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Equioptic Curves (of Conic Sections)

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outline

- $\diamond$ definition & properties
- $\diamond$ computation (in theory and practice)
- $\diamond$ bounds on algebraic degrees
- $\diamond$ conic sections
- $\diamond$ singularities
- $\diamond$ examples

definition

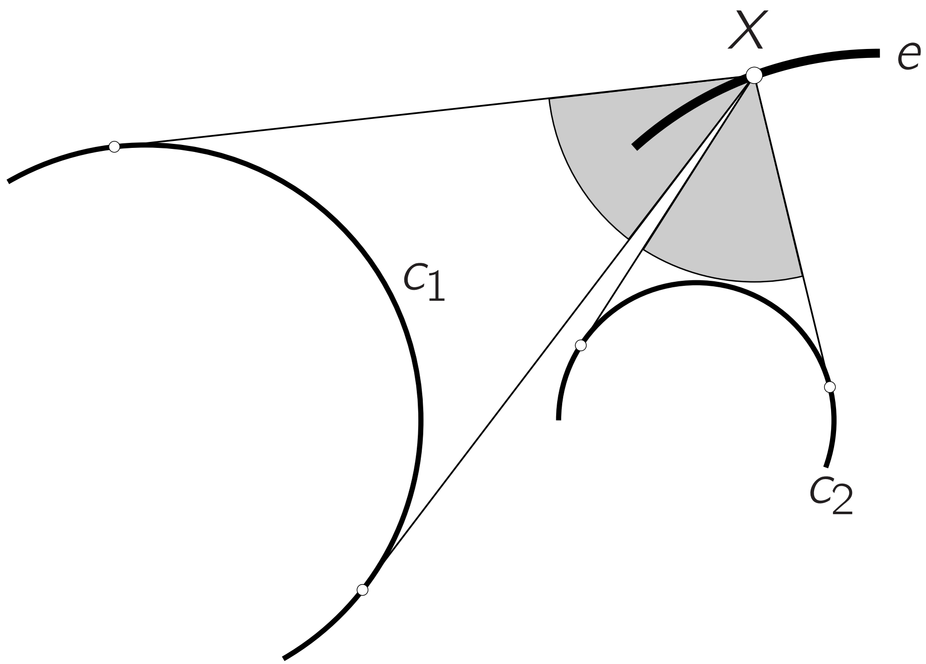
X is a point on the **equioptic curve** $e(c_1, c_2)$ of two plane curves c_1 and c_2

$$\Longleftrightarrow$$

c_1 and c_2 are **seen under equal angles** from X

$$\Longleftrightarrow$$

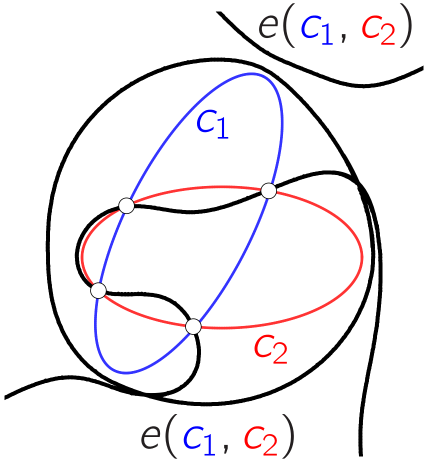
either curve c_i sends a pair (T_i, T_i') of tangents through X

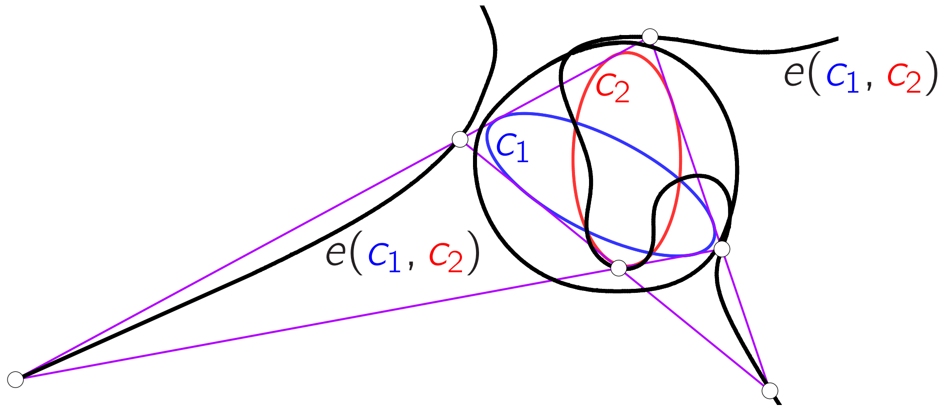


obvious properties

Common points of c_1 and c_2 are points on $e(c_1, c_2)$.

