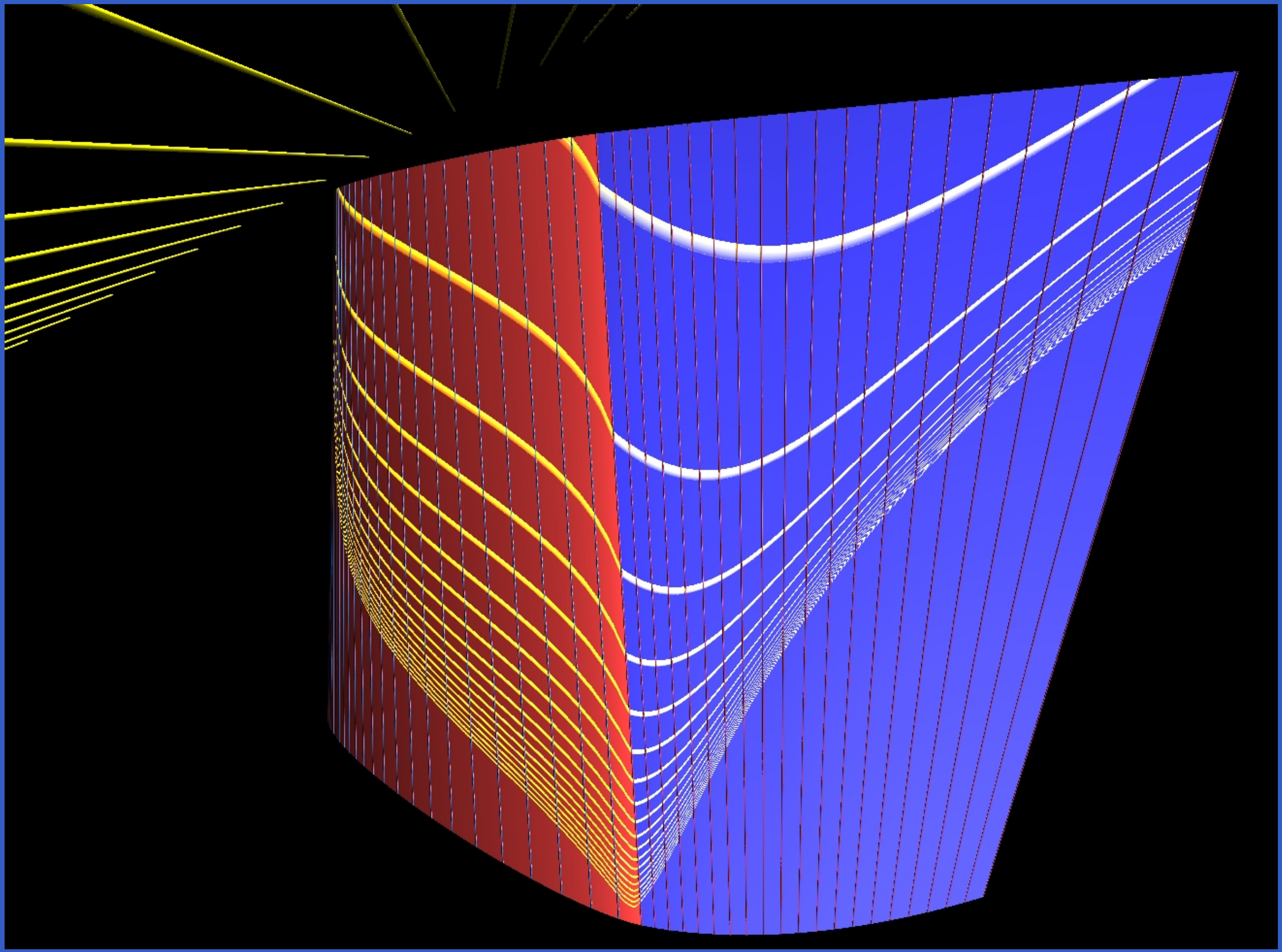
G^k interpolation with ruled surfaces

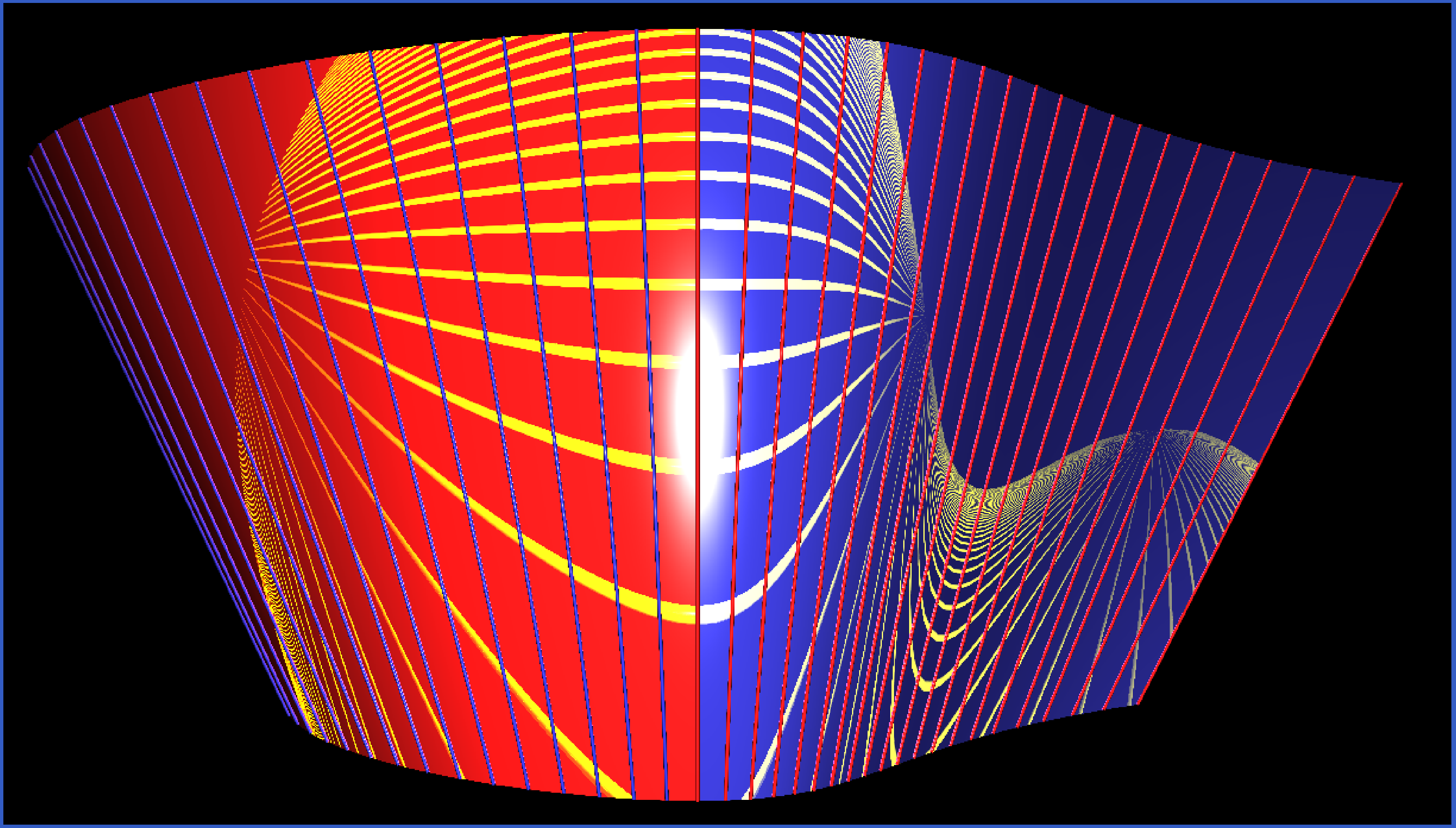
Boris Odehnal

Vienna University of Technology

1^{st} order contact



2^{nd} order contact



aims & motivation

- G^1 interpolation not sufficiently smooth for design purposes
- G^2, G^3 interpolation with rational ruled surfaces reformulated as an interpolation problem of curves
- suitable model space and some geometric ideas simplify
- conditions on the solvability of interpolation problems

related work

[1] R. Dietz, J. Hoschek, B. Jüttler: *An algebraic approach to curves and surfaces on the sphere and on other quadrics.* Comp. Aided Geom. Design **10** (1993), 211–229.

[2] J. Hoschek, H. Pottmann: *Interpolation and approximation with developable B-spline surfaces.* Mathematical Methods for Curves and Surfaces, 1995.

