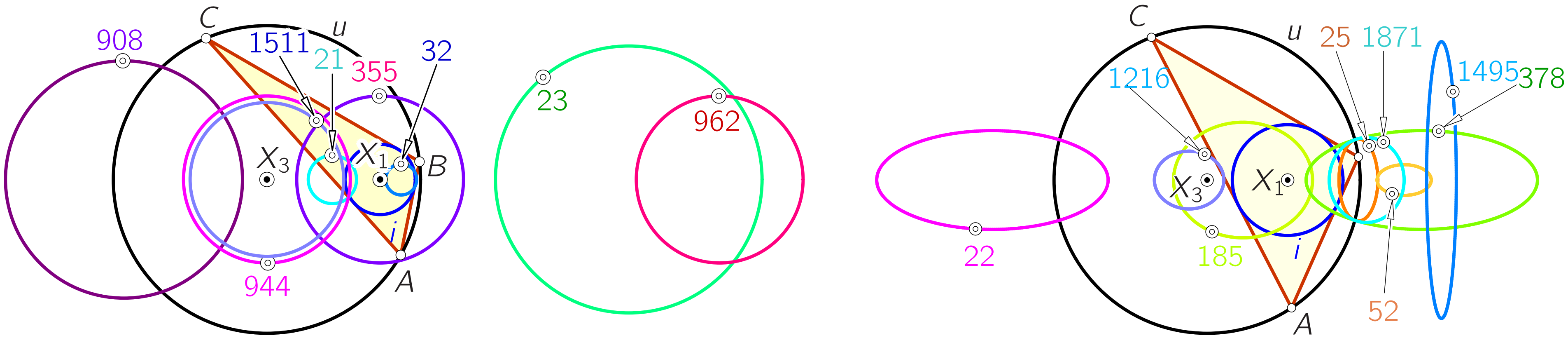
poristic traces of triangle centers I

Triangle centers on the line $\mathcal{L}_{1,3} = [X_1, X_3]$ are fixed. [12]

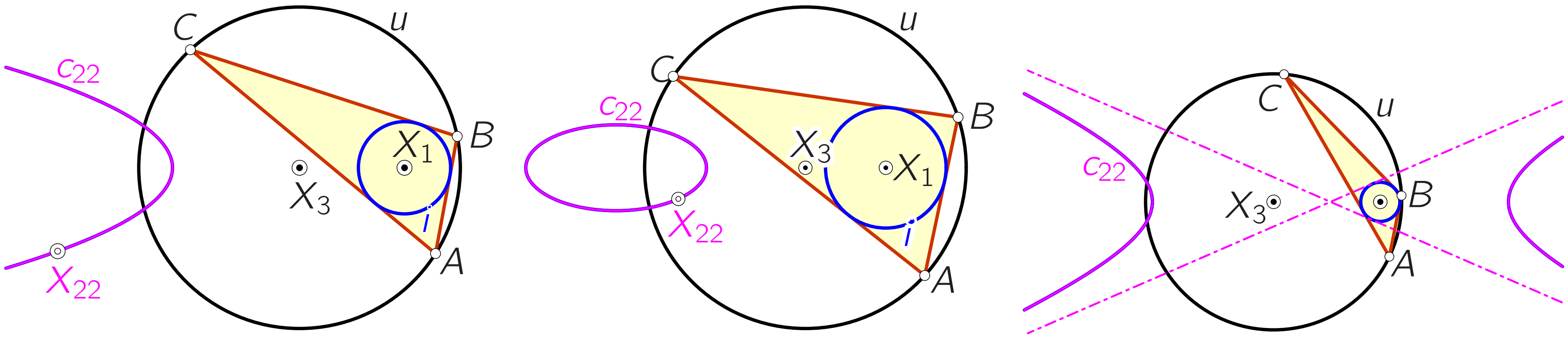


poristic traces of triangle centers II

Many centers move on circles, some on ellipses, parabolae, or hyperbolae. [12]



The shape of a poristic locus of a particular center can be of any affine type.

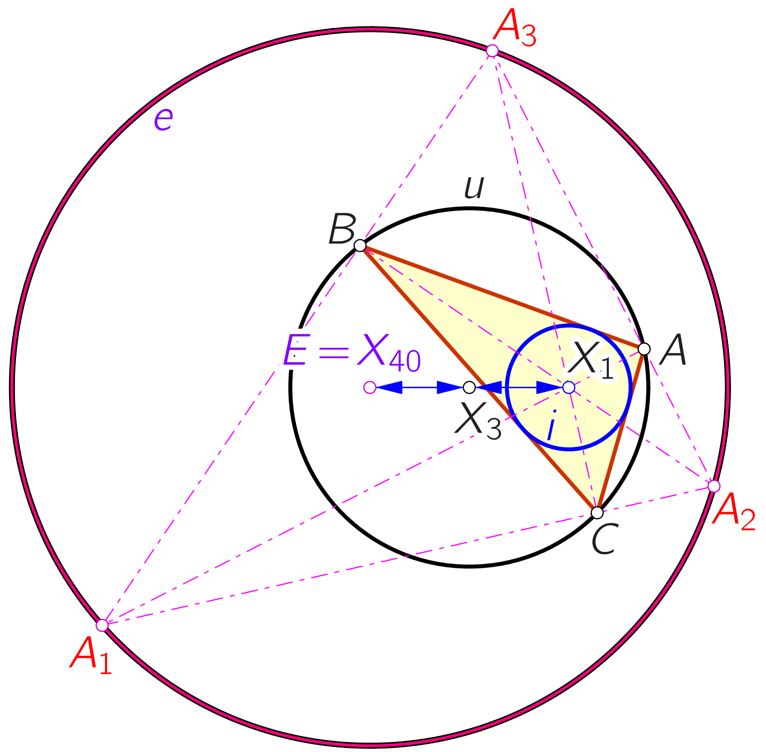


poristic traces of triangle centers III

Surprisingly, some non-centers behave pretty well:

All three excenters move on a single circle. [12]

That is the case only in Chapple's porism. [6]



closure conditions - two conics

Each triangle determines a porism.

Two circles allow for a 1-parameter family of interscribed triangles $\Longleftrightarrow$

$$d^2 = R^2 - 2rR$$ Euler triangle formula

r, R = radii, d central distance

various formulae for bicentric n-gons (Casey, Jacobi, Kerawala, Richelot, Steiner, ...)
derived either elementary or by means of Cayley's formula [1, 3, 4, 7]

There exists an n-sided polygon inscribed into u and circumscribed to i,
if and only if, the coefficients a_i in the power series

$$\sqrt{\det(t \cdot \mathbf{U} + \mathbf{V})} = a_0 + a_1 t + a_2 t^2 + \dots \quad \text{fulfill}$$

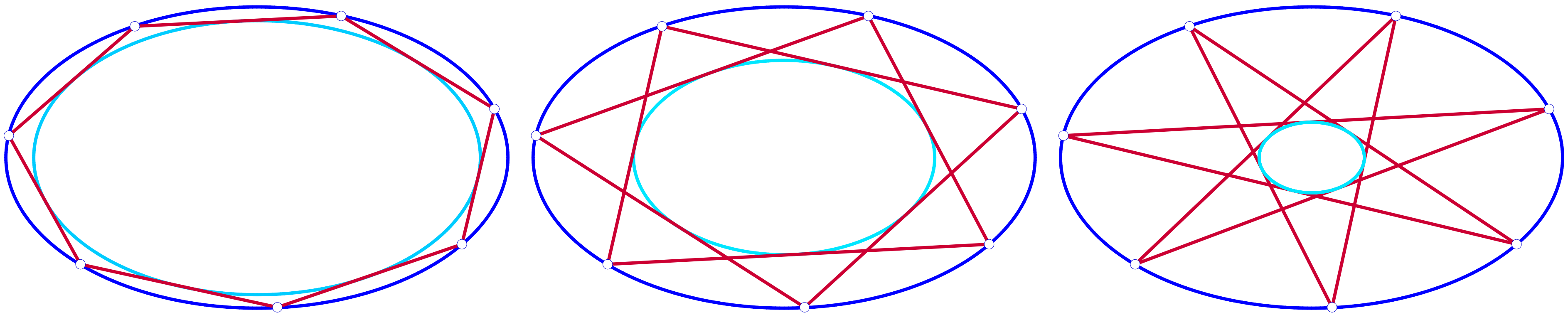
$$\begin{vmatrix} a_2 & \dots & a_{m+1} \\ \vdots & & \vdots \\ a_{m+1} & \dots & a_{2m} \end{vmatrix} = 0, \qquad \begin{vmatrix} a_3 & \dots & a_{m+1} \\ \vdots & & \vdots \\ a_{m+1} & \dots & a_{2m+1} \end{vmatrix} = 0,$$

if $n = 2m+1, \quad m \geq 1$ if $n = 2m, \quad m \geq 2.$

$\mathbf{U}, \mathbf{V} \in \mathbb{C}^{3\times 3}$ ... coefficient matrices of homogeneous equations of u and i

closure conditions - two conics

$n = 7$: Closure condition of degree **24**, ...



various elliptic billiards and their caustics

...symmetric, and delivers a bunch of opportunistic porisms.

