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Generalized Conchoids

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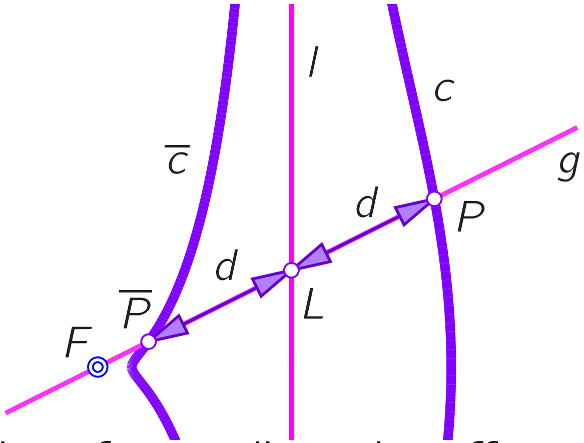
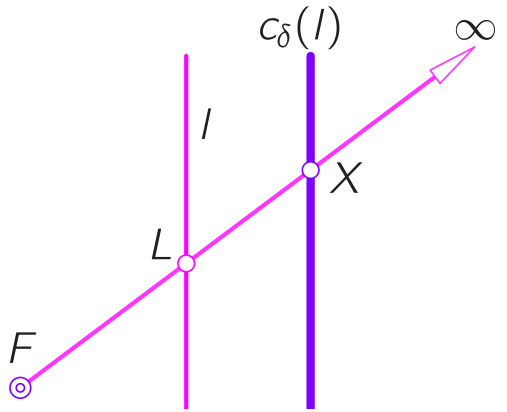
the next 15+ε minutes ($\varepsilon \gg 0$)

conchoids	|	modification of the classical definition
basic properties	|	of the conchoid transform within quadrics
line geometric version	|	linear and quadratic transformations
sphere geometric version	|	linear: three basic types
further ideas	|	Study's quadric, Möbius geometry

conchoids in the plane - definition





F, l, d ... focus, directrix, offset

c, $\overline{c}$ ... two branches of the conchoid

The conchoid is the set of points at distance d from l measured on lines through F.

F, l, δ ... focus, directrix, cross ratio

$c_\delta(l)$ ... δ-conchoid of l

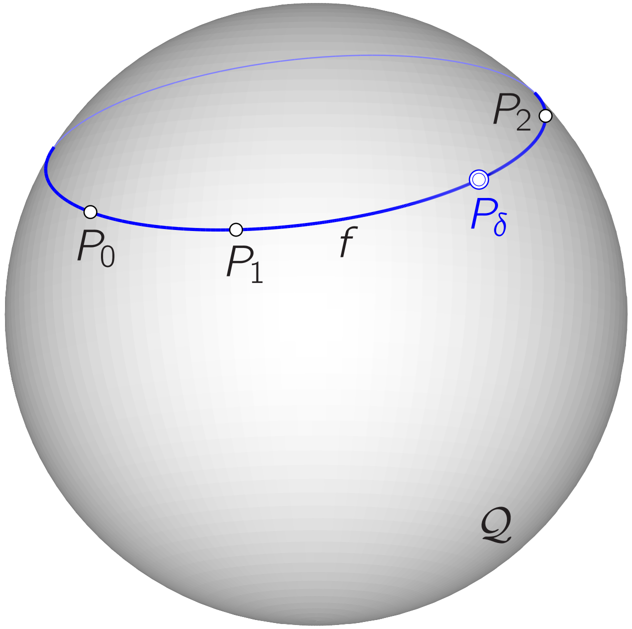
The δ-conchoid transform X of $L \in l$ is determined such that $\mathrm{cr}(F, \infty, L, X) = \delta$.

Does not look spectacular, but allows a generalization to other geometries and preserves the type of the geometric object to be transformed.

conchoids on quadrics

More general than before:

Definition: Let P_i ($i \in \{0, 1, 2\}$ be three non-collinear points on a quadric $\mathcal{Q}$ s. t. $\pi := [P_0, P_1, P_2] \not\subset \mathcal{Q}$ and π be not tangential to $\mathcal{Q}$. Then, the uniquely defined point $P_\delta \in \pi \cap \mathcal{Q}$ with $\mathrm{cr}(P_0, P_1, P_2, P_\delta) = \delta$ is called the δ-**conchoid transform** of either P_i w. r. t. any pair (P_j, P_k) $(i \neq j, k; j \neq k)$ and P_j and P_k are called the **foci** of the conchoid transform.