[3] M. Peternell: *G^1-Hermite Interpolation of Ruled Surfaces.* In: CAGD, Oslo 2000, Vanderbilt Univ. Press 2001.

[4] H. Pottmann, G. Farin: *Developable rational Bézier and B-spline surfaces.* Comput. Aided Geom. Design **12** (1995).

[5] J. Wallner: *Hopf mappings for complex quaternions.* Beitr. Algebra Geom. **44** (2003), 245–262.

line geometry

- line $L = (l, \bar{l}) \in \mathbb{R}^6 \ldots$ Plücker coordinates

  with $0 = \langle l, \bar{l} \rangle =: \frac{1}{2}\Omega(L, L)$

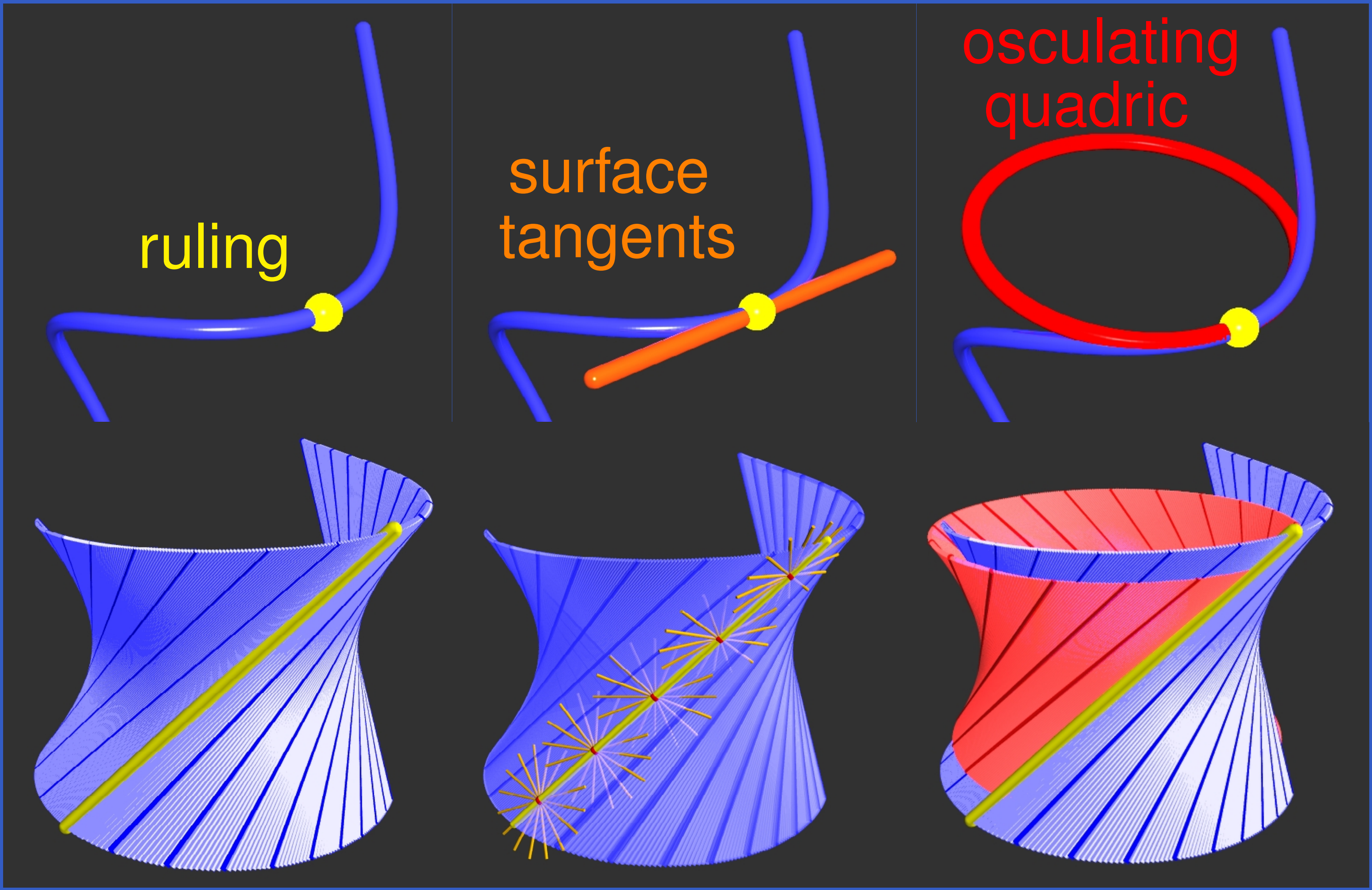
- Klein model of line space:

  lines $\subset \mathbb{P}^3 \rightarrow$ points $\in M_2^4 \subset \mathbb{P}^5$

  ruled surfaces $\subset \mathbb{P}^3 \rightarrow$ curves $\subset M_2^4$

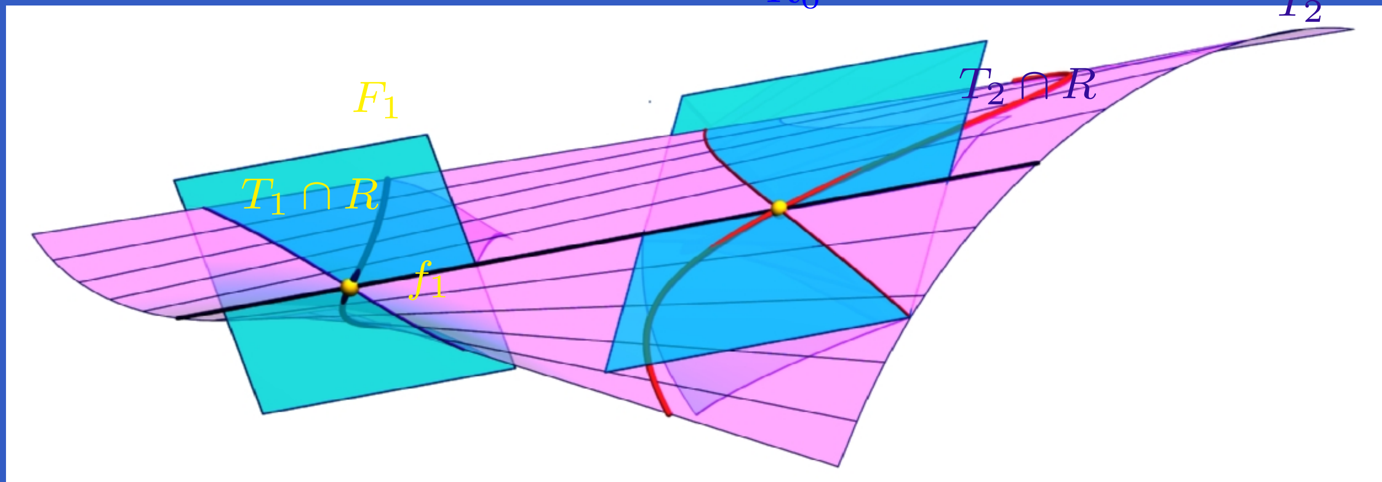
$\Longleftrightarrow$ curve design in M_2^4 = design of ruled surfaces in $\mathbb{P}^3$

differential geometric properties of ruled surfaces



flecnodes

asymptotic tangents hyperosculate the ruled surface



geometric meaning of G^0-, G^1-, G^2-, ...join

$L \subset M_2^4$, $R \subset M_2^4$ ... curves in M_2^4 representing rational ruled surfaces, joined at $L(t_0) = R(u_0)$

- G^0 ... sharing a ruling
- G^1 ... sharing a ruling and all tangent planes along the common ruling
- G^2 ... sharing a ruling, all tangent planes, and the osculating quadric (ruled quadric, Lie's quadric)
- G^3 ... sharing all from above + flecnodes

rational ruled surfaces & normal curves

- rational ruled surface $\longleftrightarrow$ rational curve in $M_2^4 \longleftrightarrow$ rational normal curve in $\mathbb{P}^5$ (or in a subspace) - at least a linear image
- Veronese variety V_k^1, parametrized by $(t_0^k : t_0^{k-1}t_1 : \ldots : t_0 t_1^{k-1} : t_0^k)$
- well known examples:
  $(t_0^2 : t_0 t_1 : t_1^2) \ldots$ conic
  $(t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3) \ldots$ twisted cubic