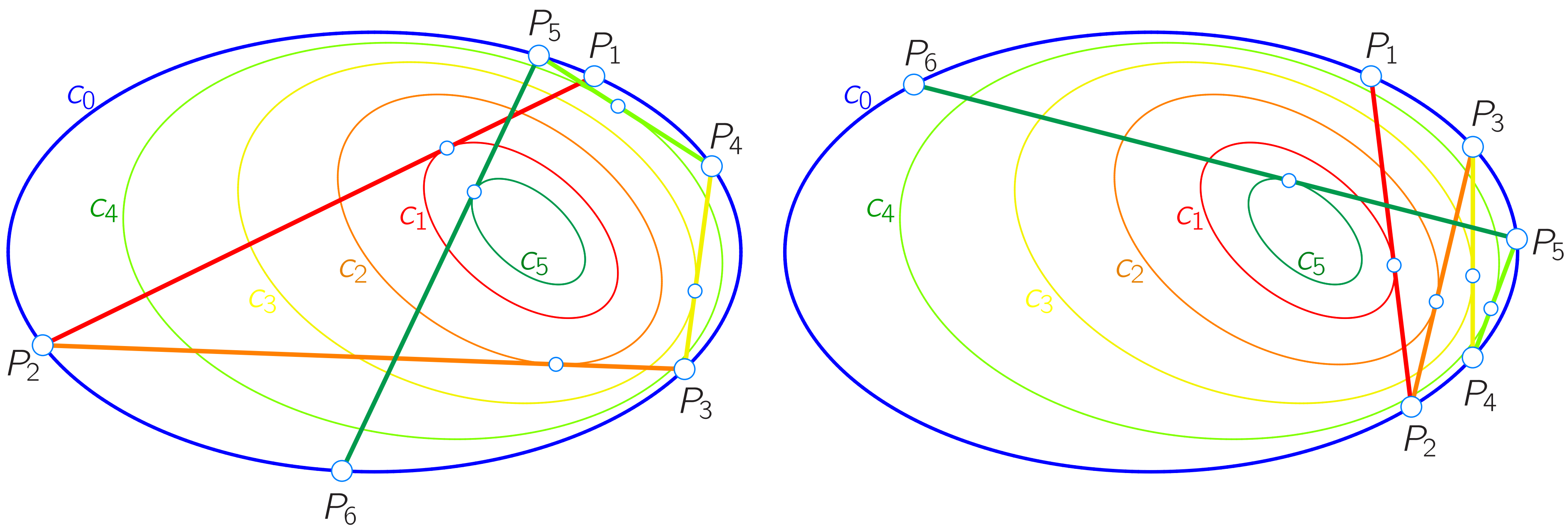
Until now: Porisms involve only two conics.

That's not the most general form of a Poncelet porism.

General Poncelet Porisms

the most general form - n conics

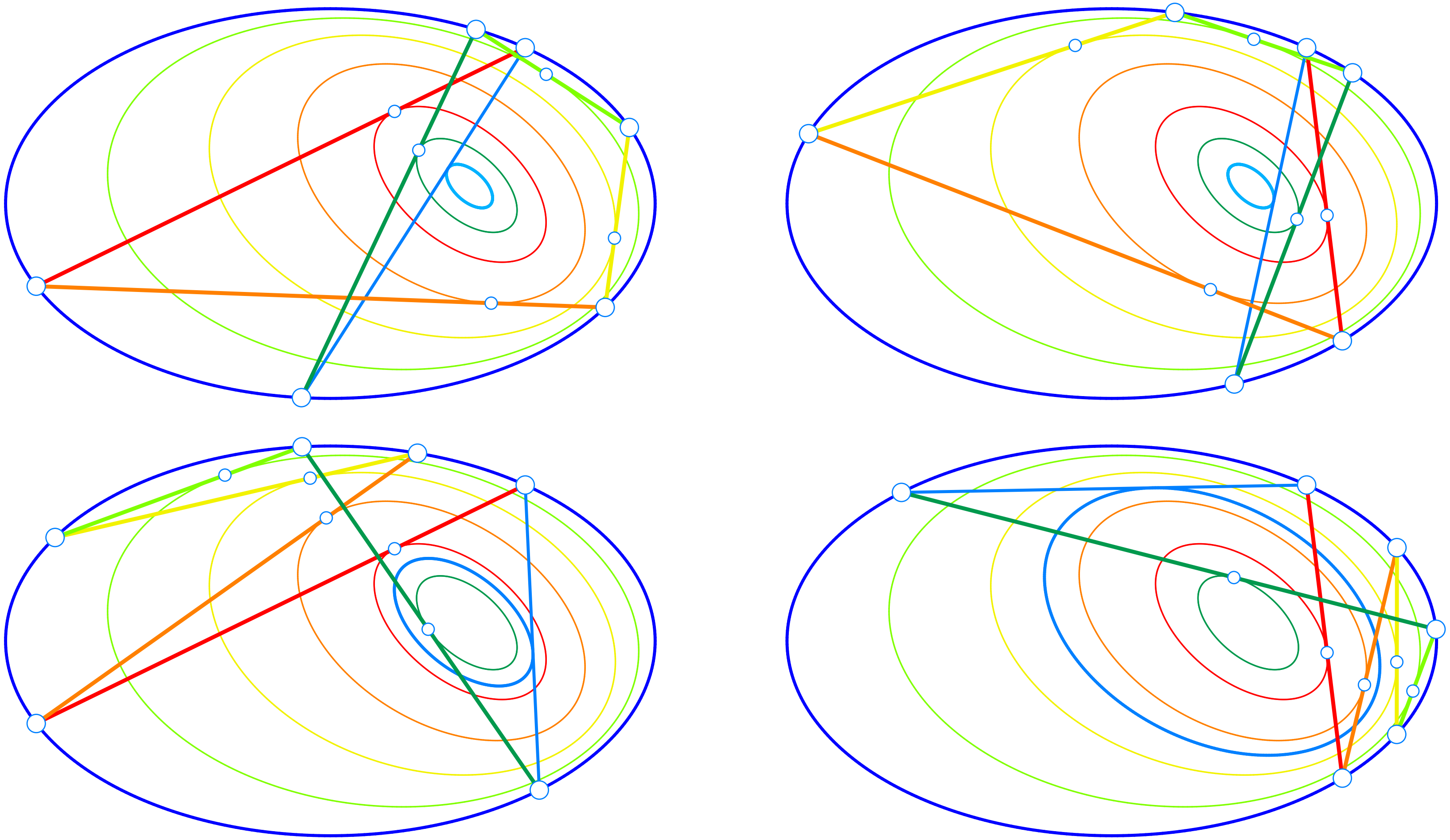
If an n-gon with vertices $P_1, \ldots, P_n$ on a conic c_0 whose side(line)s $[P_i, P_{i+1}]$ are tangent to conics c_i (with $i \in \{1, \ldots, n\}$) closes, i.e., $P_{n+1} = P_1$ *for one particular choice of $P_1 \in c_0$, then it closes for any choice of $P_1 \in c_0$, provided that the conics $c_0, c_1, \ldots, c_n$ belong to one pencil of conics.* [3]



Obviously: Cayley's formula does not apply in this case!