If we allow any permutation of (i, j, k), then **the** cross ratio is one of

$$\delta, \frac{1}{\delta}, 1-\delta, \frac{1}{1-\delta}, \frac{\delta-1}{\delta}, \frac{\delta}{\delta-1}.$$

In the following, we keep the ordering P_0, P_1, P_2; and P_δ be the conchoid transform of P_2 w. r. t. P_0 and P_1.

conchoid transformation

Theorem: Let $\Omega : \mathbb{F}^{n+1} \times \mathbb{F}^{n+1} \to \mathbb{F}$ be a symmetric bilinear form on the vector space $\mathbb{F}^{n+1}$ ($\mathbb{F}$ be a commutative field with $\operatorname{char} \mathbb{F} \neq 2$) and $\mathcal{Q} : \Omega(\mathbf{x}, \mathbf{x}) = 0$ be the equation of a quadric $\mathcal{Q} \subset \mathbb{P}^n(\mathbb{F})$ and let further $\delta \in \mathbb{F} \cup \{\infty\}$. Then, the intrinsic conchoid transform of P_2 within $\mathcal{Q}$ w.r.t. P_0, P_1 reads

$$\mathbf{p}_\delta = \delta(\delta - 1)\Omega_{12}\mathbf{p}_0 + (1 - \delta)\Omega_{02}\mathbf{p}_1 + \delta\Omega_{01}\mathbf{p}_2. \qquad (\star)$$

$\mathbf{p}_i$ ... coordinate vector of P_i, $\qquad \Omega_{ij} := \Omega(\mathbf{p}_i, \mathbf{p}_j)$

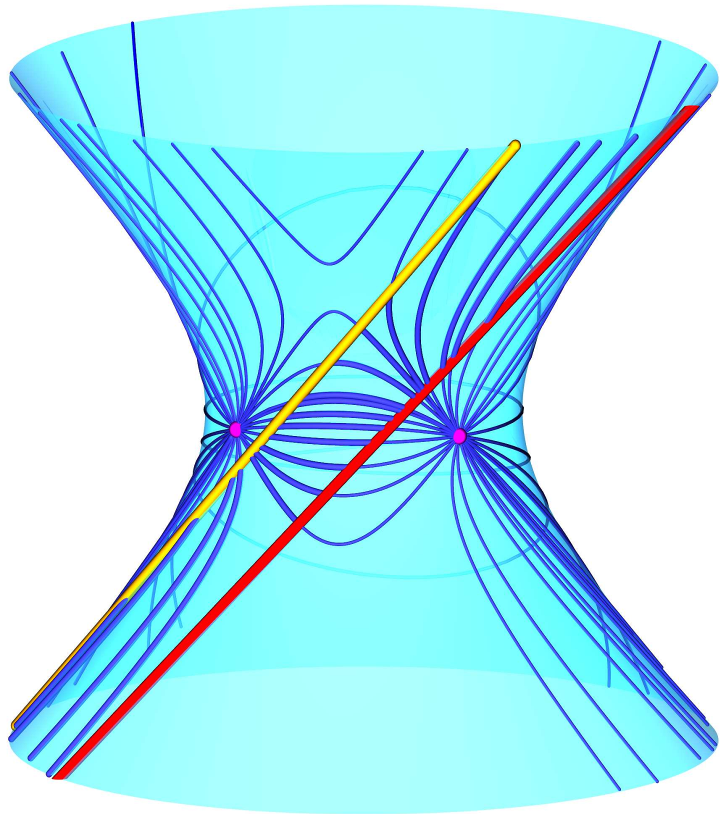
Proof: Project the line $[P_1, P_2]$ from P_0 onto $\mathcal{Q}$, observe that $(\star)$ returns P_1, P_2, P_0 for $\delta = 0, 1, \infty$, and show that $P_\delta \in \mathcal{Q}$ by showing $\Omega(\mathbf{p}_\delta, \mathbf{p}_\delta) = 0$ for all $\delta \in \mathbb{F} \cup \{\infty\}$.

The cross ratio can be replaced by a homogeneous coordinate $d_0 : d_1 \neq 0 : 0$ on the conic $[P_0, P_1, P_2] \cap \mathcal{Q}$ via $\delta = d_1 d_0^{-1}$ ($d_0 \neq 0$, otherwise $\delta = \infty$ and $P_\delta = P_0$).

conchoid transformation

Theorem: For any two fixed $P_i \neq P_j$ the conchoid transform P_δ of P_k in any regular quadric $\mathcal{Q}: \ \Omega(\mathbf{x}, \mathbf{x}) = 0$ can be extended to an automorphic collineation of $\mathcal{Q}$.

Proof: Assume that P_0, P_1 are the (fixed) foci of the conchoid transform $(\star)$ and assume $P_2 = \mathbf{x}$. Then, Ω_{02} and Ω_{12} are linear in $\mathbf{x}$ and neither is a scalar factor of $P_2 = \mathbf{x}$ in $(\star)$.



