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Isotropic Congruences of Lines in Elliptic 3-Space

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Motivation & Contents

- elegant techniques and interesting mathematics
- non-Euclidean geometry, line geometry, kinematics at the same time
- elliptic three-space: quaternions
- line geometry: Klein model, spherical kinematic mapping
- differential geometry of congruences of lines
- generation and parametrization of congruences
- isotropic & minimal congruences

Elliptic Three-Space $\mathbb{E}^3$ - quaternions

$\{1, i, j, k\}$ ... basis in $\mathbb{R}^4$

unique representation $x = x_0 + ix_1 + jx_2 + kx_3$ for any $x \in \mathbb{R}^4$

addition as usual

multiplication according to

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

$\Longrightarrow$ $\mathbb{R}^4$ becomes the skew field $\mathbb{H}$ of quaternions.

conjugate quaternion $\widetilde{x} = x_0 - ix_1 - jx_2 - kx_3$

norm of a quaternion $N(x) = x\widetilde{x}$, multiplicative, *i.e.*, $N(xy) = N(x)N(y)$

inverse quaternion $x^{-1} = \widetilde{x}/N(x)$, if $N(x) \neq 0$

unit quaternion x with $N(x) = 1$

Elliptic Three-Space - quaternions

We use unit quaternions as coordinates of points in projective three-space $\mathbb{P}^3$. [1,3,12]

elliptic distance d between points X, Y

$$\cos d = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 =: \langle X, Y \rangle_e$$

with unit quaternions x, y

$d = \frac{\pi}{2} \Longleftrightarrow X$, Y are orthogonal points

metric can be expressed in terms of quaternions as

$$2\langle X, Y \rangle_e = x\widetilde{y} + y\widetilde{x} = \widetilde{x}y + \widetilde{y}x$$

$\Longrightarrow \mathbb{P}^3$ becomes an elliptic three-space $\mathbb{E}^3$.

as a Cayley-Klein space [6]: a space with the absolute quadric

$$\Omega: \; x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$$

Elliptic Three-Space - spherical kinematic mapping

oriented line $L \subset \mathbb{E}^3$: either $X \vee Y$ or $Y \vee X$ (order pair of points)

w.l.o.g. $X \perp_e Y$; parametrization of a line

$$L(t) = x \cos t + y \sin t \quad t \in [0, \pi)$$

define left and right spherical kinematic image points [1,9,12,16]

$$L^l = \widetilde{x} y \quad \text{and} \quad L^r = y \widetilde{x}$$

$L^l, L^r \in \operatorname{im} \mathbb{H} := i\mathbb{R} + j\mathbb{R} + k\mathbb{R} \cong \mathbb{R}^3$ and $\langle L^l, L^l \rangle = \langle L^r, L^r \rangle = 1$

L^l, L^r are independent on the choice of $X \perp_e Y$ on L.

$Y \vee X \longrightarrow (-L^l, -L^r)$

$\Longrightarrow$ Clifford's spherical kinematic mapping $\vec{\mathcal{L}} \to S^2 \times S^2$ [4,6], one-to-one and onto

Elliptic Three-Space - Kinematics

elliptic motions $\beta: \mathbb{E}^3 \to \mathbb{E}^3$

$$\beta(x) = x' = \widetilde{q} x p$$

with unit quaternions p, q

left and right images of $L' = X' \vee Y'$

$$L'^l = \widetilde{p} L^l p \quad \text{and} \quad L'^r = \widetilde{q} L^r q$$

show that β induces Euclidean motions in the spherical kinematic image

$\Longrightarrow$ $\text{isom}\mathbb{E}_3 = \text{SO}_3 \times \text{SO}_3$ [1,12]

spherical Euclidean kinematics: $x \in \text{im}\,\mathbb{H}$ [1,12,15]