rational normal curves

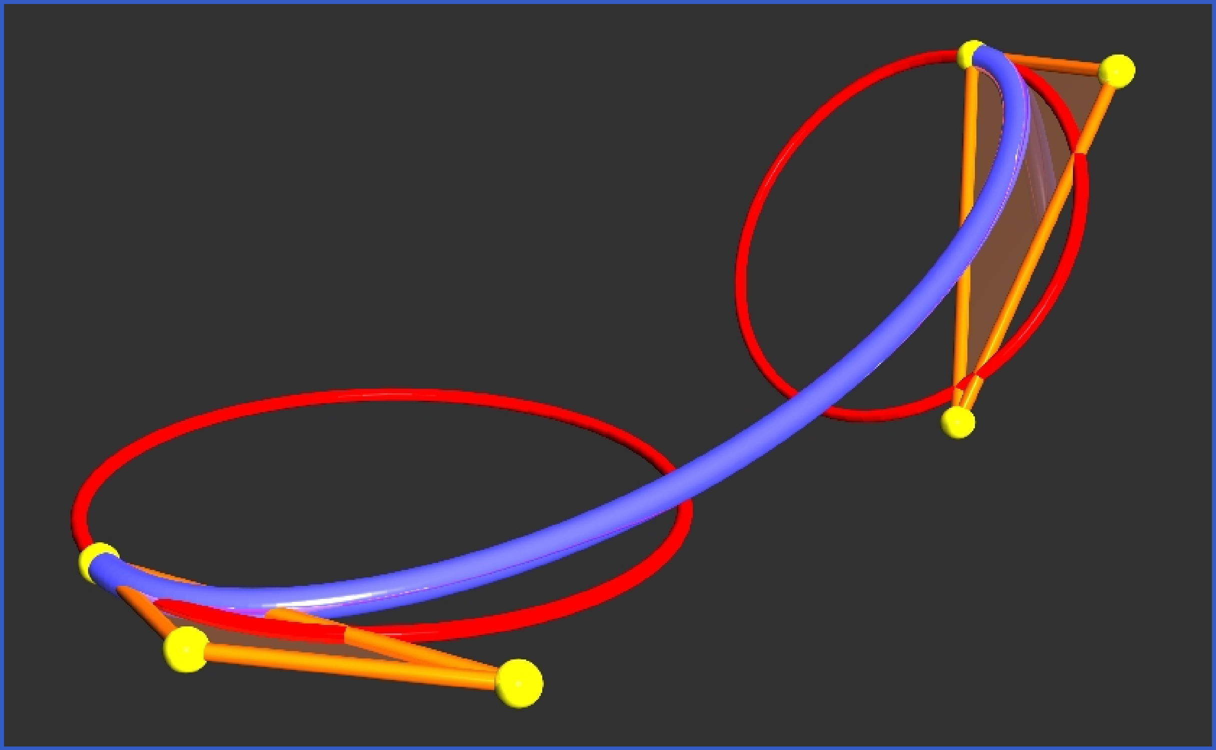
- normal curve $N_n = (t, t^2, \ldots, t^n)$, affine parameter $t = t_0 t_1^{-1}$
- N_n spans an n-space
- osculating subspaces $O^k(t_1)$ and $O^{n-k}(t_2)$ intersect at a point (for any $t_1 \neq t_2$)
- intersection of some quadrics
- admits a projective generation
- possess a group of automorphic collineations

rational normal curves: low degree examples

deg	span	object in line space
2	plane	regulus (ruled quadric, quadr. cone, tangents of a conic)
3	3-space	linear line congruence (spread)
4	hyperplane	linear line complex
5	$\mathbb{P}^5$	lines + linear line complexes

how to interpolate I

- hermite data $O_P^k := [P, \dot{P}, \ldots, P^{(k)}]$, O_Q^k, *osculating elements* of order k



how to interpolate II

- rewrite rational curve as Bézier curve

$$R(t) = \sum_{i=0}^{n} A_i b_i^n, \;\; A_i \in \mathbb{R}^6, \;\; b_i^n = \binom{n}{i}(1-t)^{n-i}t^i$$

- endpoint interpolation yields

$$A_0 = P, \;\; A_1 = \frac{1}{n}((n+1)P + \pi_0 \dot{P}), \;\; A_2 = \dots$$

how to interpolate III

- $R \subset M_2^4 \iff \Omega(R, R) = 0$
- $\Longrightarrow$ equations for the unknowns A_i, π_i, ...