The generalized conchoid transform is involutive if, and only if, $\delta = -1$.

conchoid transformation - line geometry

The line geometric conchoid transform takes place within Plücker's quadric

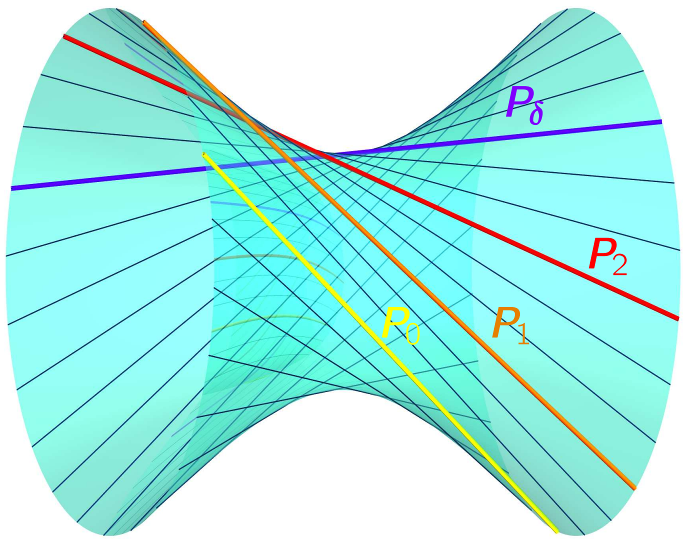
$$M_2^4 : \ \langle \mathbf{l}, \bar{\mathbf{l}} \rangle = l_1 l_4 + l_2 l_5 + l_3 l_6 = 0.$$

$\mathbf{l} = (l_1, l_2, l_3)$, $\bar{\mathbf{l}} = (l_4, l_5, l_6)$ ... direction, momentum of a line L,
$(l_1, \dots, l_6)$ ... Plücker coordinates of L

The line geom. conchoid transform preserves ruled surfaces, congruences, and even complexes of lines.

The fibres of the line geom. conchoid transform correspond to the conics in M_2^4, and are, thus, reguli:

that family of rulings on the "fibre quadric" that contains P_0, P_1, and P_2.



conchoid transformation - line geometry

Two types of line geometric conchoid transforms:

(1) $P_0 \cap P_1 = \emptyset \iff \Omega_{01} \neq 0$ and **(2)** $P_0 \cap P_1 = S \iff \Omega_{01} = 0$

in the followig: $B_i = i^{\text{th}}$ canonical base point

(1) $P_0 = B_1,\ P_1 = B_4,\ P_2 = L = (l_1, \ldots, l_6)$

$L_\delta = (\delta^2 l_1, \delta l_2, \delta l_3, l_4, \delta l_5, \delta l_6) \ \ldots$ collineation

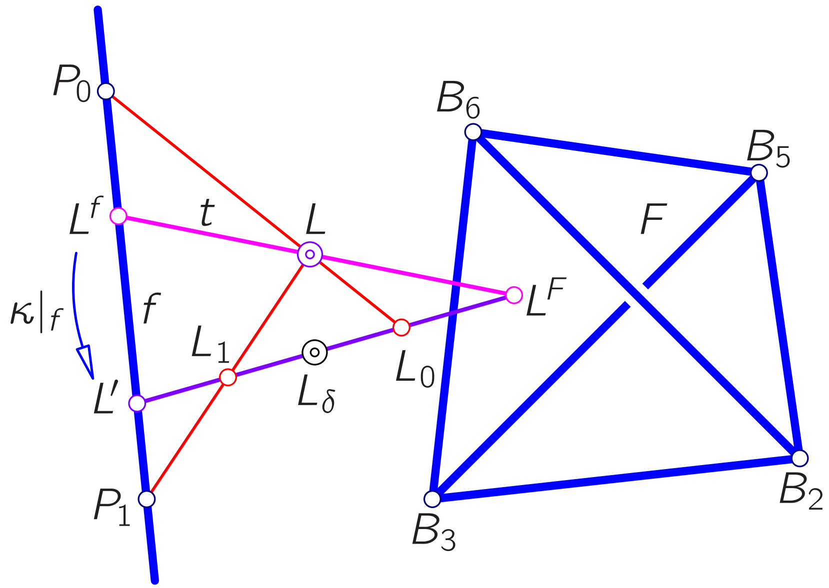
κ with matrix $M = \mathrm{diag}(\delta^2, \delta, \delta, 1, \delta, \delta)$

$f := [P_0, P_1] \ \ldots$ fixed, but not point wise

$\kappa|_f : L^f \to L',\ (\kappa|_f)_{\{P_0,P_1\}} = \mathrm{diag}(\delta^2, 1)$

$F := [B_2, B_3, B_5, B_6] \ \ldots$ fixed point wise

$t = [L^f, L^F] \ni L \ \ldots$ unique "Treffgerade"



$L_0, L_1 \ \ldots$ projections of L from P_0, P_1 to $[L', L^F]$

$\mathrm{cr}(L', L^F, L_0, L_\delta) = \delta^{-1}$ or $\mathrm{cr}(L', L^F, L_1, L_\delta) = \delta$.

(2) κ is just a projection onto the 1-dimensional subspace $[P_0, P_1]$.