Line Geometry - Klein Model

$L = X \vee Y$, $X = (x_0, x_1, x_2, x_3)$, $Y = (y_0, y_1, y_2, y_3)$

Plücker coordinates of L [16]:

determinants of the 2×2-matrices built by the i-th and j-th column of

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix}$$

with $(i, j) \in ((0, 1), (0, 2), (0, 3), (2, 3), (3, 1), (1, 2))$

$$(l_{01}, l_{02}, l_{03}; l_{23}, l_{31}, l_{12}) = (L_1, L_2, L_3; L_4, L_5, L_6)$$

"direction vector" $l = (L_1, L_2, L_3)$, momentum vector $\bar{l} = (L_4, L_5, L_6)$ satisfy

$$M_2^4 : \ \langle l, \bar{l} \rangle = L_1 L_4 + L_2 L_5 + L_3 L_6 = 0$$

M_2^4 Klein quadric, Plücker quadric, point model for the set of lines in $\mathbb{P}^3$

$L^l = l - \bar{l}$, $L^r = l + \bar{l}$ linear in $L_i \Longrightarrow$ linear line mapping [2,8,11,13]

Line Geometry - Klein Model

L^l, L^r ... unit vectors $\Longrightarrow$

$$\langle L^l, L^l \rangle = \langle L^r, L^r \rangle = \langle l, l \rangle \pm 2\langle l, \bar{l} \rangle + \langle \bar{l}, \bar{l} \rangle = \langle l, l \rangle + \langle \bar{l}, \bar{l} \rangle = 1$$

$\Longrightarrow$ line geometry in $\mathbb{E}^3$ is geometry on a 4-manifold

$$\langle l, \bar{l} \rangle = 0 \quad \text{and} \quad \langle l, l \rangle + \langle \bar{l}, \bar{l} \rangle = 1$$

admits a rational parametrization!

Congruences of lines

Congruences of lines are two-dimensional submanifolds of the Klein quadric. [7,10]

$x = x(u^1, u^2)$, $y = y(u^1, u^2)$ with $x \perp_e y$ defined over $D \subset \mathbb{R}^2$

Lines in a congruence C can be parametrized by

$$L(u^1, u^2; t) = x(u^1, u^2)\cos t + y(u^1, u^2)\sin t \quad t \in [0, \pi].$$

attach a simplex $\Sigma = (P_0, P_1, P_2, P_3)$ to the lines of C,

such that $L = P_0 \vee P_3$ and Σ is a polar simplex with regard to Ω [1,9,12]:

$$\langle P_i, P_j \rangle_e = \delta_{ij} \quad \Longrightarrow \quad \mathrm{d}P_i = \omega_i^s P_s$$

obviously: $\omega_j^i = -\omega_i^j$ and $\omega_i^j = \langle \mathrm{d}P_i, P_j \rangle_e$

integrability conditions: $\mathrm{d}\omega_i^j + \omega_i^s \wedge \omega_s^j = 0$

Congruences of lines

The left and right spherical kinematic images of Σ's edges $E_i = P_0 \vee P_i$ build an orthonormal frame

$$2\langle E_i^l, E_j^l \rangle_e = \widetilde{p}_0 p_i \widetilde{\widetilde{p}_0 p_j} + \widetilde{p}_0 p_j \widetilde{\widetilde{p}_0 p_i} = \widetilde{p}_0 \langle P_i, P_j \rangle_e p_0 = \delta_{ij}.$$

similar for $E_i^r \implies \mathrm{d}E_i^l = \rho_i^k E_k^l$ and $\mathrm{d}E_i^r = \tau_i^k E_k^r$

with $\rho_i^j = -\rho_j^i$, $\tau_i^j = -\tau_i^j$, and the integrability conditions

$$\mathrm{d}\rho_i^j + \rho_k^j \wedge \rho_i^k = 0 \quad \text{and} \quad \mathrm{d}\tau_i^j + \tau_k^j \wedge \tau_i^k = 0$$