$$\Omega(A_0, A_0) = \Omega(A_0, A_1) = 0,$$

$$(n-1)\Omega(A_0, A_2) + n\Omega(A_1, A_1) = 0, \ldots$$

- equations at the beginning/end are fulfilled automatically
- $\pi_0, \pi_1, \ldots$ serve as shape parameters

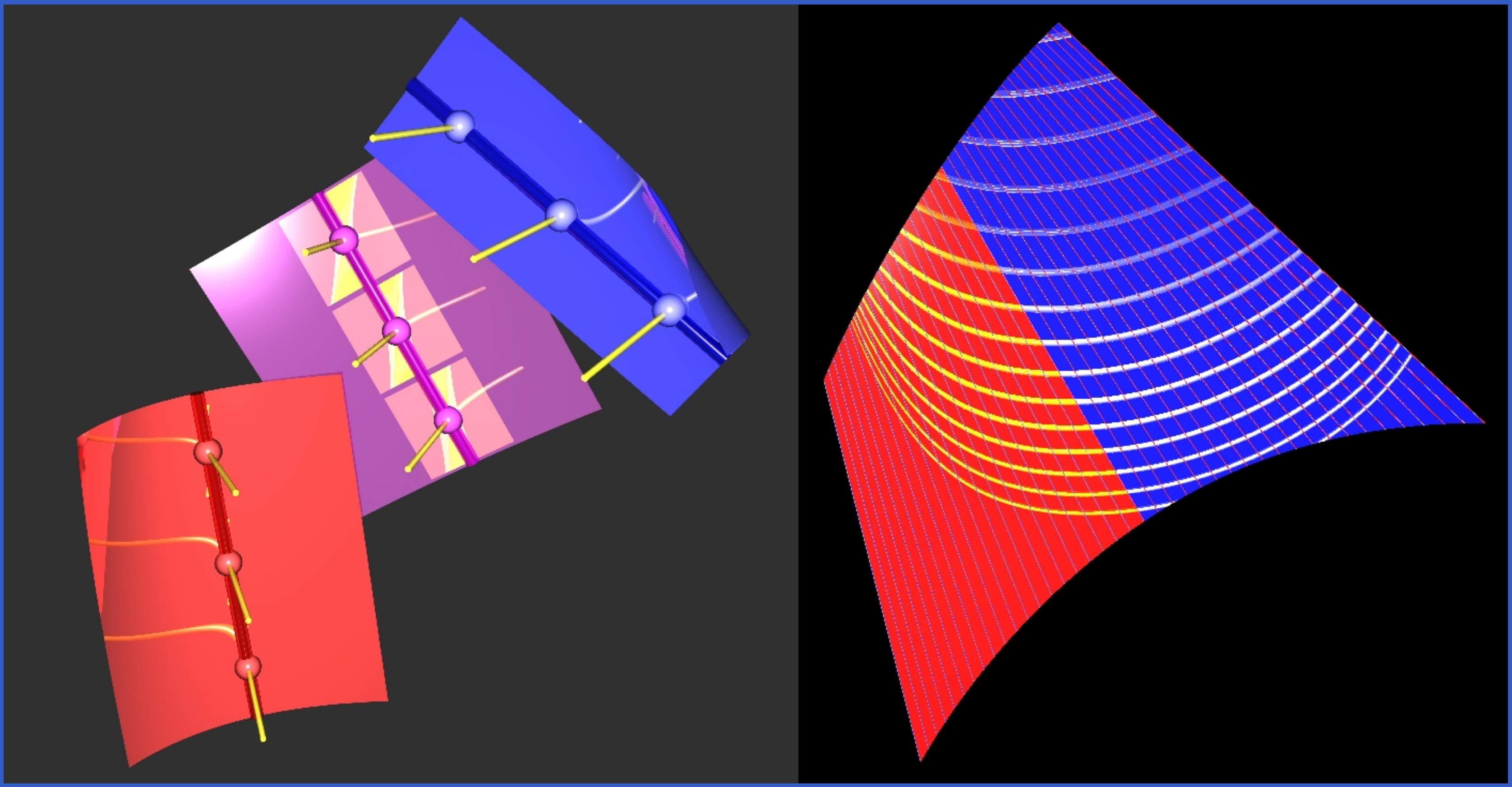
low degree?

