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Equioptic Points of a Triangle

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outline

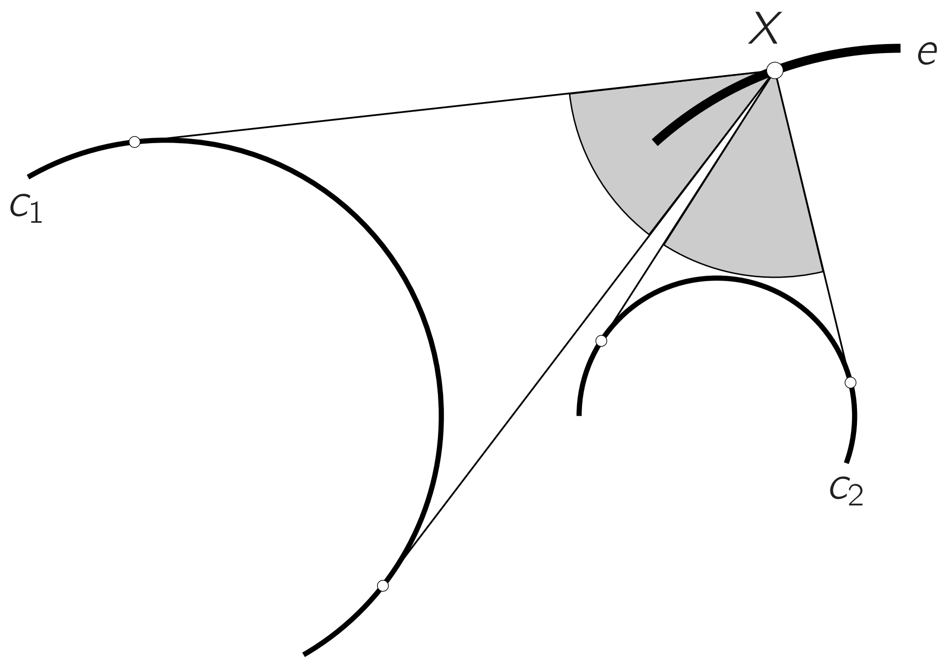
- ◇ equioptic point / curve
- ◇ equioptic circle
- ◇ triangle
- ◇ equioptic points of a triangle
- ◇ special cases
- ◇ the common radical axis and some related triangle centers
- ◇ more equioptic points

equioptic curve

X is a point on the equioptic curve e of two curves c_1 and c_2

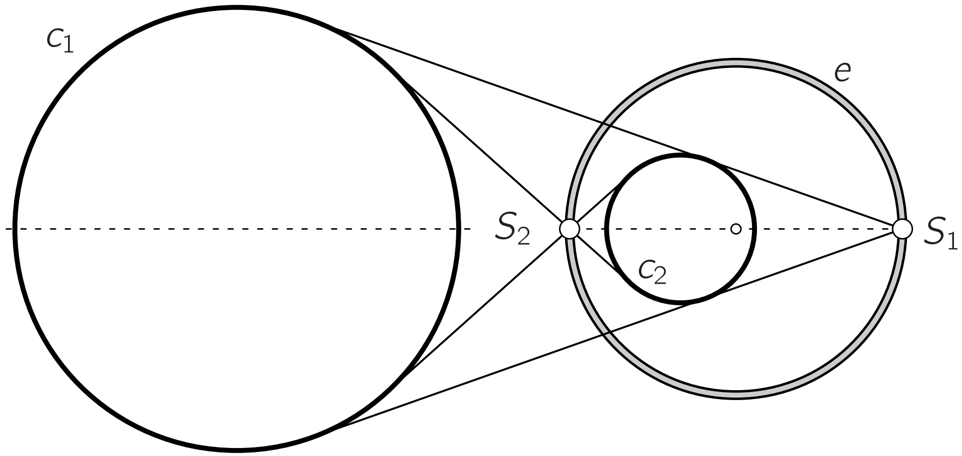
$\Longleftrightarrow$

c_1 and c_2 are seen under equal angles from X



equioptic circle

The **equioptic curve of two circles** ist the **Thales circle** of the segment bounded by the internal and external center of similitude, cf. [2].