Intersection points of common tangents of c_1 and c_2 are points on $e(c_1, c_2)$.





computation

assumption: c_1 and c_2 are algebraic, given by equations $F_1 = 0$ and $F_2 = 0$

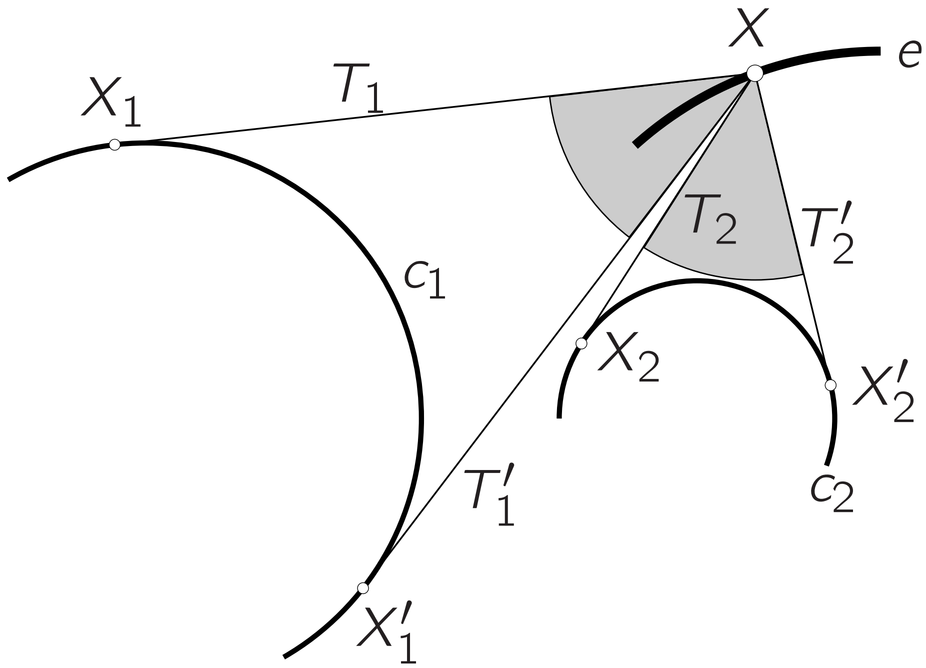
$$X \in e(c_1, c_2) = [x, y]^\mathrm{T}$$

(1) $\quad X_i \in c_i, \quad X_i' \in c_i$

(2) $\quad [X, X_i] = T_i, \quad X_i = [x_i, y_i]^\mathrm{T}$

$\quad [X, X_i'] = T_i', \quad X_i' = [\xi_i, \eta_i]^\mathrm{T}$

(3) $\quad \angle(T_1, T_1') = \angle(T_2, T_2')$



computation - equations

$$
\begin{array}{lrl}
(1.1,2) & F_1(X_1) = 0, & F_1(X_1') = 0, \\
(1.3,4) & F_2(X_2) = 0, & F_2(X_2') = 0, \\
(2.1,2) & \langle g_1, X - X_1 \rangle = 0, & \langle g_1', X - X_1' \rangle = 0, \\
(2.3,4) & \langle g_2, X - X_2 \rangle = 0, & \langle g_2', X - X_2' \rangle = 0, \\
(3) & \langle g_1, g_1' \rangle^2 \cdot \langle g_2, g_2 \rangle \cdot \langle g_2', g_2' \rangle \; = & \langle g_2, g_2' \rangle^2 \cdot \langle g_1, g_1 \rangle \cdot \langle g_1', g_1' \rangle,
\end{array}
$$

where $g_i = \mathrm{grad}\ \mathrm{F_i}(\mathrm{X_i})$, $g_i' = \mathrm{grad}\ \mathrm{F_i}(\mathrm{X_i'})$