- biarcs of reguli: cf. [3]
- concept of [3] does not apply to G^k, if $k > 1$
- cubic, quartic, quintic (rational) ruled surfaces for G^1 (certain constraints to input)
- sextic, septic, ... allow G^2 interpolation

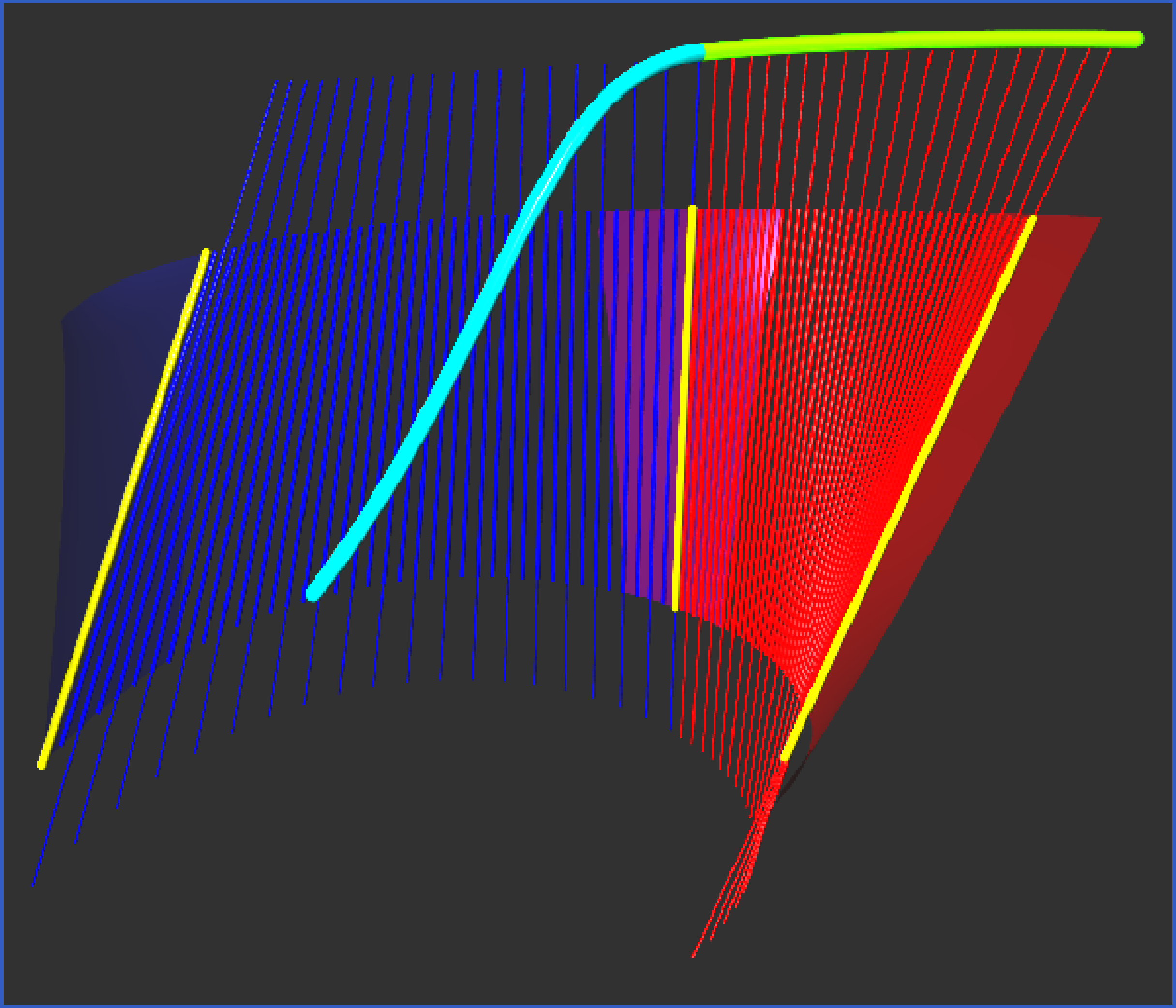
possible or impossible?