triangle - notations

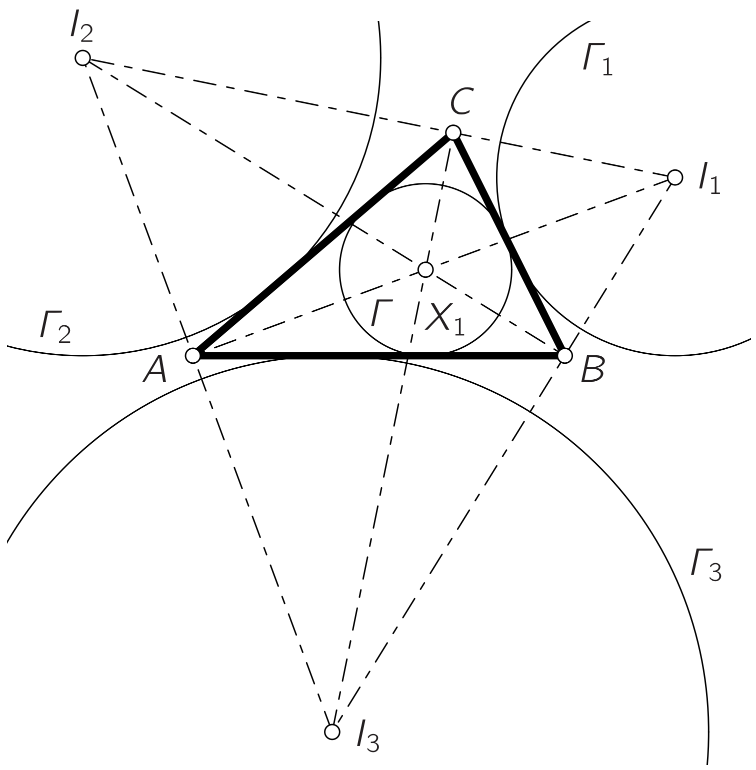
triangle $\Delta = (A, B, C)$

incenter X_1, excenters I_j

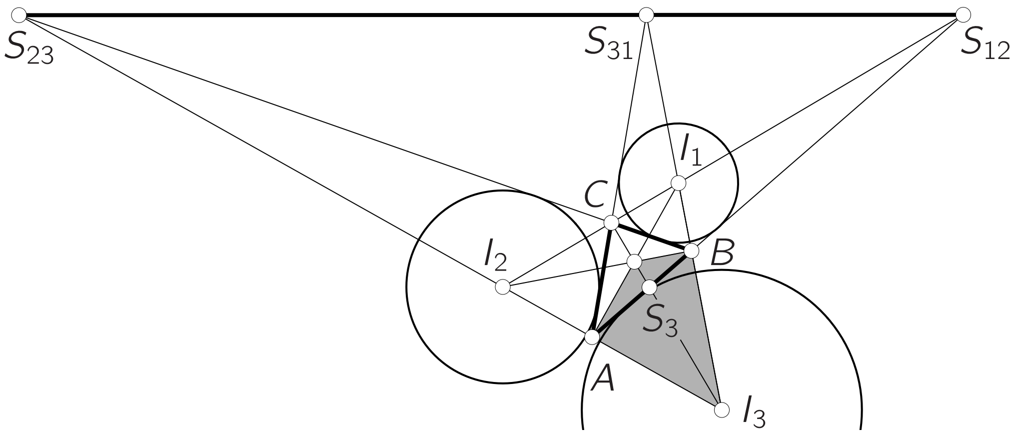
incircle Γ, excircles Γ_j

vertices of $\Delta =$

internal centers of similitude of excircles



external centers of similitude



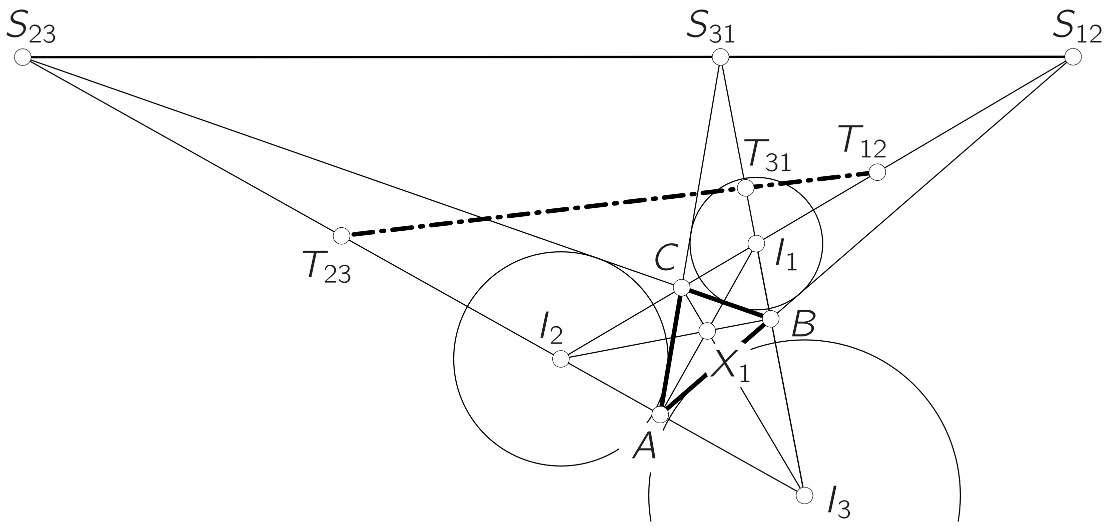
Theorem.
The external centers of similitude of the excircles are collinear.

Proof.