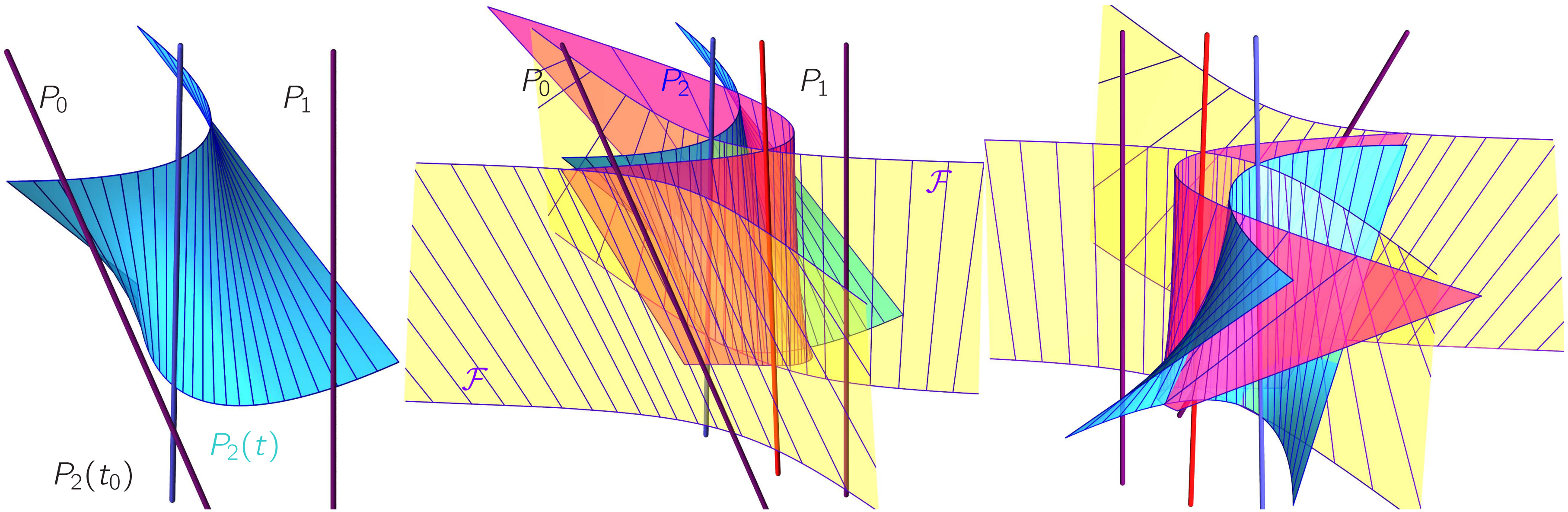
conchoid transformation - line geometry

Example 1:

focal lines: $P_0 = (0, -1, 2, 0, -2, -1)$, $P_1 = (0, 0, 1, 1, 0, 0)$, cross ratio: $\delta = -\frac{1}{2}$

ruled surface: $P_2(t) = (t^2 + t, t - t^2, 2, -t^2 - t, t^2 - t, t^4 + t^2)$ with $t \in \mathbb{R}$