- G^1 hermite with cubic arc:
  only if $[P, Q] \subset [P, \dot{P}, Q, \dot{Q}] \cap M_2^4$
- G^2 hermite with quartic arc:
  in general O_P^2 and O_Q^2 are skew $\Longrightarrow$ there is no quartic arc interpolating arbitrary G^2 hermite data
- lowest posssible degree for G^2 hermite interpolation is six

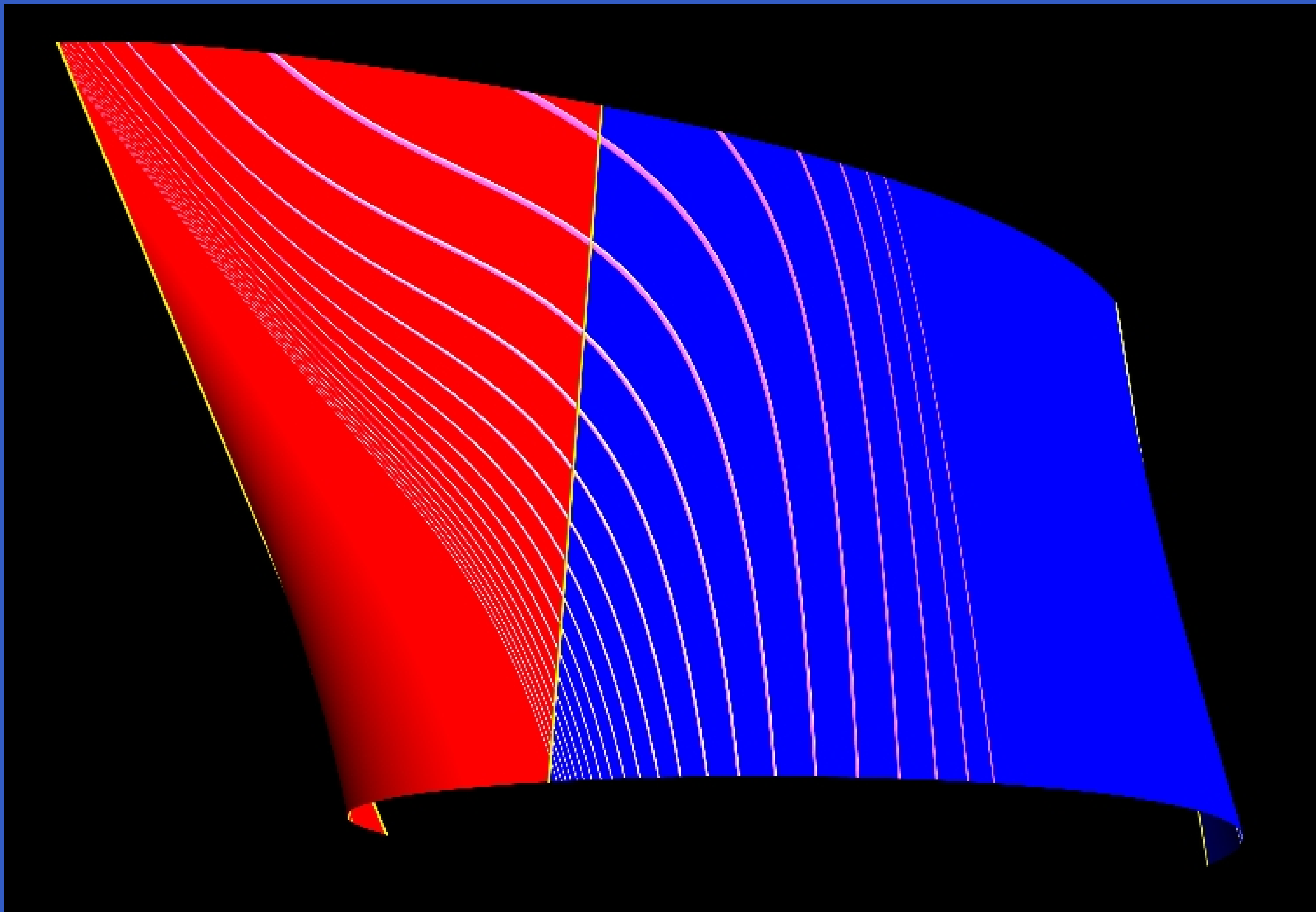
example: ruled surface patches joined with G^2 continuity



example: ruled surface patches joined with G^3 continuity



example: ruled surface patches joined with G^3 continuity



future work

- projective generation of ruled surfaces
- work out analogies to the projective generation of spatial cubics and other norm curves
- solutions of alegbraic equations are not unique
- what are useful shape parameters?

Thank you for your attention!