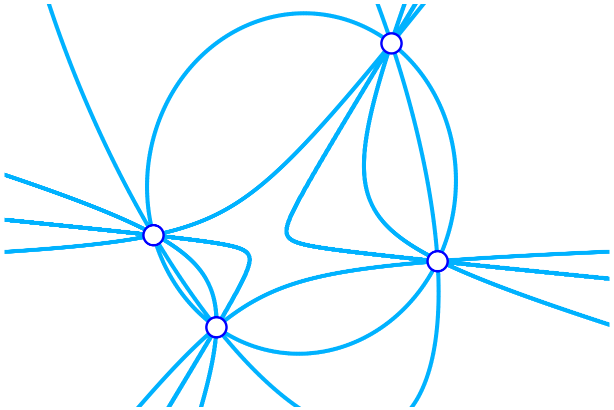
the most general form - n conics

The polygons close in any case: Just draw the missing segment P_nP_1.
Surprise: While the points P_i move along c_0 with $[P_i, P_{i+1}]$ tangent to c_i, the line $[P_n, P_1]$ envelopes a conic from the pencil of the c_i's. [3]

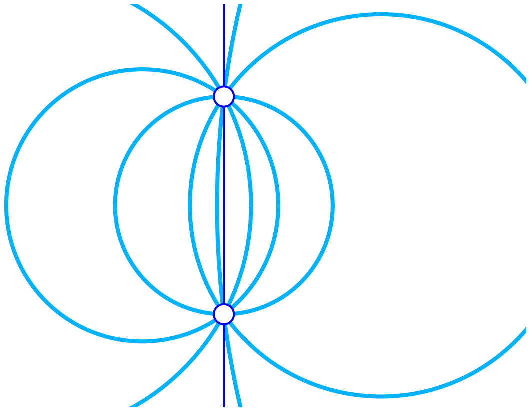


Poncelet's idea - pencils of circles

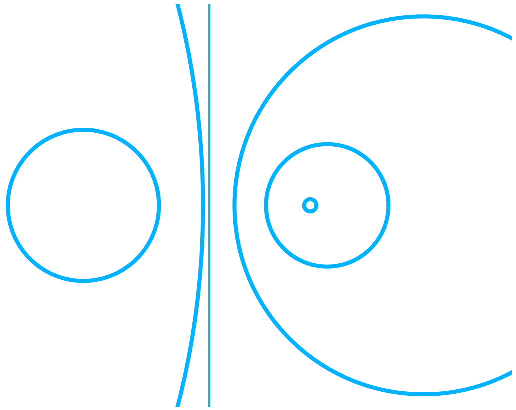
Complex proj. collineations may transform some pencils of conics pencils into circle pencils.



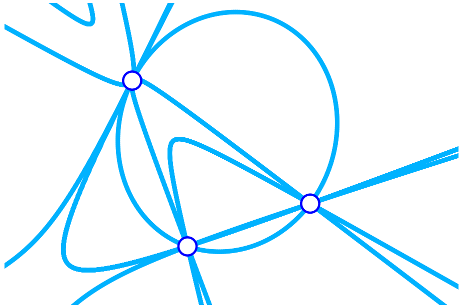
conic pencil type 1



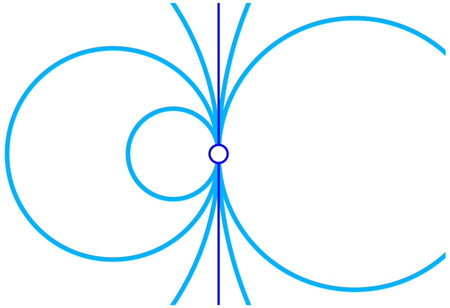
elliptic circle pencil



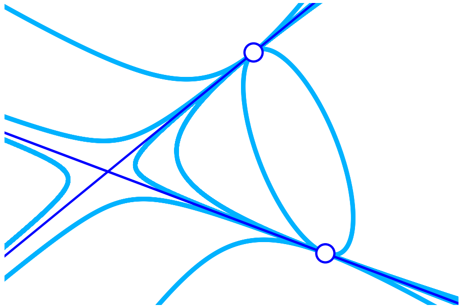
hyperbolic circle pencil



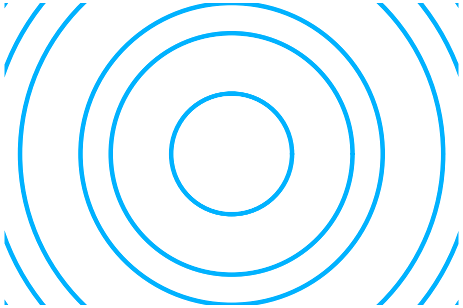
conic pencil type 2



parabolic circle pencil

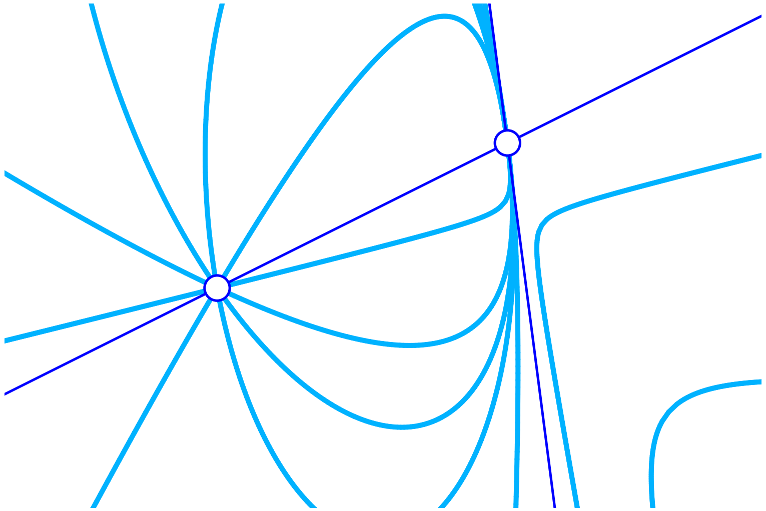


conic pencil type 3



concentric circles

no corresponding pencil of circles - osculating conics



There are no n-gons interscribed between a pair of osculating conics.

Use Cayley's criterion with

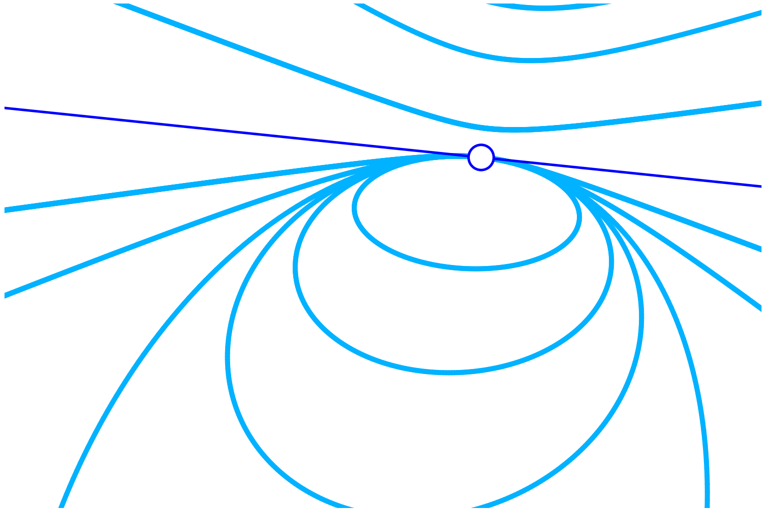
$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

and show that

$$\sqrt{\det(t \cdot \mathbf{U} + \lambda \mathbf{U} + \mu \mathbf{V})} = \sqrt{2\lambda}\left(\lambda + \tfrac{3}{2}t + \tfrac{3}{8\lambda}t^2 - \tfrac{1}{16\lambda^2}t^3 + \dots\right)$$

Hint: Look at the coefficients ...

no corresponding pencil of circles - hyperosculating conics



There are no n-gons interscribed between a pair of **hyper**osculating conics.