fibre quadric $\mathcal{F}$: at $t_0 = \frac{1}{2}$



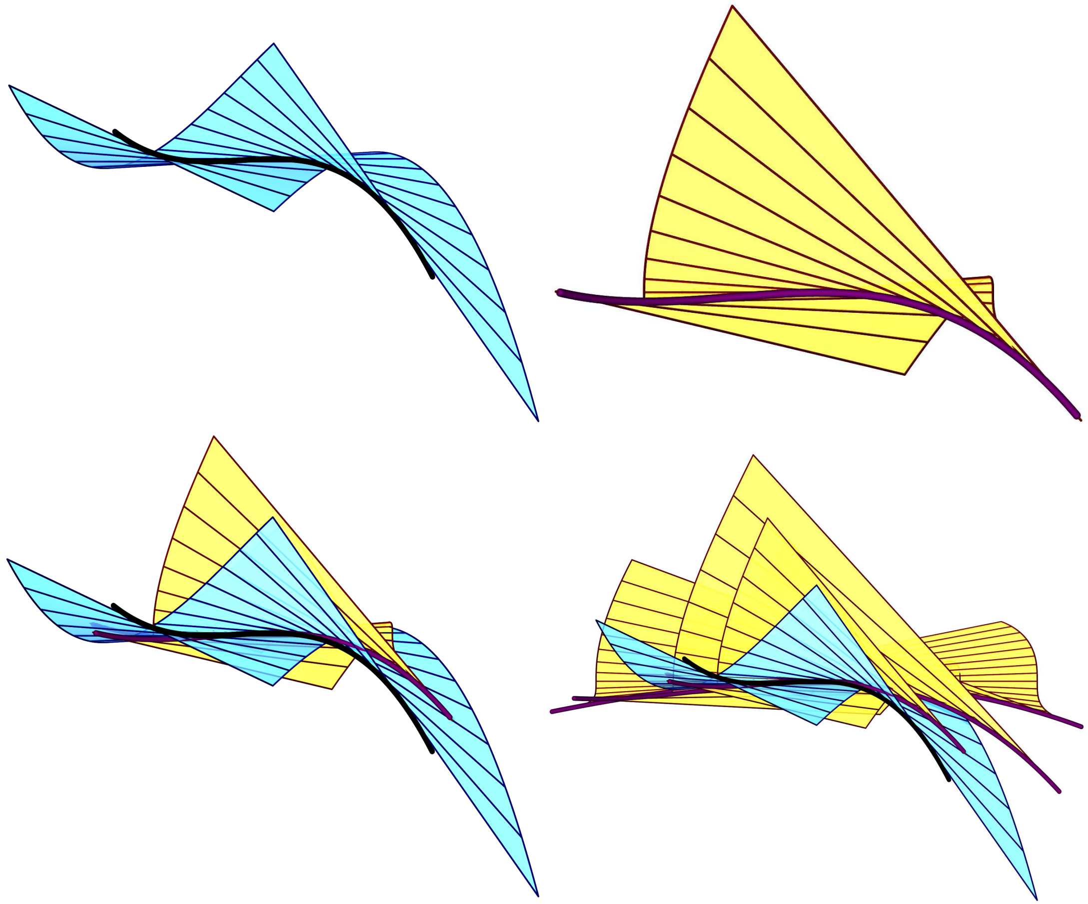
conchoid transformation - line geometry

Example 2:

Torsality is preserved under the line geometric conchoid transform.

The tangent developable of $\left(t, \frac{1}{2}t^2, \frac{1}{3}t^3\right)$ transforms via (1) to the tangent developable of $\left(\frac{1}{\delta}t, \frac{1}{2\delta}t^2, \frac{1}{3}t^3\right)$ for any admissible cross ratio $\delta \in \mathbb{R}^\star$.

Only x- and y-coordinates are scaled with δ^{-1}.



The relative position of P_0 and P_1 w. r. t. the ruled surface matters.

conchoid transformation - a quadratic transformation in line space

$P_0 = (0, 1, k, 0, -ek, e)$ with $(e, k \in \mathbb{R})$,
$P_1 = (l_1, \ldots, l_6)$, $P_2 = (0, 0, 0, l_1, l_2, l_3)$

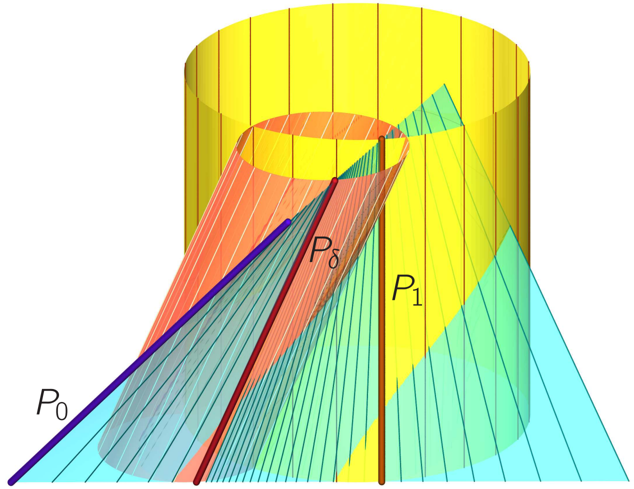
$l_i : \ I \subset \mathbb{R} \to \mathbb{R} \iff P_1(t) : \ I \to M_2^4$ is a ruled surface.

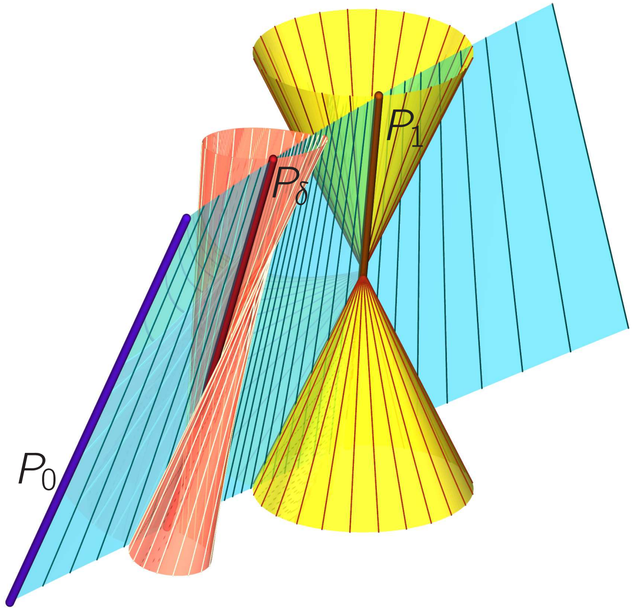
The focal line P_1 traverses a ruled surface and the line geometric conchoid transform is applied to P_1's **absolute polar** (line) (polar w. r. t. the absolute quadric).

$P_2 \mapsto P_\delta$ is obviously quadratic in the l_is.