9 equations in **10 unknowns**: $x_1, y_1, x_2, y_2, \xi_1, \eta_1, \xi_1', \eta_1', x, y$

eliminate all but x and $y \Longrightarrow$ equation of $e(c_1, c_2)$

computation - bounds on degrees

Let $\deg c_1 = d_1$ and $\deg c_2 = d_2$:

Theorem.
The algebraic degree d of the equioptic curve $e(c_1, c_2)$ is at most

$$d_1^2 d_2^2 (d_1 - 1)^2 (d_2 - 1)^2. \tag{4}$$

Proof. Computing the equation of $e(c_1, c_2)$ = successive elimination of variables from Eqs. (1.1)–(3). Compare degrees of all equations considered as polynomials in x and y. □

disadvantages of algebraic definition

- actual degrees will be lower in most cases
- ideal line splits off (sometimes with high multiplicity)
- $\cos^2(\varphi) = \cos^2(\varphi + \pi)$
- parasitic branches may appear

conic sections

Assume: c_1, c_2 are conic sections, i.e., $d_1 = d_2 = 2$

$\deg e(c_1, c_2) \leq 16$ according to (4)

Moreover: Computation of resultants fails because of memory shortage!

$\Longrightarrow$ intersect isoptics (of conic sections)

computation - in practice

X is a point on the **equioptic curve** $e(c_1, c_2)$

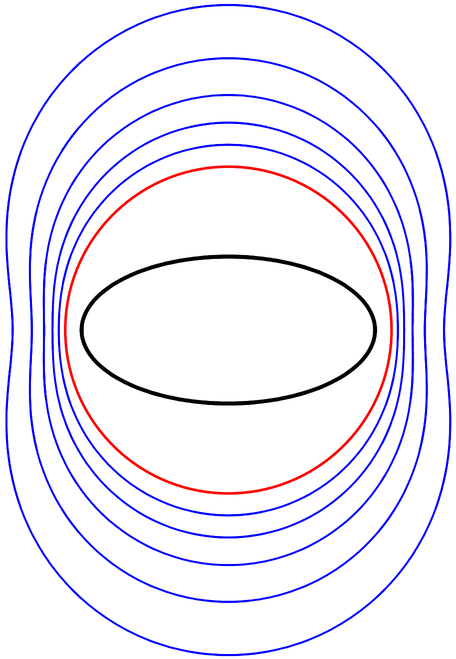
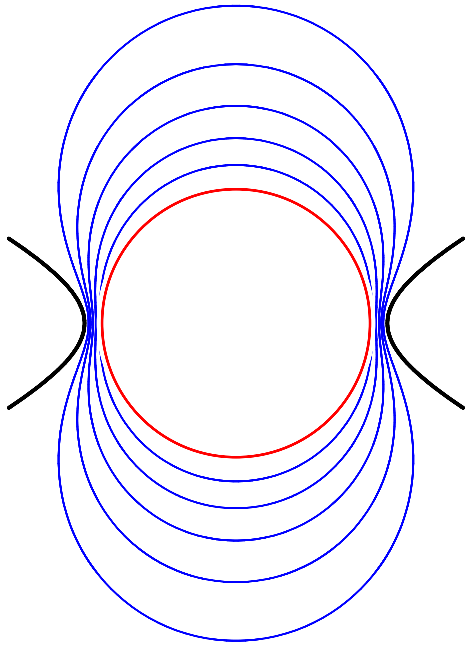
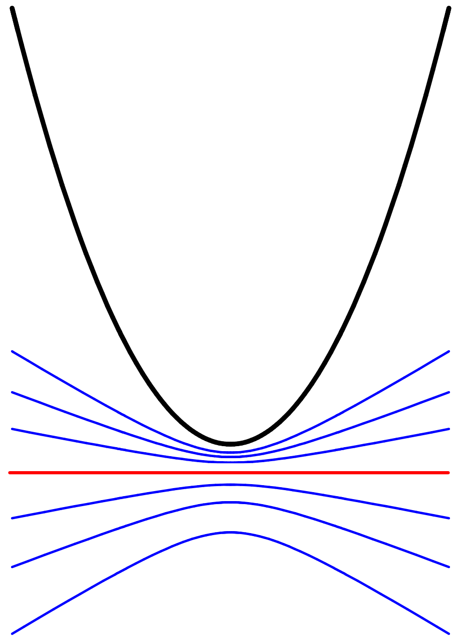
$$\Longleftrightarrow$$