Differential forms ω_i^j, ρ_i^j, τ_i^j are not independent (eg.):

$$\mathrm{d}E_1^l = d(\widetilde{p}_0 p_1) = \omega_0^2 \widetilde{p}_2 p_1 + \omega_0^3 \widetilde{p}_3 p_1 + \omega_1^2 \widetilde{p}_0 p_2 + \omega_1^3 \widetilde{p}_0 p_3 = \rho_1^2 E_2^l + \rho_1^3 E_3^l$$

with $E_1^l = \widetilde{p}_0 p_1$, $E_i^\star = P_j \vee P_k$, $E_i^{\star l} = \widetilde{p}_k p_j$, $E_i^{\star r} = p_j \widetilde{p}_k$ (i, j, k cyclically ordered)

$$-\omega_0^3 E_3^l - \omega_0^3 E_2^l + \omega_1^2 E_2^l + \omega_1^3 E_3^l = \rho_1^2 E_2^l + \rho_1^3 E_3^l$$

$E_i \to \pi_\Omega(E_i) \implies E_i^l \mapsto -E_i^l$, $E_i^r \mapsto E_i^r$ with $i \in (1, 2, 3)$ absolute polarity

Congruences of lines

dependencies of differential forms of left and right kinematical image

$$\rho_2^1 = \omega_2^1 + \omega_0^3, \quad \tau_2^1 = \omega_2^1 - \omega_0^3,$$
$$\rho_3^1 = \omega_1^3 + \omega_0^2, \quad \tau_3^1 = \omega_3^1 - \omega_0^2,$$
$$\rho_2^3 = \omega_3^2 + \omega_0^1, \quad \tau_2^3 = \omega_2^3 - \omega_0^1$$

simplify $\quad \mathrm{d}E_3^l = \rho_3^1 E_1^l + \rho_3^2 E_2^l$ and $\mathrm{d}E_3^r = \tau_3^1 E_1^r + \sigma_3^2 E_2^r$

by letting $\quad \rho := \rho_3^2 + i\rho_3^1$ and $\tau = \tau_3^2 + i\tau_3^1$ with $i^2 = -1$

The mapping $\lambda' : E_3^l \to E_3^r$ is conformal if ρ and τ are $\mathbb{R}$-linear dependent, *i.e.*,

$$\rho \wedge \tau = (\rho_3^2 + i\rho_3^1) \wedge (\tau_3^2 + i\tau_3^1) = \rho_3^2 \wedge \tau_3^2 - \rho_3^1 \wedge \tau_3^1 + i(\rho_3^1 \wedge \tau_3^2 + \rho_3^2 \wedge \tau_3^1) = 0.$$

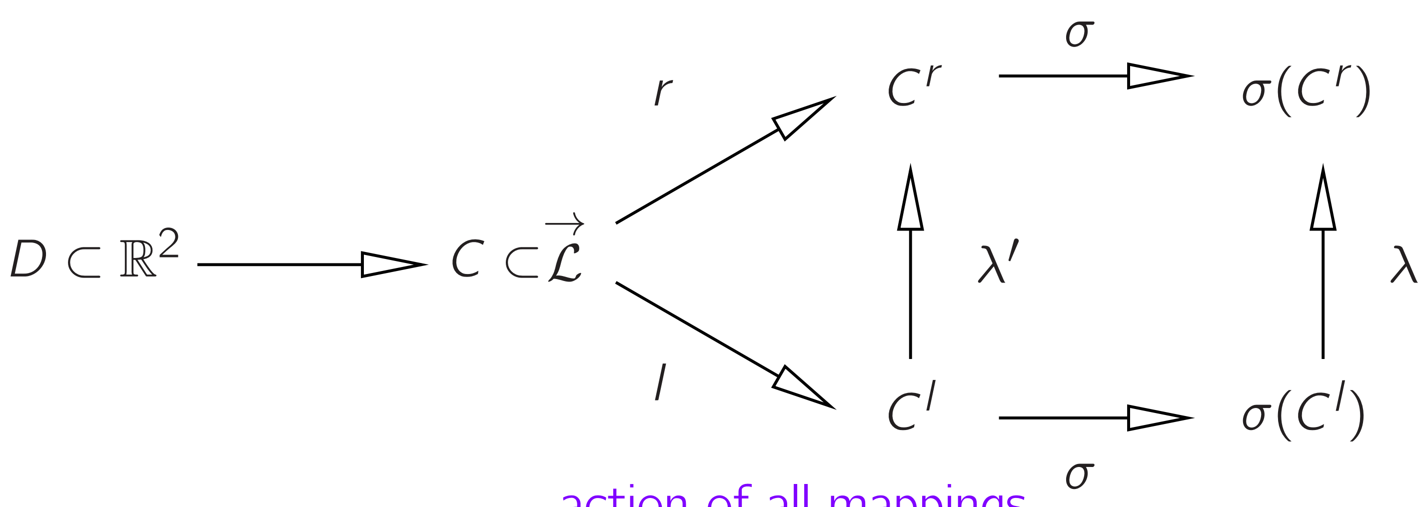
A congruence C is called isotropic if $\rho \wedge \tau \equiv 0$ in D. [1,9,12]