Obviously S_{12}=$[A, B] \cap [I_1, I_2]$.

We see that (I_2, I_1, C, S_{12}) and (A, B, S_3, S_{12}) form harmonic quadrupels.

Cyclically reordering $\Longleftrightarrow$ $S_{31} \in [S_{12}, S_{23}]$ and $[S_{12}, S_{23}]$ is the polar of X_1 with respect to Δ and its excentral triangle. $\square$

centers of equioptic circles

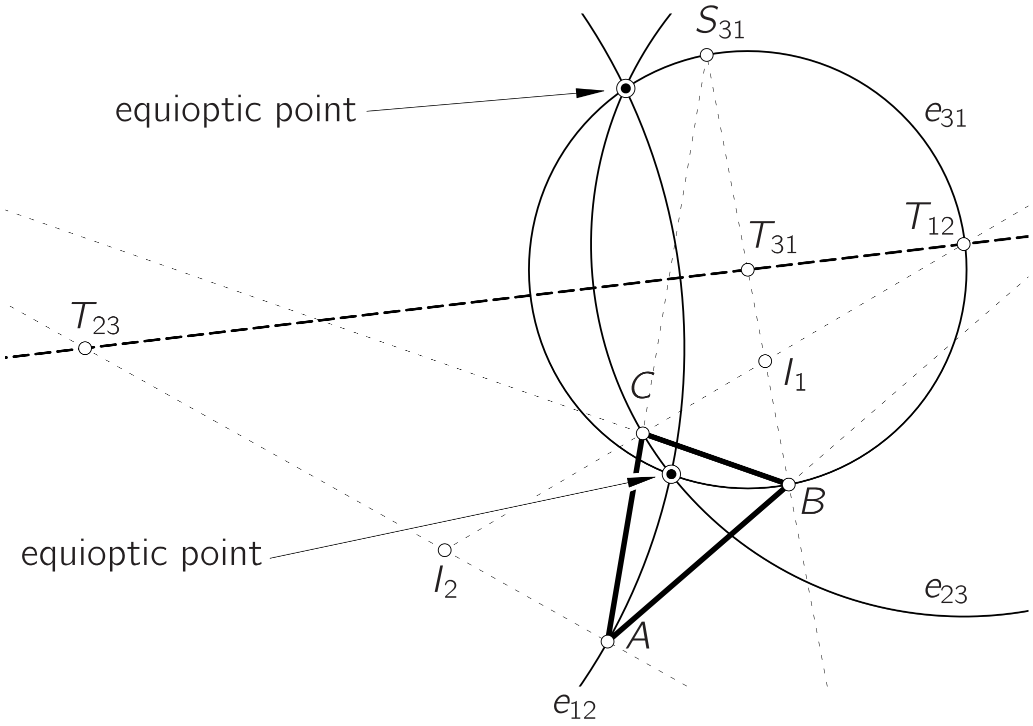


Theorem.
The centers T_{ij} of the equioptic circles are collinear.