both curves c_1 and c_2 are seen under φ from X

$$\Longleftrightarrow$$

X is a common point of the **isoptic curves** $i(c_1, \varphi)$ and $i(c_2, \varphi)$

isoptics of conic sections

curve	**ellipse**	**hyperbola**	**parabola**
			
isoptic	quartic	quartic	hyperbola
orthoptic	circle	circle	directrix

isoptics of conic sections - equations

conic section with center	$\alpha x^2 + \beta y^2 = 1,\ \alpha\beta \neq 0$
isoptic	$(\alpha + \beta - \alpha\beta(x^2 + y^2))^2 \sin\varphi$
	$-4\alpha\beta(\alpha x^2 + \beta y^2 - 1)\cos^2\varphi = 0$
parabola	$2py = x^2,\ p \neq 0$
isoptic	$(4x^2 + (p - 2y)^2)\cos^2\varphi - (p + 2y)^2 = 0$

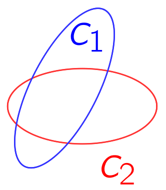
write down isoptics of c_1 and c_2 (probably after suitable change of coordinates) and eliminate φ (or $\cos\varphi$ and $\sin\varphi$, respectively)

$$\Longrightarrow$$

yields equation of $e(c_1, c_2)$ and keeps degrees low

example

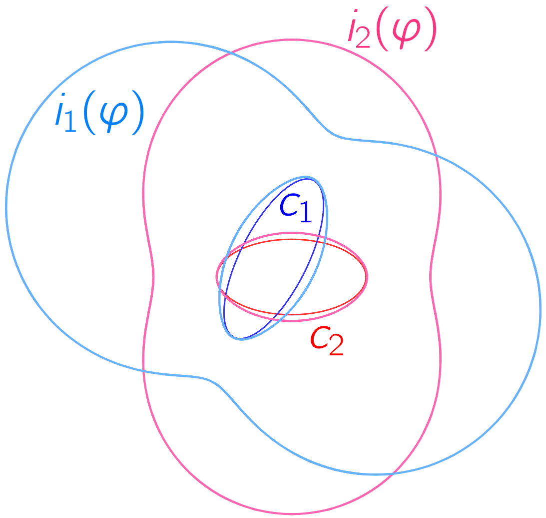