Parametrizations [14]

$C: \ D \to \overrightarrow{\mathcal{L}}$ ...parametrization of C, we are looking for a mapping $\lambda': \ C^l \to C^r$

$\sigma: \ S^2 \setminus S \to \mathbb{R}^2$ ...stereographic projection from the south pole $S = (0, 0, -1)$

$u = u^1 + iu^2 \ / \ \lambda = \lambda^1 + i\lambda^2$ ...coordinates in $\sigma(C^l)(D) \ / \ \sigma(C^r)(D)$



action of all mappings

Parametrizations [14]

apply stereographic projection to left and right image

$$L^l = (L_1 - L_4, L_2 - L_5, L_3 - L_6), \quad L^r = (L_1 + L_4, L_2 + L_5, L_3 + L_6)$$

$$\begin{aligned} \sigma(L^l) &= \frac{L_1 - L_4}{1 + L_3 - L_6} + i\frac{L_2 - L_5}{1 + L_3 - L_6} &= u^1 + iu^2 &= u, \\ \sigma(L^r) &= \frac{L_1 + L_4}{1 + L_3 + L_6} + i\frac{L_2 + L_5}{1 + L_3 + L_6} &= \lambda^1 + i\lambda^2 &= \lambda \end{aligned}$$

solve for L_1, L_2, L_4, L_5

$$L_1 = \frac{u^1}{2}(1+L_3-L_6) + \frac{\lambda^1}{2}(1+L_3+L_6), \quad L_2 = \frac{u^2}{2}(1+L_3-L_6) + \frac{\lambda^2}{2}(1+L_3+L_6),$$

$$L_4 = \frac{\lambda^1}{2}(1+L_3+L_6) - \frac{u^1}{2}(1+L_3-L_6), \quad L_5 = \frac{\lambda^2}{2}(1+L_3+L_6) - \frac{u^2}{2}(1+L_3-L_6)$$

Parametrization [14]

eliminate L_3 and L_6 with $\langle l, l\rangle + \langle \bar{l}, \bar{l}\rangle = 1$ and $\langle l, \bar{l}\rangle = 0$

$$\begin{aligned} \lambda\bar{\lambda}(1+L_3+L_6)^2 &+ u\bar{u}(1+L_3-L_6)^2 &=& \; 2(1-L_3^2-L_6^2), \\ \lambda\bar{\lambda}(1+L_3+L_6)^2 &- u\bar{u}(1+L_3-L_6)^2 &=& \; -4L_3L_6 \end{aligned}$$

yields

$$\lambda\bar{\lambda} = \frac{1-L_3-L_6}{1+L_3+L_6}, \qquad u\bar{u} = \frac{1-L_3+L_6}{1+L_3+L_6}$$

and finally

$$L_3 = \frac{1-u\bar{u}\lambda\bar{\lambda}}{(1+\lambda\bar{\lambda})(1+u\bar{u})}, \qquad L_6 = \frac{u\bar{u}-\lambda\bar{\lambda}}{(1+\lambda\bar{\lambda})(1+u\bar{u})}$$