Use Cayley's criterion with

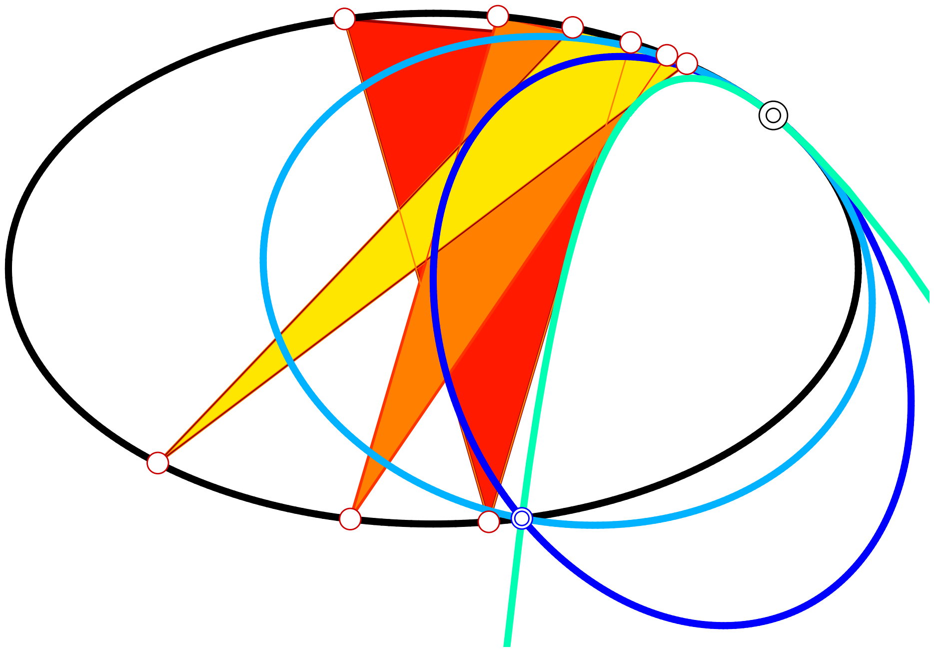
$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and show that

$$\sqrt{\det(t \cdot \mathbf{U} + \lambda\mathbf{U} + \mu\mathbf{V})} = \sqrt{2\lambda}\left(\lambda + \tfrac{3}{2}t + \tfrac{3}{8\lambda}t^2 - \tfrac{1}{16\lambda^2}t^3 + \tfrac{3}{128\lambda^3}t^4 - \tfrac{3}{256\lambda^4}t^5 + \ldots\right)$$

Hint: Look at the coefficients . . .

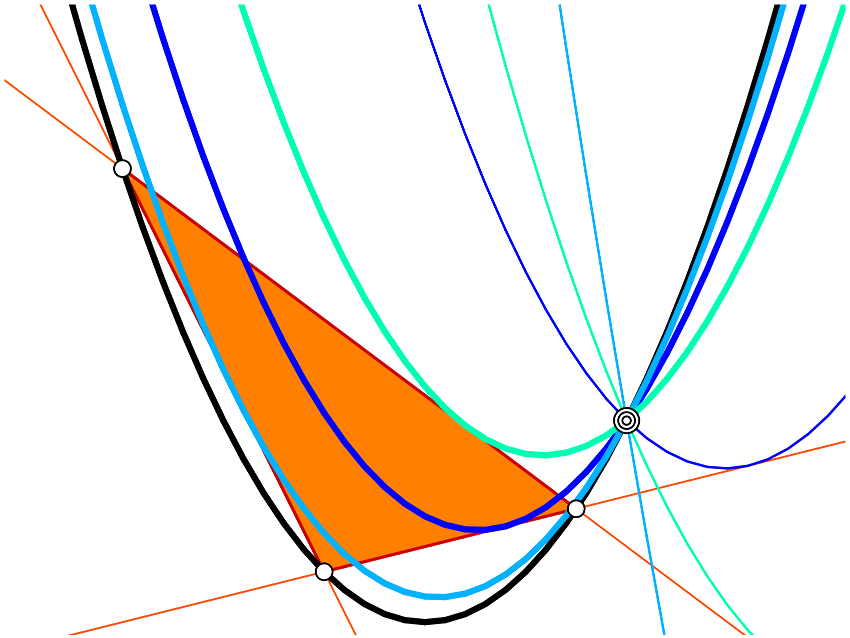
general Poncelet porisms in pencils of osculating conics



only general Poncelet families possible

The Poncelet triangle families interscribed between osculating conics consist of real and complex triangles.

general Poncelet porisms in pencils of osculating conics



pencil of isotropic circles with one common point

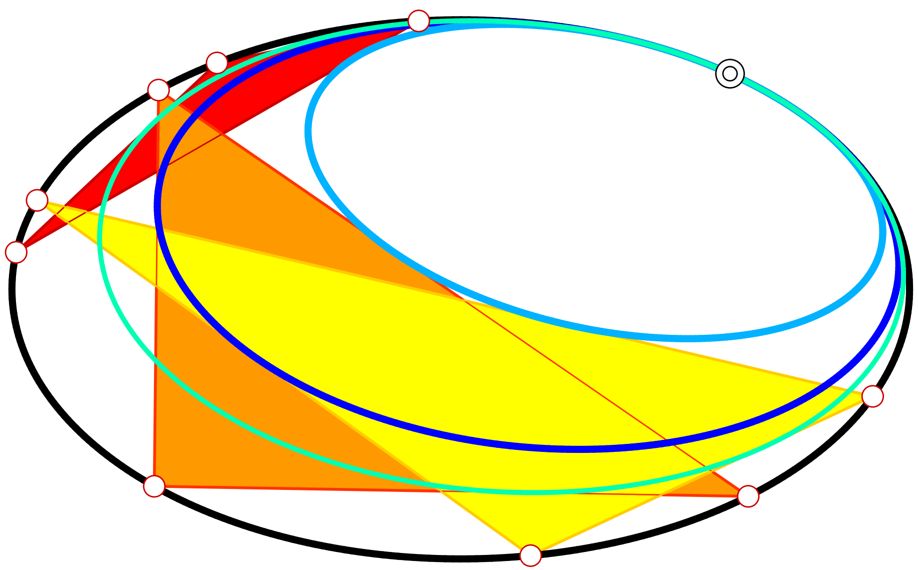
$\mathcal{B}(\lambda): \; x^2 - y + \lambda(x - p) = 0, \quad \lambda, p \in \mathbb{R}$

The closure condition for threee isotropic circles $\mathcal{B}(l_i)$ ($i \in \{1, 2, 3\}$) is a quadratic cone.

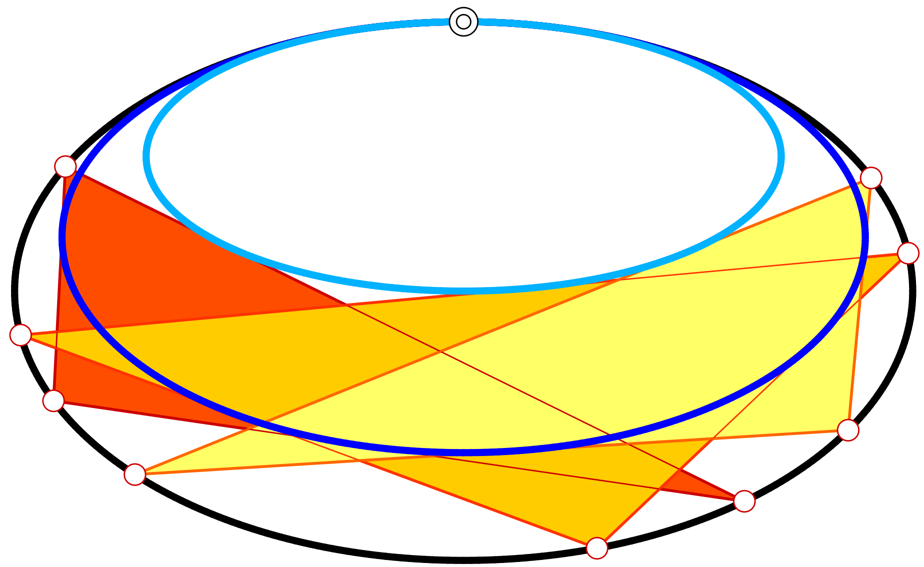
$$\sum_{i=1}^{3} l_i^2 - \sum_{i \neq j} l_i l_j = \lambda_1^2 - 4\lambda_2 = 0$$

λ_1, λ_2 ... elementary symmetric functions in l_i

general Poncelet porisms in pencils of hyperosculating conics



not symmetric, 3 caustics



symmetric, 2 caustics

Closure conditions will be relations between the conics' coordinates in the pencil.