given: two conic sections c_1 and c_2



example

given: two conic sections c_1 and c_2

1. compute isoptics $i_1(\varphi)$ and $i_2(\varphi)$

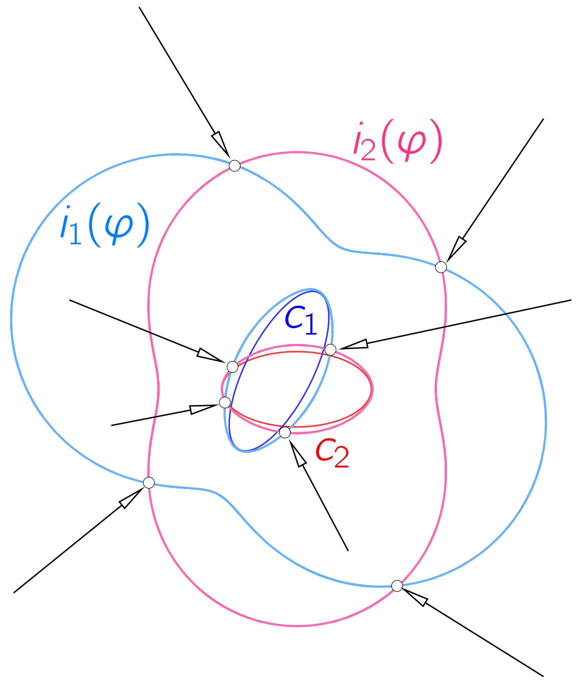


example

given: two conic sections c_1 and c_2

1. compute isoptics $i_1(\varphi)$ and $i_2(\varphi)$

2. compute intersection $i_1(\varphi) \cap i_2(\varphi)$



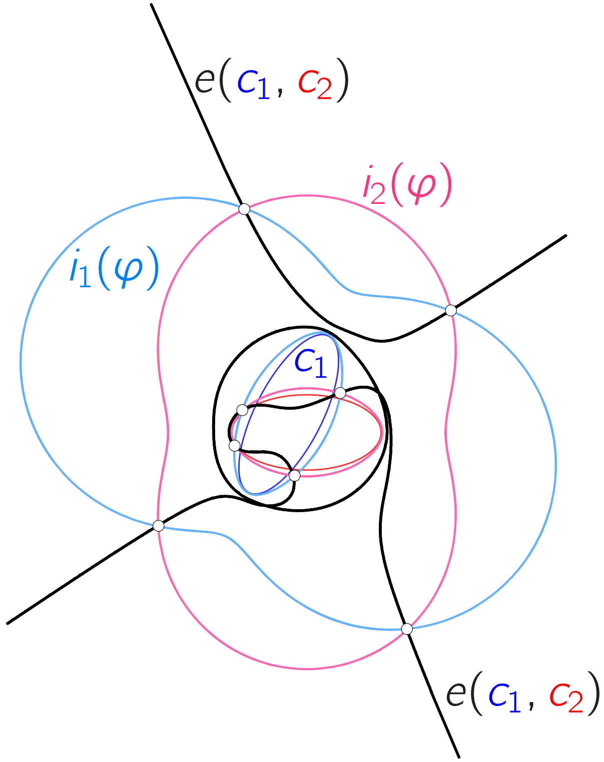
example

given: two conic sections c_1 and c_2

1. compute isoptics $i_1(\varphi)$ and $i_2(\varphi)$

2. compute intersection $i_1(\varphi) \cap i_2(\varphi)$

3. $\forall \varphi \iff$ elimination of $\varphi \implies$ equation of $e(c_1, c_2)$

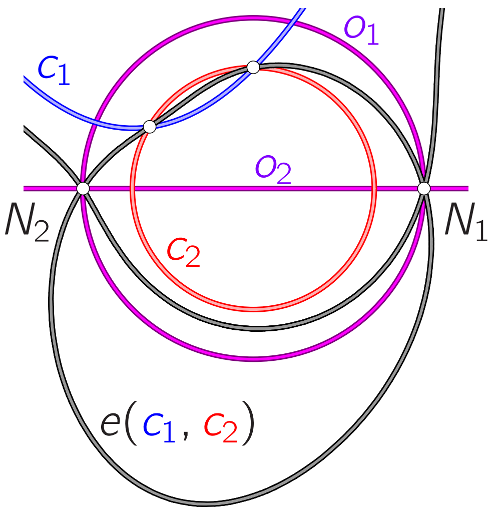


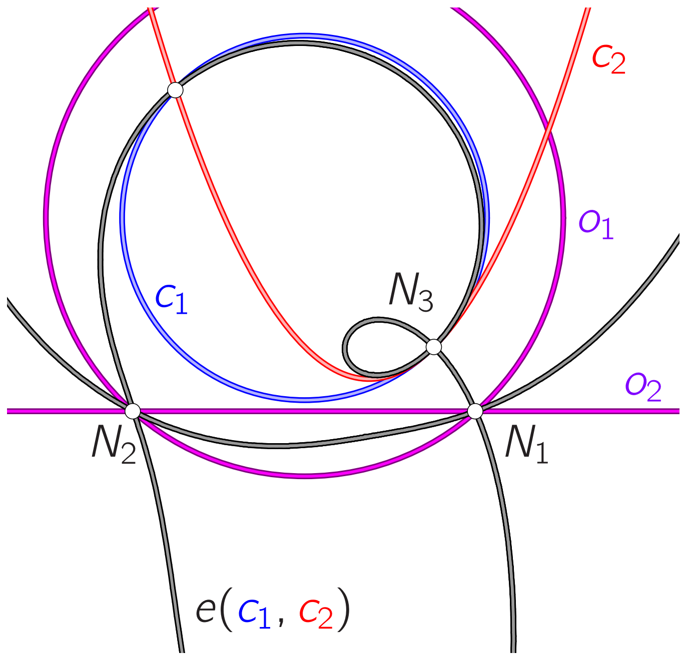
equioptics of conic sections - degrees

Theorem.
1. The degree of the equioptic of two conic sections with center or of a conic section with center and a parabola equals 6.
2. The equioptic of two parabolae is of degree 4.