Cylinders are mapped to cylinders. App. 1

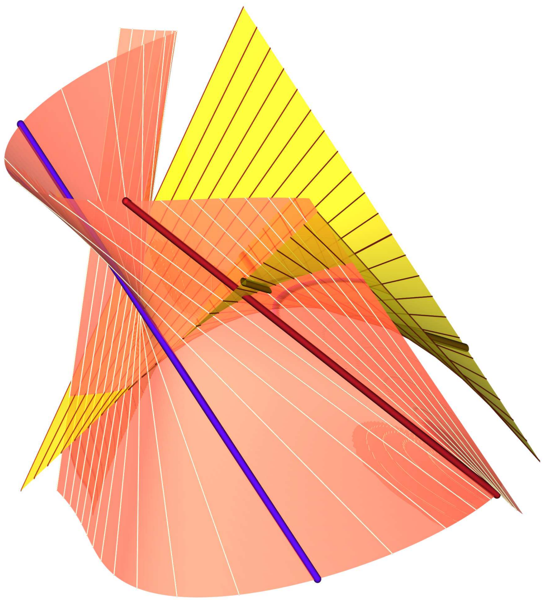
Cylinders of revolution are mapped to elliptic cylinders.





conchoid transformation - a quadratic transformation in line space

The two ruled surfaces on one ruled quadric are mapped to two different ruled surfaces.



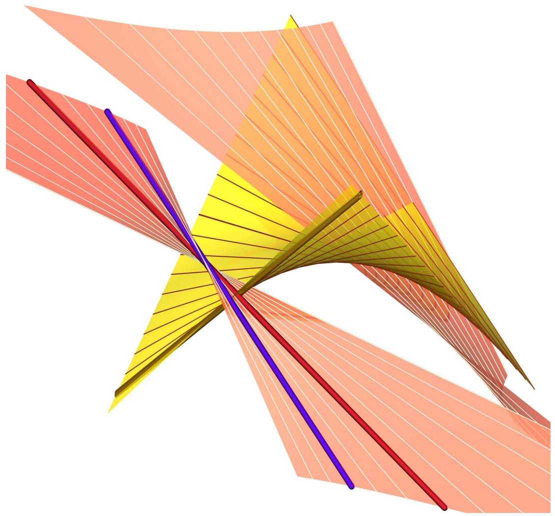
Example 3:

The hyperbolic paraboloid $xy = z$ carries

$\mathbf{x}_1(u, v) = (u, 0, 0) + w(0, 1, u)$,

$\mathbf{x}_2(u, v) = (0, u, 0) + w(1, 0, u)$.

With $P_0 = (0, 1, k, 0, -ek, e)$, $e, k \in \mathbb{R}$, they are mapped to two different cubic ruled surfaces.



Ruled quadrics are mapped to quartic ruled surfaces (in general).

conchoid transformation - sphere geometry

The sphere geometric conchoid transform is a conchoid transform within Lie's quadric

$$L_2^4: \ s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_5^2 - s_6^2 = 0.$$

$(s_1, \ldots, s_6) \ldots$ Lie coordinates of a sphere S: $\frac{1}{s_6 - s_4}(s_1, s_2, s_3) \ldots$ center, $\frac{s_5}{s_6 - s_4} \ldots$ radius of the sphere $(s_6 - s_4)(x^2 + y^2 + z^2) - 2(s_1 x + s_2 y + s_3 z) + (s_6 + s_4) = 0$

The sphere geometric conchoid transforms preserves channel surfaces, ...

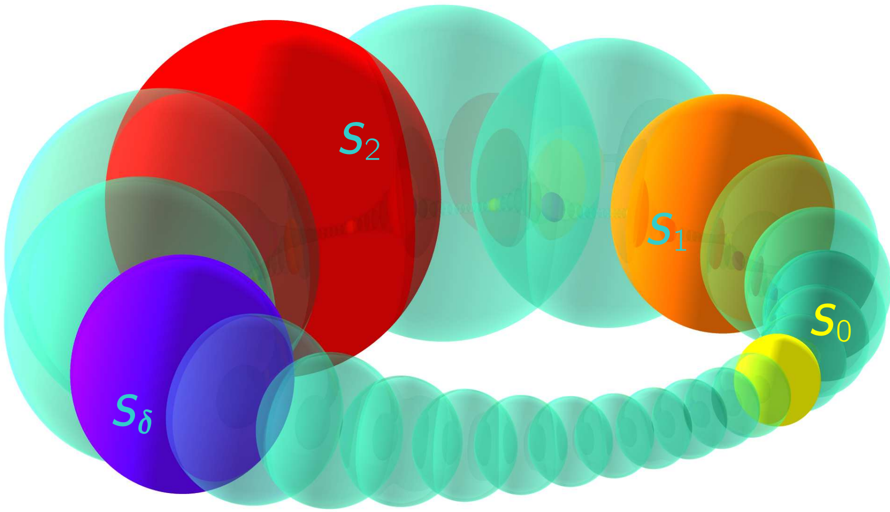
The fibres of the sphere geom. conchoid transform are one family of spheres that envelope a Dupin cyclide.

distinguished spheres

$s_6 - s_4 = 0 \ldots$ planes $(r = \infty)$,

$s_5 = 0 \ldots$ points $(r = 0$, isotropic cone$)$

$(0, 0, 0, 1, 0, 1) \ldots$ plane at infinity



conchoid transformation - sphere geometry

Some special spheres can serve as focal spheres of the linear sph. conchoid transform:

$\Gamma_0: \ x^2+y^2+z^2=0$	$(0,0,0,1,0,-1)$	isotropic cone with vertex $(0,0,0)$
$\omega: \ x_0=0$	$(0,0,0,1,0,1)$	plane at infinity = flat sphere
$\mathrm{S}_0^2: \ x^2+y^2+z^2=1$	$(0,0,0,1,1,0)$	Euclidean unit sphere

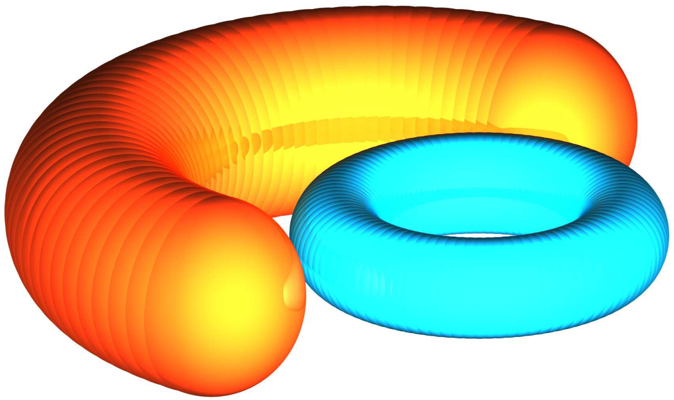
Choosing the focal pair (S_0, S_1) as (Γ_0, ω), $(\Gamma_0, \mathrm{S}_0^2)$, (ω, S_0^2) yields three special types of linear sphere conchoid transforms.

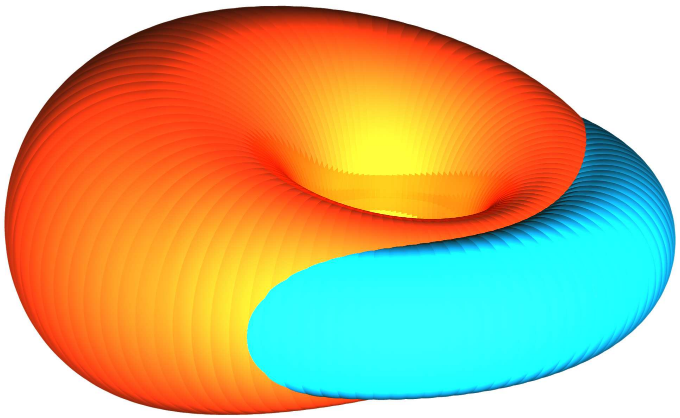
They all show up as automorphic collineations of L_2^4 with a fixed line f (not point wise) and a fixed three space F (pointwise) and induce sphere preserving contact transformations.

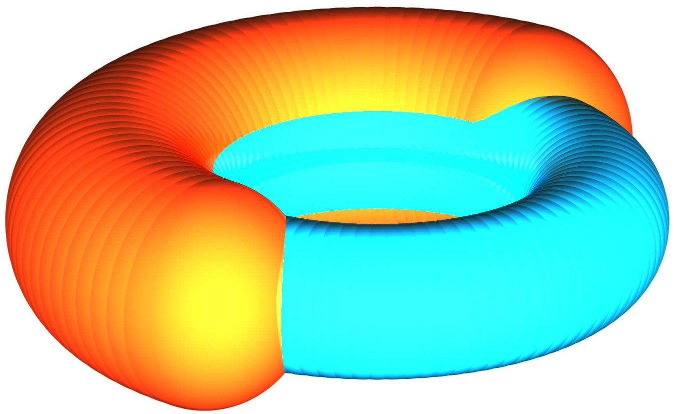
Their action is similar to that of the collineation (1) given on slide 6.

conchoid transformation - sphere geometry

S_0	S_1	**transformation**
Γ_0	ω	equiform transformation, scaling factor δ^{-1} $\rightarrow$
Γ_0	S_0^2	inversion $\downarrow$
ω	S_0^2	Laguerre transformation, $\mathbf{m} \mapsto \delta\mathbf{m}$, $r \mapsto \delta r + \delta - 1$ $\searrow$