Exponential Pencils of Conics

from linear to exponential pencils

$\mathbf{B}$, $\mathbf{B}_0$, $\mathbf{B}_1$, ... real symmetric 3×3 matrices, coefficient matrices of conics

$\mathbf{C} = \mathrm{e}^{\mathbf{B}}$, $\mathbf{C}_0 = \mathrm{e}^{\mathbf{B}_0}$, $\mathbf{C}_1 = \mathrm{e}^{\mathbf{B}_1}$, ... their exponentials

linear pencil - the usual notion

$$\mathbf{B}(t) = (1-t) \cdot \mathbf{B}_0 + t \cdot \mathbf{B}_1, \; t \in \mathbb{R}$$

exponentiation yields the exponential pencil

$$\mathrm{e}^{t \cdot \mathbf{B}_1 + (1-t) \cdot \mathbf{B}_0} = \mathrm{e}^{t \cdot \mathbf{B}_1} \cdot \mathrm{e}^{(1-t) \cdot \mathbf{B}_0} =$$

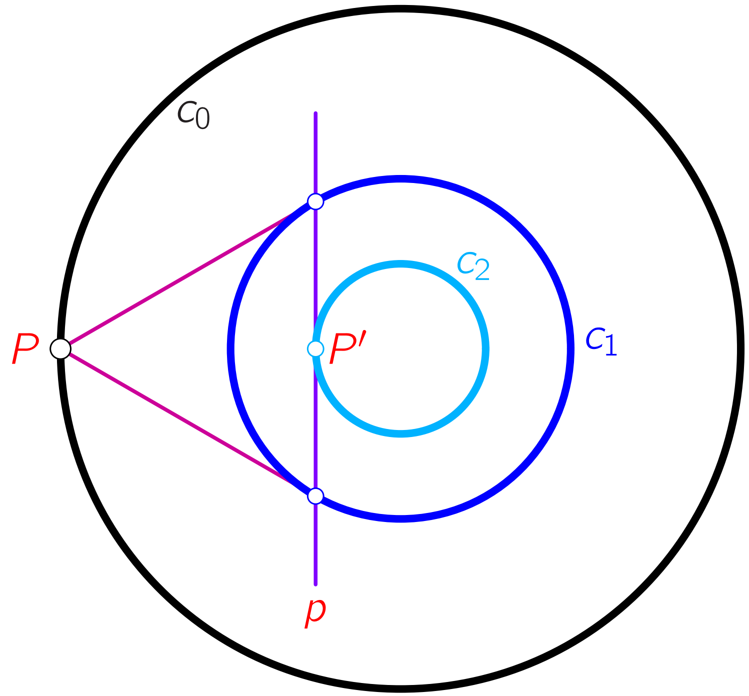
$$\mathbf{C}_1 \cdot \mathbf{C}_1^{-1} \cdot \mathbf{C}_1^{t} \cdot \mathbf{C}_0^{1-t} = \mathbf{C}_1 \cdot \mathbf{C}_1^{-1+t} \cdot \mathbf{C}_0^{1-t} =$$

$$= \mathbf{C}_1 \cdot \left(\mathbf{C}_0^{-1} \cdot \mathbf{C}_1 \right)^{t-1} = \mathbf{C}(t)$$

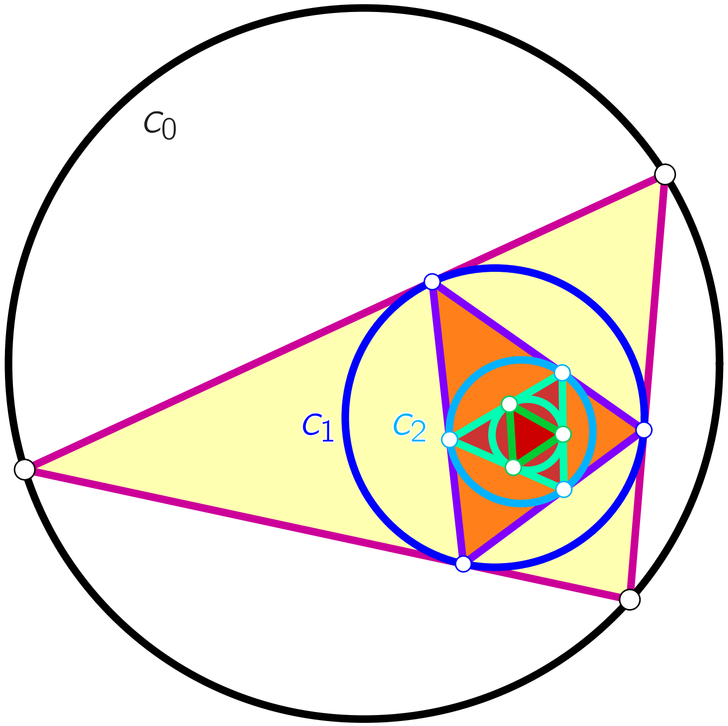
closed under conjugation and dualization

$$\mathbf{C}(2) = \mathbf{C}_2 = \mathbf{C}_1 \cdot \mathbf{C}_0^{-1} \cdot \mathbf{C}_1$$

c_2 is the conjugate conic of c_1 w.r.t. c_0. [8]



exponential pencils - sequences of Poncelet intouch triangles



assume: conics $c_1 : \mathbf{x}^\top \mathbf{B}_1 \mathbf{x} = 0,\ c_0 : \mathbf{x}^\top \mathbf{B}_0 \mathbf{x} = 0,$
$\mathbf{C}, \mathbf{C}_0, \mathbf{C}_1$ exponentials of $\mathbf{B}, \mathbf{B}_0, \mathbf{B}_1$,
real symmetric 3×3 matrices,
conics are nested (either $\partial c_0 \subset \partial c_1$,
or $\partial c_1 \subset \partial c_0$)

then: The limit of $\mathbf{C}(t) = \mathbf{C}_1 \cdot \left(\mathbf{C}_0^{-1} \cdot \mathbf{C}_1\right)^{t-1}$
for $t \to \infty$ is a point. [8]

for example: $\mathbf{C}_0$ = circumcircle, $\mathbf{C}_1$ = incircle

limit for $t \to \infty$ = X_{3513}=1$^{\text{st}}$ dilation center [10]

That's not the incenter limit!

Derivation of some Closure Conditions

closure conditions - elliptic pencil of circles [5]

$c(t): \; x^2 - 2tx + y^2 - 1 = 0$ … normal form of the elliptic pencil of circles
(real) base points $B_{1,2} = (0, \pm 1)$, parameter $t \in \mathbb{R}$ [7]

replace t with $\frac{u^2-1}{2u}$ … technical detail: rationality preferred

circles $c_i: \; x^2 - \frac{u_i^2-1}{u_i} x + y^2 - 1 = 0$ with centers $\left(\frac{u_i^2-1}{2u_i}, 0\right)$ and radii $r_i = \frac{u_i^2+1}{2u_i}$

Vertices P_1, P_2, P_3 correspond to 3 pw. different parameter values $w = U, V, W$ in

$$\mathbf{c}_1(w) = r_1 \left(\frac{1-w^2}{1+w^2}, \frac{2w}{1+w^2}\right) + (m_1, 0).$$

c_1 = circumcircle; c_2 tangent to $[P_1, P_2]$ and $[P_2, P_1]$, c_3 tangent to $[P_2, P_3]$ (2 caustics)

chords $[P_1, P_2]: \; u_1(UV-1)x - u_1(U-V)y + UV + u_1^2 = 0, \; \ldots$

with $1 \to 2 \to 3 \to 1$ and $U \to V \to W \to U$

derive tangency conditions of circles and lines

closure conditions - elliptic pencil of circles [5]

Elimination of V, W (could also be U, V or U, W) from tangency conditions for the pairs $([P_1, P_2], c_2)$, $([P_2, P_3], c3)$, and $([P_3, P_1], c_2)$ yields a polynomial

$$\mathcal{R} = 2^8 \prod_{i=1}^{8} f_i^{\mu_i}$$

in u_1, u_2, u_3, U of resp. degrees $[32, 32, 16, 16]$ / factors' multiplicities $\mu = (2, 2, 2, 2, 2, 2, 1, 1)$.

for a porism: $\mathcal{R}$ has to be **annihilated by** u_i **indenpent of** U

$f_1 = u_1 u_3 + 1,\ f_2 = (u_1 - u_3)^2 \ldots c_1 = c_3 \longrightarrow$ cancel

$f_3 = U^2 u_3 + u_1 \ \ldots$ indep. of U if $u_3 = 0 \Longrightarrow r_3 = \infty \longrightarrow$ cancel