Proof.
T_{12} = midpoint of C and S_{12}.
Use trilinear coordinates in order to compute T_{ij} and show linear dependencies. $\square$

equioptic points

Observation.
The three equioptic circles of the excircles have up to two (real) common points.

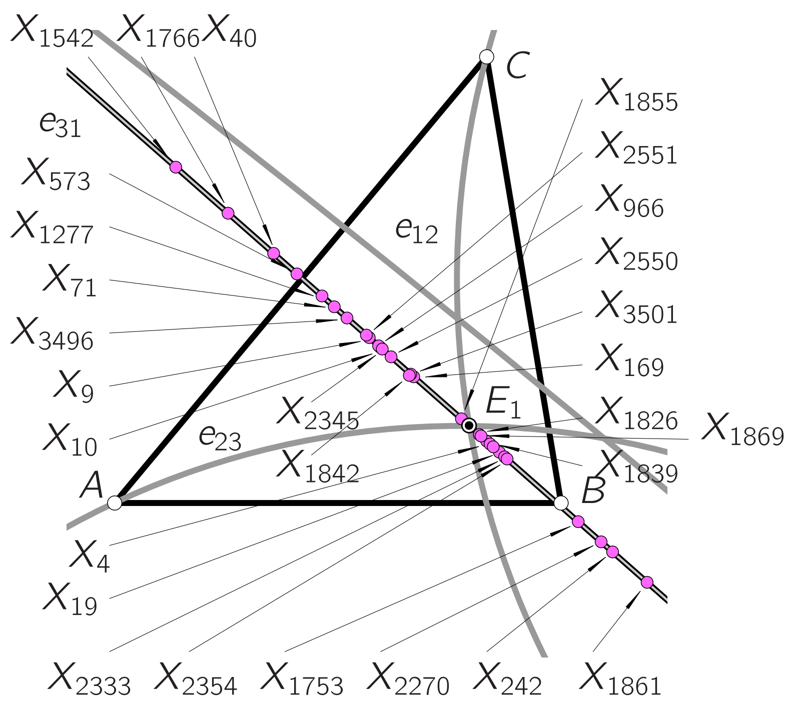


some centers

Theorem.
The three equioptic circles e_{ij} of Δ's excircles Γ_i and Γ_j belong to a pencil of circles and thus there are up to two (real) equioptic points.

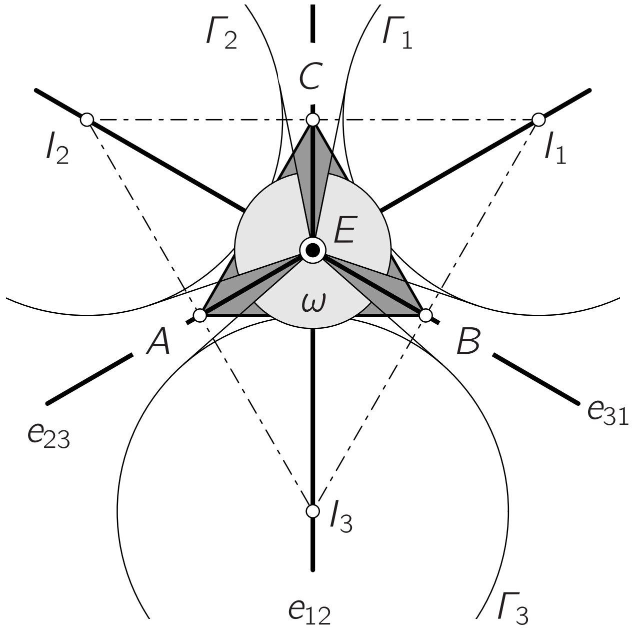
Theorem.
The common radical axis carries Δ's centers with Kimberling number, cf. [1]:

4, 9, 10, 19, 40, 71, 169, 242, 281, 516, 573, 966, 1276, 1277, 1512, 1542, 1544, 1753, 1766, 1826, 1839, 1842, 1855, 1861, 1869, 1890, 2183, 2270, 2333, 2345, 2354, 2550, 2551, 3496, 3501.



two special cases – unique equioptic points

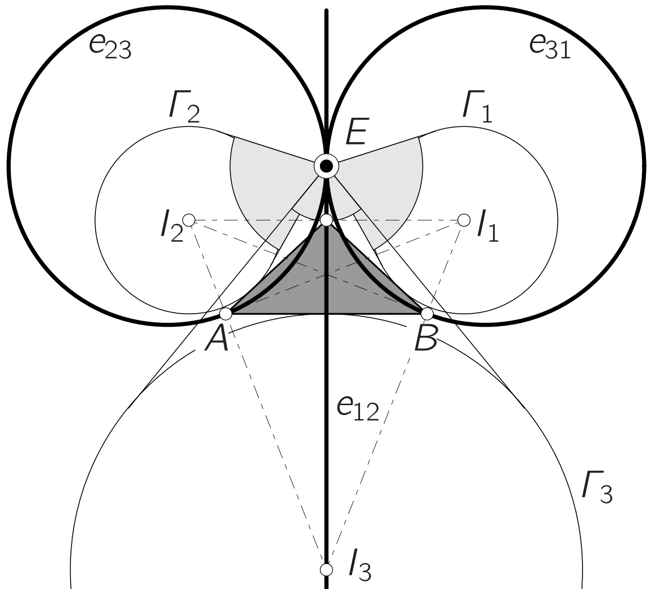
equilateral triangle



$$E = X_1$$

$$\omega = \arccos\left(-\tfrac{1}{8}\right) \approx 97.181^\circ$$

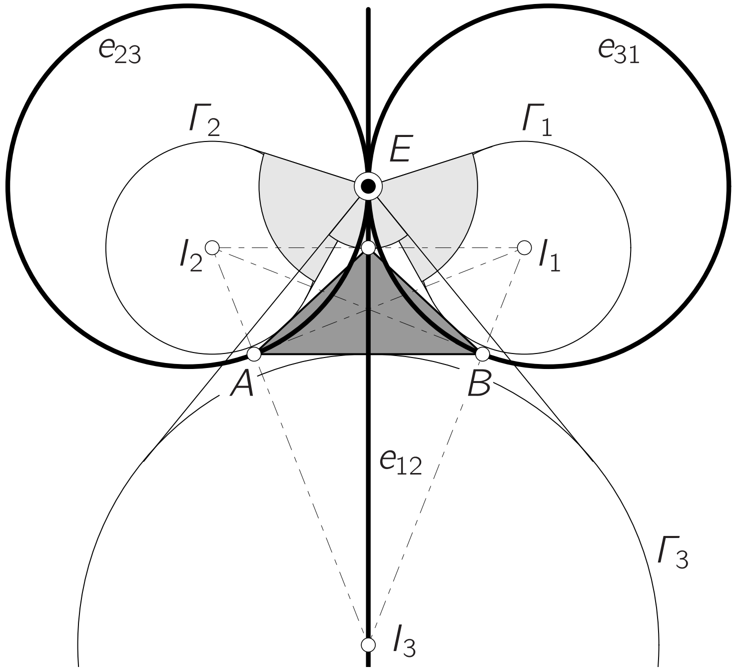
isosceles triangle