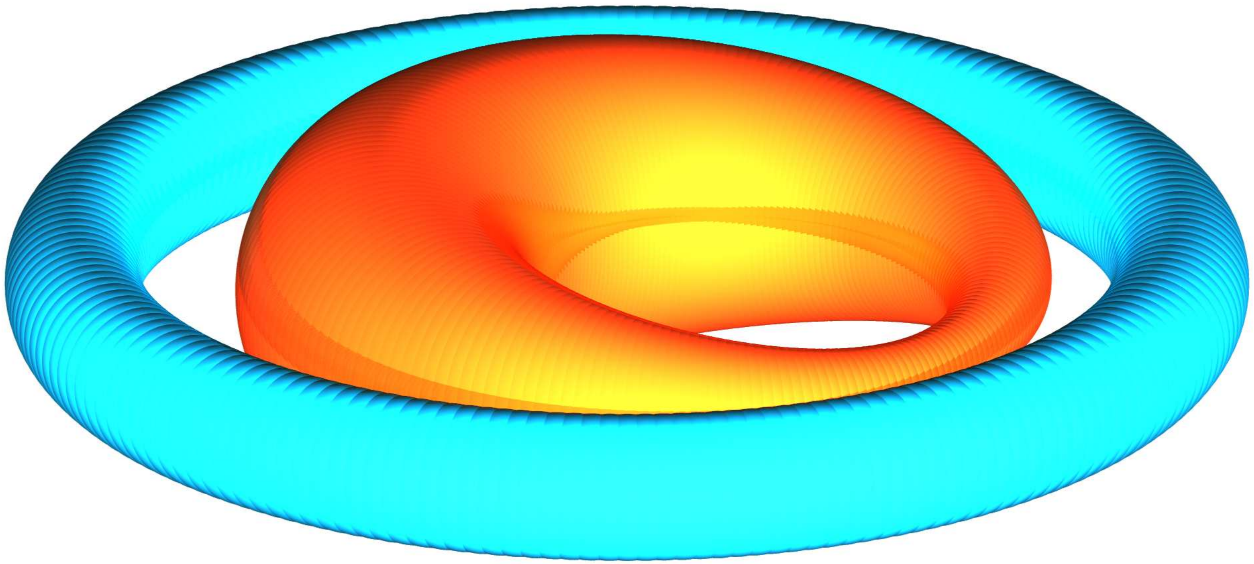
Note: Dupin cyclides are the sphere geometric analoga to ruled quadrics.

conchoid transformation - sphere geometry - some quadratic transformations

We replace the focal line S_1 by some sphere that is related to the sphere S_2 to be transformed. If $S_2 \mapsto S_1$ is linear in s_i, then $S_2 \mapsto S_\delta$ is quadratic in s_i.

S_0 ... one of Γ_0, ω, S_0^2

S_1 ... *e.g.* the polar plane of S_2 w. r. t. some point



finally . . .

Study's quadric is a model for the set of Euclidean motions in $\mathbb{R}^3$.

What is the conchoid transform of a Euclidean motion?

Möbius geometry in the plane or on the sphere:

Cross ratios of four complex numbers can characterize concyclic points.

A conchoid transform is straight forward.

Singular quadrics (like the Blaschke cylinder (cone)) . . .

More or less related work

[1] A. Albano, M. Roggero: *Conchoidal transform of two plane curves.* Applicable Algebra in Engineering, Communication and Computing **21**/4 (2010), 309–328.

[2] M. Hamann, B. Odehnal: *Conchoidal ruled surfaces.* Proc. 15^{th} Internat. Conf. Geometry & Graphics, Aug. 1–5, 2012, Montreal/Canada, article No. 089.

[3] M. Peternell, D. Gruber, J. Sendra: *Conchoid surfaces of rational ruled surfaces.* CAGD **28** (2011), 427–435.

[4] M. Peternell, D. Gruber, J. Sendra: *Conchoid surfaces of spheres.* CAGD **30** (2013), 35–44.

[5] M. Peternell, L. Gotthart, J. Sendra, J.R. Sendra: *Offsets, Conchoids and Pedal Surfaces.* J. Geom. **106** (2015), 321–339.

[6] D. Gruber, M. Peternell: *Conchoid surfaces of quadrics.* J. Symb. Computation **59** (2013), 36–53.

[7] J. Sendra, J.R. Sendra: *Rational parametrization of conchoids to algebraic curves.* Appl. Algebra Eng. Comm. Comp. **21**/4 (2010), 413–428.

Thank You For Your Patience!

Appendix 1

The quadratic conchoid transform maps cylinders to cylinders.

Proof: Let $P_0 = (0, 1, k, 0, -ek, e)$ with $e, k \in \mathbb{R}$ (const.) and $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6)$ be the Plücker representation of a cylinder, *i.e.*, $\mathbf{p}_1 = (v_1, v_2, v_3) \in \mathbb{R}^3$ s. t. $\|\mathbf{p}_1\| = 1$ and $\mathbf{p}_1 =$ const., further $\overline{\mathbf{p}}_1, \dot{\overline{\mathbf{p}}}_1 : I \subset \mathbb{R} \to \mathbb{R}^3$ are not constant. Then, $P_2 = (\mathbf{o}, \mathbf{p}_1) =$ const. and $\Omega_{12} = 1$. Since $\mathbf{p}_0 =$ const., $\Omega_{02} =$ const. We insert into (1) and compute only $\mathbf{p}_\delta = \delta(\delta - 1)\mathbf{p}_0 + (1 - \delta)\langle \mathbf{p}_0, \overline{\mathbf{p}_1} \rangle =$ const. which makes $P_\delta : I \subset \mathbb{R} \to M_2^4$ the Plücker representation of cylinder since $\overline{\mathbf{p}}_\delta : I \subset \mathbb{R} \to \mathbb{R}^3$ is not constant.