$f_4 = U^2 - u_1 u_3 \ \ldots$ depends on U anyhow $\longrightarrow$ cancel

$f_5 = u_1(2u_1 u_2 - u_2^2 + 1)^2 u_3 - (u_1 u_2^2 - u_1 + 2u_2)^2 \longrightarrow$ good one

$f_6 = u_1(2u_1 u_2 - u_2^2 + 1)^2 + (u_1 u_2^2 - u_1 + 2u_2)^2 u_3 \longrightarrow$ good one

$f_7 = U^4(\ldots) + U^2(\ldots) + U^0(\ldots) \ \ldots$ vanishes if, and only if, $u_1 = 0, \pm \mathrm{i} \longrightarrow$ cancel

$f_8 = U^4(\ldots) + U^2(\ldots) + U^0(\ldots) \ \ldots$ like $f_7 \longrightarrow$ cancel

closure conditions - elliptic pencil of circles [5]

f_5 and f_6 are not substantially different: $u_3 \to -u_3^{-1}$ causes $f_5 \to f_6$

condition on pencil parameters:
eliminate u_i with $u_i^2 - 2t_i u_1 - 1 = 0$ from f_5 and f_6 (yields one polynomial):

$$(2t_1 t_2 - t_2^2 + 1)^2 t_3 + 4t_1^3 - 8t_1^2 t_2 - (t_2^4 - 6t_2^2 - 3)t_1 - 4t_2^3 - 4t_2 = 0 \qquad (\star)$$

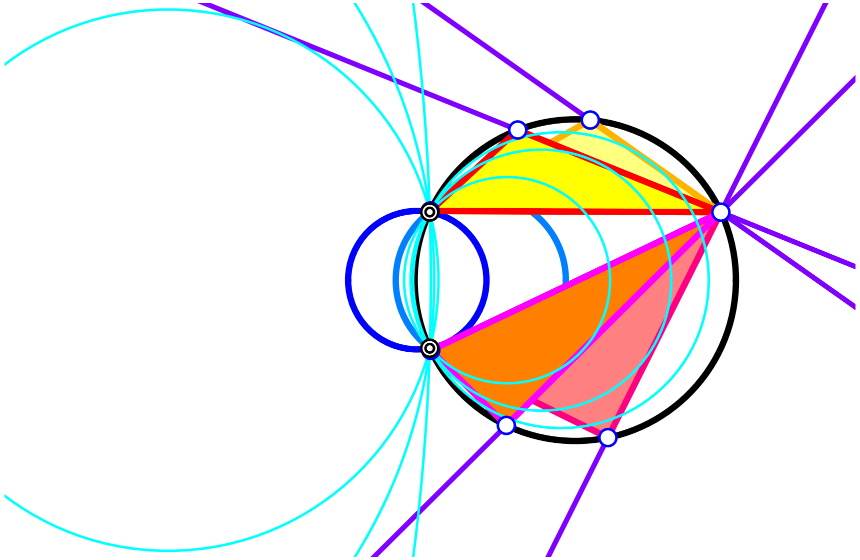
(factor 2^{20} and multiplicity 4 ignored)

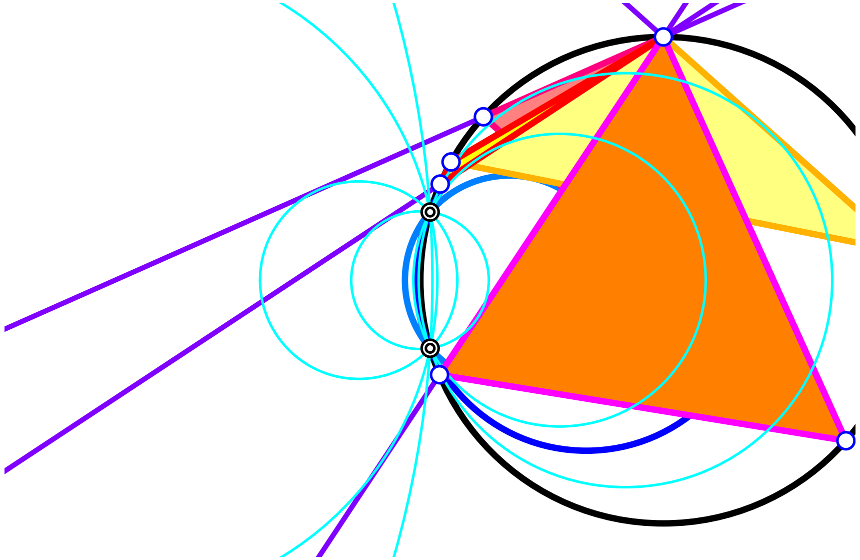
condition on circle radii: eliminate t_i with $r_i^2 - t_i^2 - 1 = 0$ from $(\star)$ yields

$$\begin{aligned}&\Big((4r_1^2 r_2^2 - r_2^4 - 4r_1^2)^2 r_3^2 + 2r_1(r_2^4 + 4r_1^2 - 4r_2^2)(4r_1^2 r_2^2 + r_2^4 - 4r_1^2)r_3 + \\ &\quad + r_1^2(r_2^8 + 8r_1^2 r_2^4 + 8r_2^6 + 16r_1^4 - 32r_1^2 r_2^2)\Big) \cdot \\ &\cdot \Big((4r_1^2 r_2^2 - r_2^4 - 4r_1^2)^2 r_3^2 - 2r_1(r_2^4 + 4r_1^2 - 4r_2^2)(4r_1^2 r_2^2 + r_2^4 - 4r_1^2)r_3 + \\ &\quad + r_1^2(r_2^8 + 8r_1^2 r_2^4 + 8r_2^6 + 16r_1^4 - 32r_1^2 r_2^2)\Big) = 0\end{aligned}$$

closure conditions - elliptic pencil of circles [5]

Degrees are not equal to the actual numbers of circles touching the last side(s).





Each third line is touched by two circles of the third kind, but
some circles touch two of the third lines.

closure conditions - hyperbolic/parabolic pencil of circles [5]

$x^2 - 2tx + y^2 + 1 = 0$ normal form
of the hyperbolic pencil of circles with
(complex) base points $B_{1,2} = (0, \pm \mathrm{i})$ and parameter $t \in \mathbb{R} \neq 0, \pm 1$ [7]

$x^2 - 2tx + y^2 = 0$ normal form
of the parabolic pencil of circles with
one real base point $(0, 0)$ and one base tangent $x = 0$, parameter $t \in \mathbb{R} \setminus \{0\}$ [7]

assume c_1 is the circumcircle and
c_2 tangent to $[P_1, P_2]$ and $[P_2, P_1]$ (the symmetric case); c_3 tangent to $[P_2, P_3]$

Computations do not differ.

closure conditions - hyperbolic / elliptic / parabolic pencil of circles [5]

the non-symmetric case with three caustics (parameters t_1, t_2, t_3)
and circumcircle c_0 (parameter t_0):

$$
\begin{aligned}
h:&\ 4t_0^4 - 4(\tau_1+\tau_3)t_0^3 + (\tau_2^2+6\tau_2-3)t_0^2 - 2(\tau_2-1)(\tau_1+\tau_3)t_0 + (\tau_1+\tau_3)^2 - 4\tau_2 = 0,\\
e:&\ 4t_0^4 - 4(\tau_1-\tau_3)t_0^3 - (\tau_2^2-6\tau_2-3)t_0^2 - 2(\tau_2+1)(\tau_1-\tau_3)t_0 - (\tau_1-\tau_3)^2 + 4\tau_2 = 0,\\
p:&\ 4\tau_3 t_0^3 - \tau_2^2 t_0^2 + 2\tau_2\tau_3\, t_0 - \tau_3^2 = 0.
\end{aligned}
$$