$$\angle ACB = 2\arcsin(\sqrt{3}-1) \approx 94.117^\circ$$

$$\omega = 2\arcsin\left(\frac{3-\sqrt{3}}{2}\right) \approx 76.688^\circ$$

a unique equioptic point



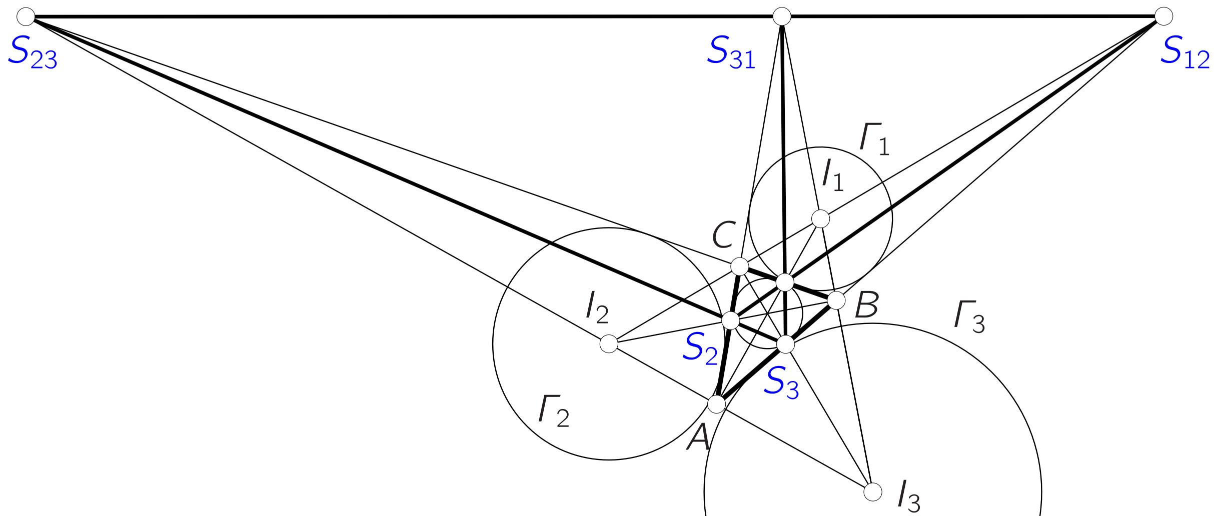
A **generic triangle** has a unique equioptic point if and only if

$$6a^2b^2c^2 + 4 \sum_{cyclic} (a^3 b(b^2 - \hat{a}c)) =$$

$$= \sum_{cyclic} (a^6 + 2\hat{a}a^5 - a^4(\hat{a}^2 + 4bc)).$$

Right angled triangles cannot have unique equioptic points.