Proof. Compute resultants of pencils of isoptics with respect to φ. (Computing with homogeneous equations: equation of ideal line splits off twice.) $\square$

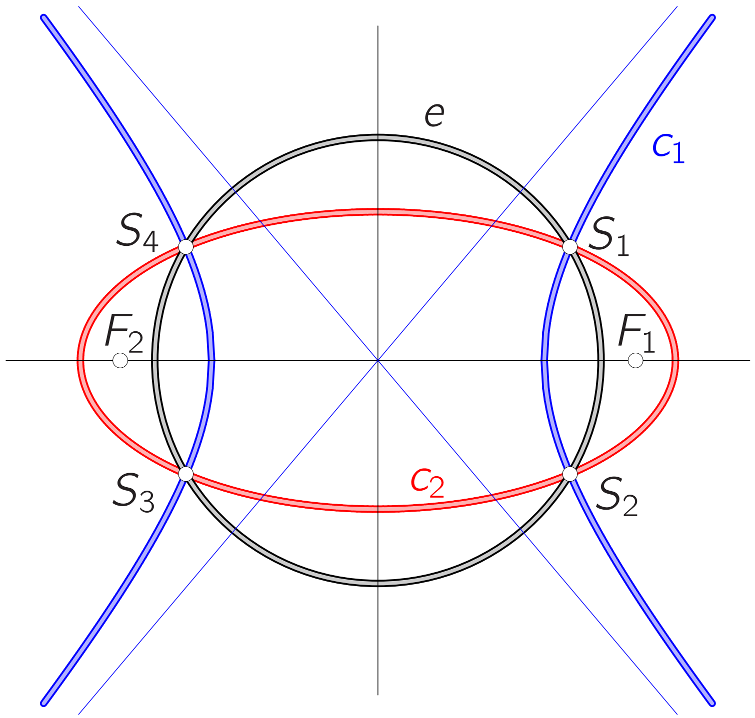
singularities . . .

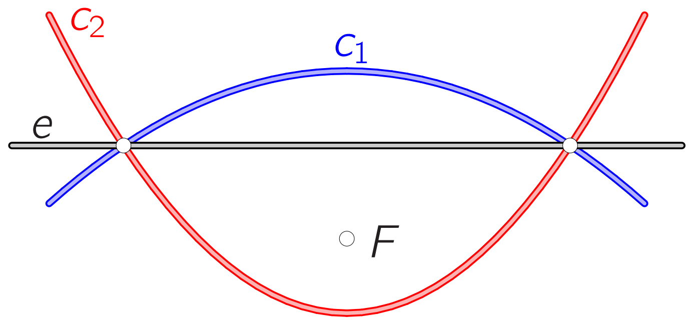




. . . occur at the **ideal points**, at **intersections of orthoptics**, and at **higher order contacts**.

equioptics of confocal pairs of conic sections . . .





. . . are circles or lines.

references

[1] E. Brieskorn, H. Knörrer: *Ebene algebraische Kurven.* Birkhäuser, 1981.

[2] J.L. Coolidge: *A treatise on algebraic plane curves.* Dover Publications, 1959.

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[4] B. O.: *Equioptic curves of conic sections.* J. Geom. Graph. **14**/1 (2010), 29–43.

[5] F.H. Siebeck: *Über eine Gattung von Curven vierten Grades, welche mit den elliptischen Funktionen zusammenhängen.* J. Reine Angew. Math. **57** (1860), 359–370; **59** (1861), 173–184.

[6] H. Wieleitner: *Spezielle ebene Kurven.* Göschen'sche Verlagshandlung, 1908.

Thank You for Your Attention!