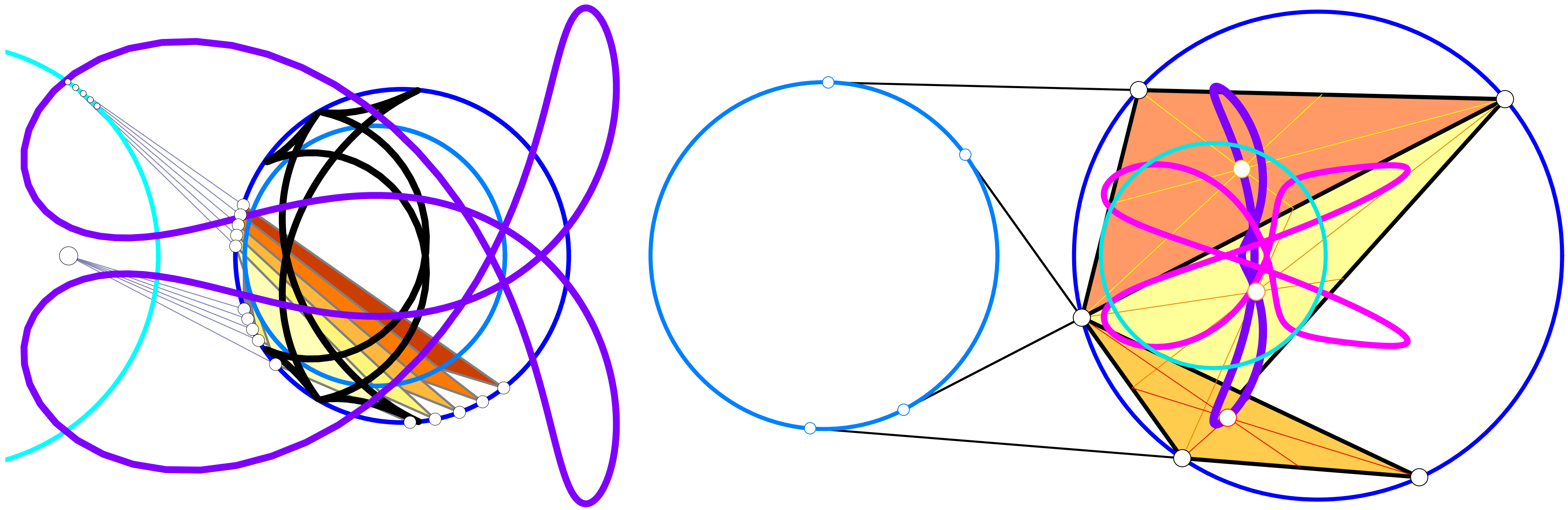
τ_i ... elementary symmetric functions in t_1, t_2, t_3

Their equivalents in terms of radii are of degree 8 (e,h) or 2 (p) in each radius and of degree 24 (e,h) or 6 (p) in total.

Paths of Triangle Centers and other Points

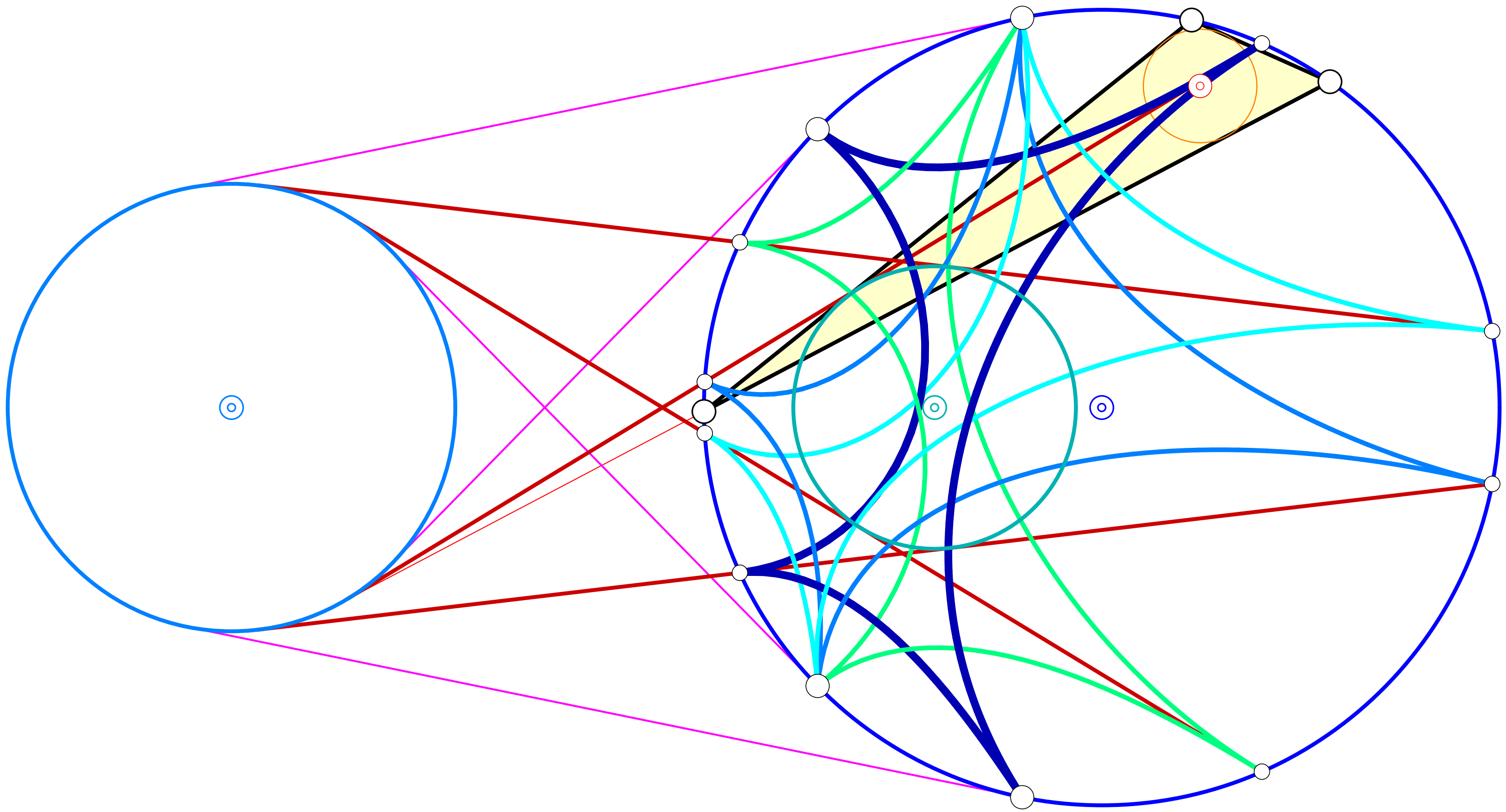
traces of points / centers in general Poncelet porisms [5,6]

No direct computation of parametrizations of traces possible.
Everythings is formulated in terms of implicit equations.
Elimination of unnecessary parameters with constraint equations yields the complete algebraic picture including opportunistic stuff.



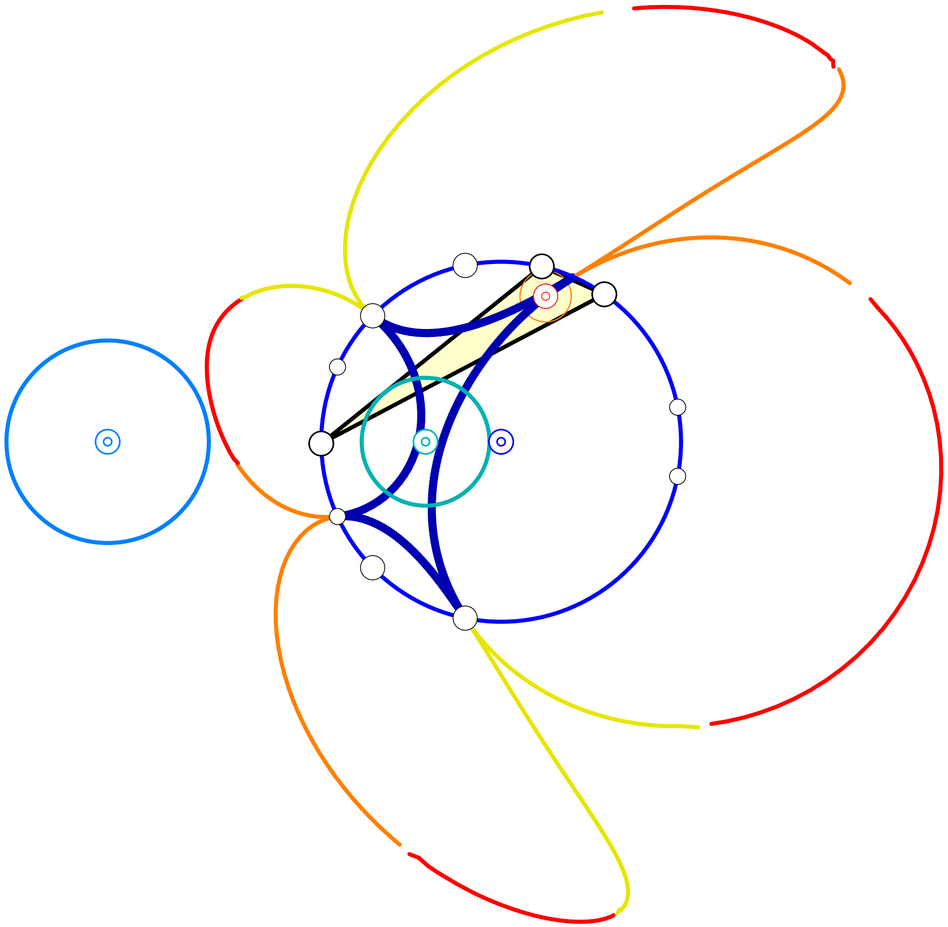
path of the incenter in general Poncelet porisms [5,6]

the various incenter paths for the totality of poristic triangle families interscribed between three circles of a hyperbolic pencil:



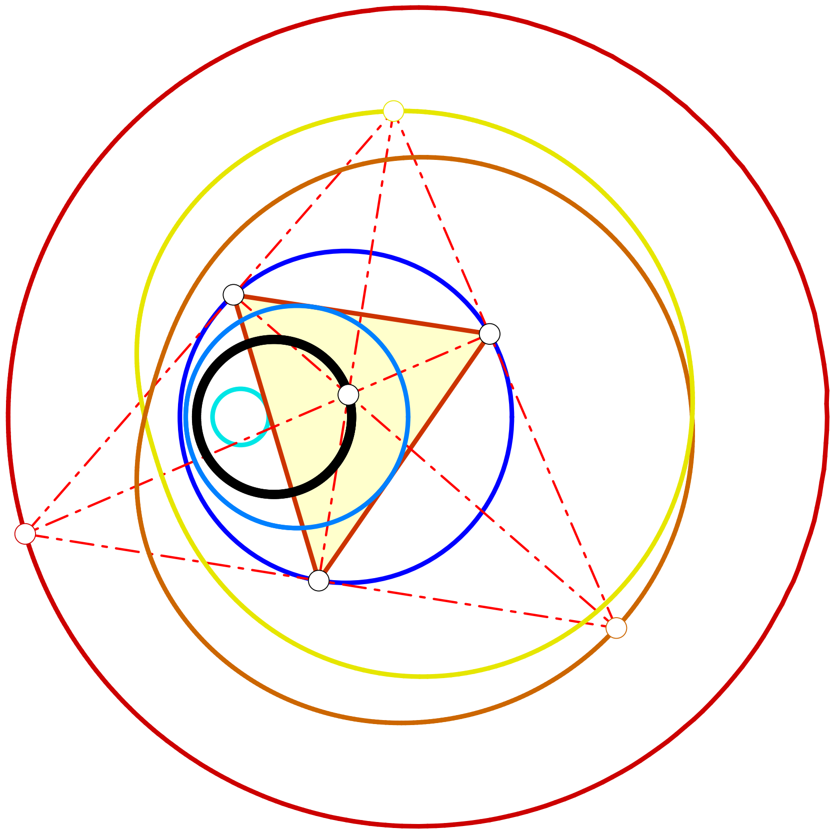
path of the tritangent circles' centers in general Poncelet porisms [5,6]

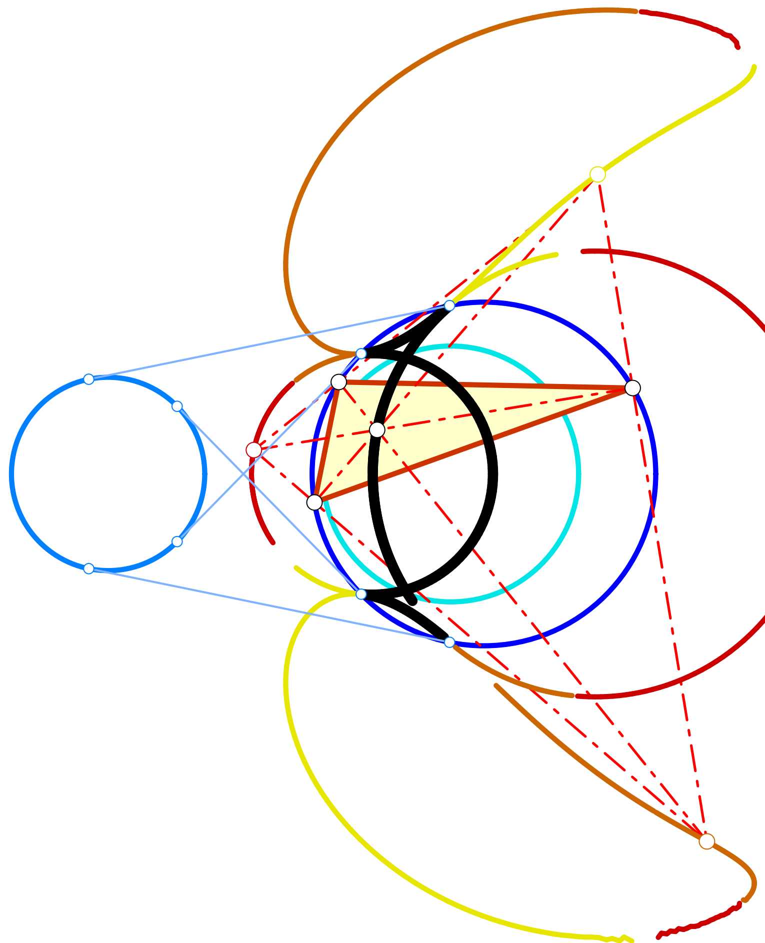
The paths of the centers of the tritangent circles change their geometric meaning at the cusps of the incenter trace.



Is there a difference between incenter and excenters? [5,6,11,14]

In porisms with symmetry poses, at least one changeling moves on a circle.





concluding remarks

- confocal conics: see D. Reznik's simulations
- paths of centers of tritangent circles in the general case: equations?
- algebraic problems treated algebraically, no elliptic functions
- numerical invariants
- ellipticity of poristic traces (shape, not genus = 1)
- systematic / automatic computation of closure conditions

literature I

[1] A. Cayley: *Developments on the porism of in-and-circumscribed polygon.* Philosophical Magazine **7** (1854), 339–345.

[2] W. Chapple: *An essay on the properties of triangles inscribed in and circumscribed about two given circles.* Misc. Curiosa Math. **4** (1746), 117–124.

[3] A. del Centina: *Poncelet's Porism: a long story of renewed discoveries I.* Arch. Hist. Exact Sci. **70**/1 (2016), 1–122.

[4] L. Flatto: *Poncelet's theorem.* American Mathematical Society, Providence, RI, 2008.

[5] R.A. Garcia, B. Odehnal, D. Reznik: *Poncelet porisms in hyperbolic pencils of circles.* J. Geom. Graphics **25**/2 (2021), 205–225.

[6] R.A. Garcia, B. Odehnal, D. Reznik: *Loci of Poncelet Triangles in the General Closure Case.* J. Geom. **113**:17 (2022), 1–17.

[7] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Concis. From the ancient Greeks to 21st century developments.* Springer-Spektrum, Springer-Verlag, Heidelberg, 2016.

literature II

[8] L. Halbeisen, N. Hungerbühler: *The exponential pencil of conics.* Beitr. Algebra Geom. **59** (2018), 549–571.

[9] C. Kimberling: *Triangle Centers and Central Triangles*. Congressus Numerantium **129**, Winnipeg, Canada, 1998.

[10] C. Kimberling: *Encyclopedia of Triangle Centers*. Available at: `http://faculty.evansville.edu/ck6/encyclopedia`

[11] B. Odehnal: *Generalized Gergonne and Nagel points.* Beitr. Algebra Geom. **51**/2 (2010), 477–491.

[12] B. Odehnal: *Poristic Loci of Triangle Centers.* J. Geom. Graphics **15**/1 (2011), 45–67.

[13] B. Odehnal: *Two convergent triangle tunnels.* KoG **22**/22 (2018), 20–28.

[14] N. Le, N.J. Wildberger: *Universal affine triangle geometry and four-fold incenter symmetry.* KoG **16**/16 (2012), 63–80.

Thank You For Your Attention!