more circles - more centers of similitude

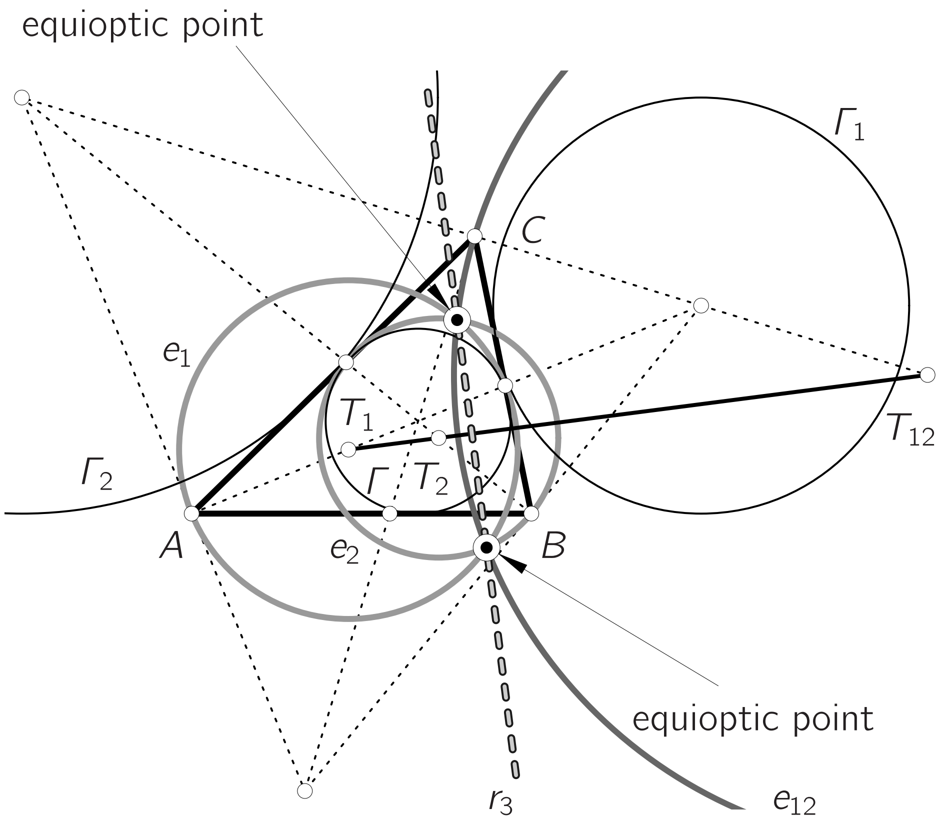


S_i = internal center of similitude of Γ_i and Γ

Δ's i-th vertex = external center of similitude of Γ and Γ_i

S_i, S_j, and S_{ij} are collinear

triplets of incircle and two excircles

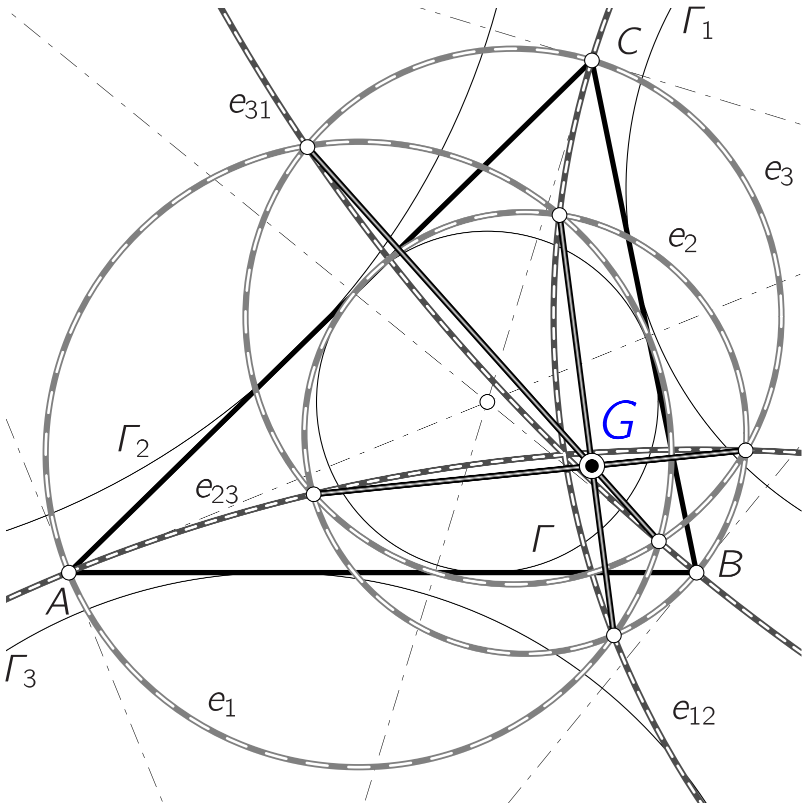


Theorem.
The centers T_i, T_j, T_{ij} of the three equioptic circles e_i, e_j, e_{ij} of any triplet $(\Gamma_i, \Gamma_j, \Gamma)$ are collinear.

Theorem.
Any triplet (e_i, e_j, e_{ij}) of equioptic circles has a common radical axis r_k.

Theorem.
Any triplet $(\Gamma_i, \Gamma_j, \Gamma)$ has up to two equioptic points.

all together



Theorem.
The three radical axes r_k are concurrent in a new triangle center G.

references

[1] C. Kimberling: *Encyclopedia of triangle centers.* Available at:
`http://www.faculty.evansville.edu/ck6/encyclopedia/ECT.hml`

[2] B. Odehnal: *Equioptic curves of conic sections.* J. Geom. Graphics **14**/1 (2010), 29–43.

Thank You for Your Attention!