Parametrization [14]

the "rational" parametrization of C generated by $\lambda' : C^l \to C^r$

$$L(u^1, u^2) = \begin{bmatrix} \dfrac{u^1}{1+u\overline{u}} + \dfrac{\lambda^1}{1+\lambda\overline{\lambda}} \\ \dfrac{u^2}{1+u\overline{u}} + \dfrac{\lambda^2}{1+\lambda\overline{\lambda}} \\ \dfrac{1-u\overline{u}\lambda\overline{\lambda}}{(1+u\overline{u})(1+\lambda\overline{\lambda})} \\ \dfrac{\lambda^1}{1+\lambda\overline{\lambda}} - \dfrac{u^1}{1+u\overline{u}} \\ \dfrac{\lambda^2}{1+\lambda\overline{\lambda}} - \dfrac{u^2}{1+u\overline{u}} \\ \dfrac{u\overline{u}-\lambda\overline{\lambda}}{(1+u\overline{u})(1+\lambda\overline{\lambda})} \end{bmatrix}$$

(truly) rational parametrizations: insert a rational function λ

rational parametrization of the manifold of lines in $\mathbb{E}^3$

(coordinates u^1, u^2, $u^3 = \lambda^1$, $u^4 = \lambda^2$)

Isotropic and minimal congruences in $\mathbb{E}^3$ [14]

Theorem: The congruences of lines in $\mathbb{E}^3$ generated by holomorphic functions $\lambda: \; D \subset \mathbb{C} \to \mathbb{C}$ are isotropic congruences. [14]

Proof: Compute $\rho \wedge \tau$ and find

$$\rho_3^2 = -\sqrt{\Phi}\mathrm{d}u^1, \;\; \rho_3^2 = \sqrt{\Phi}\mathrm{d}u^2 \quad \text{and} \quad \tau_3^2 = -\sqrt{\Psi}\mathrm{d}\lambda^1, \;\; \tau_3^1 = \sqrt{\Psi}\mathrm{d}\lambda^2$$

with $\Phi = 4/(1+u\overline{u})^{-2}$ and $\Psi = 4/(1+\lambda\overline{\lambda})^{-2}$ yields

$$\rho \wedge \tau = \sqrt{\Phi\Psi}(\lambda^2_{,1} + \lambda^1_{,2} + i(\lambda^1_{,1} - \lambda^2_{,2}))\mathrm{d}u^1 \wedge \mathrm{d}u^2$$

Real and imaginary part are the Cauchy-Riemann differential equations and $\rho \wedge \tau = 0$ is equivalent to λ is holomorphic.

Minimal congruences [14]

$\mathbb{C} \cong \mathbb{R}^2$ is a Riemannian manifold with metric $g = \mathrm{diag}(1, 1)$.

$\overset{[5]}{\Longrightarrow}$ energy density of λ

$$e(\lambda) = \frac{1}{2}\Big((\lambda^1_{,1})^2 + (\lambda^1_{,2})^2 + (\lambda^2_{,1})^2 + (\lambda^2_{,2})^2\Big)$$

$\overset{[5]}{\Longrightarrow}$ energy of λ

$$E(\lambda) = \int_D e(\lambda) \star 1$$

with $\star 1$ being the Riemannian volume element on $\mathbb{R}^2$

$$e(C) = e(\lambda), \quad E(C) = E(\lambda) \quad [14]$$

$\lambda : \ D \subset \mathbb{C} \to \mathbb{C}$ is harmonic $\Longleftrightarrow$ λ satisfies the Laplace equation

$$\Delta\lambda^i = \lambda^i_{,11} + \lambda^i_{,22} = 0$$

and minimizes E

Minimal congruences [14]

Definition: A congruence C of lines in $\mathbb{E}^3$ is called minimal if the generating function λ is harmonic.

Theorem: A congruence C of lines in $\mathbb{E}^3$ is minimal if the generating function λ is holomorphic or anti-holomorphic.

Theorem: A minimal congruence C in $\mathbb{E}^3$ is isotropic if the generating function λ is holomorphic.

The generation of minimal congruences is invariant under elliptic motions.

In analogy to isotropic congruences in $\mathbb{R}^3$:
Any holomorphic function defines an isotropic congruence.

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Thank You For Your Attention!