Higher dimensional geometries: What are they good for?

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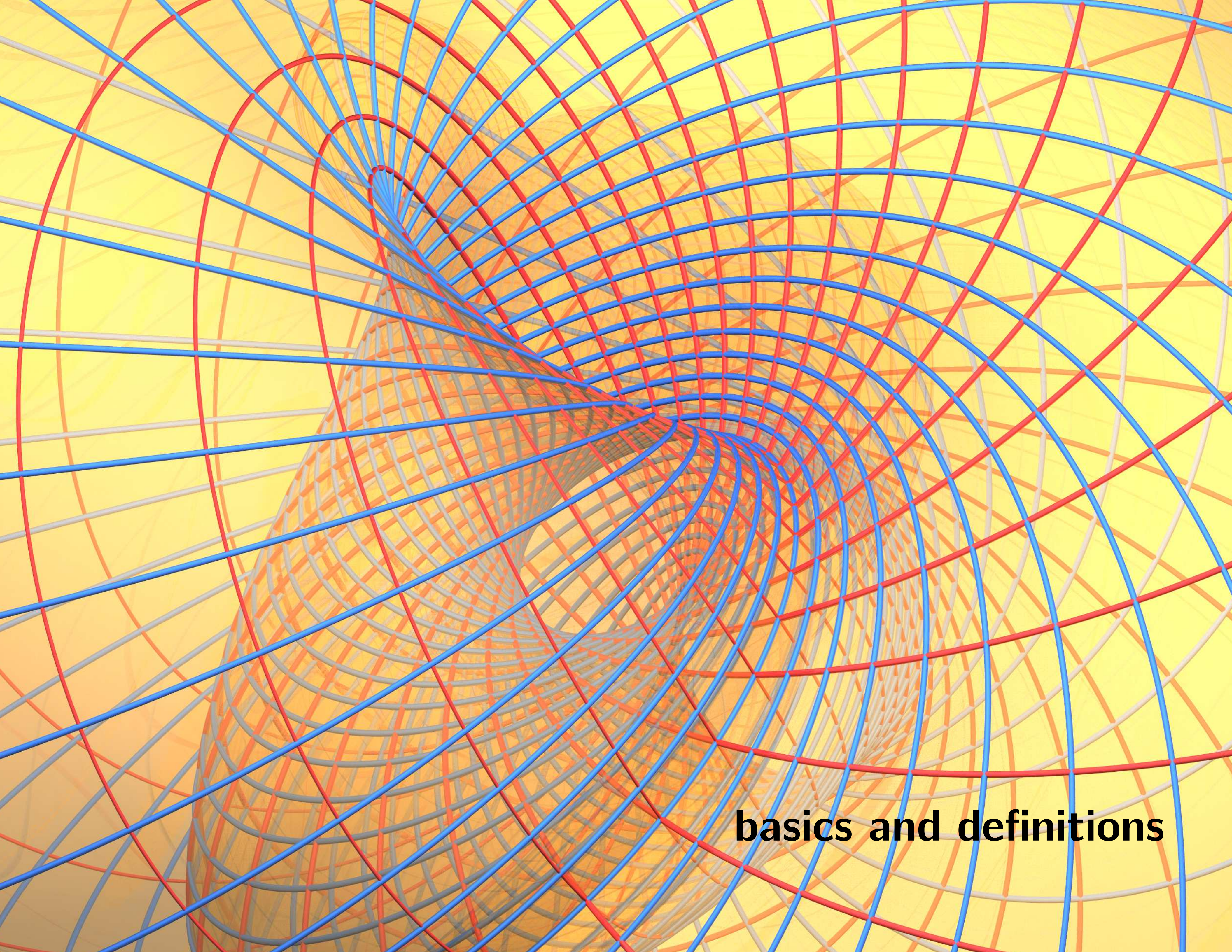
the next 30+ε minutes ($\varepsilon \gg 0$)

dimension(s)	|	basics and definitions - not too mathematical
examples	|	natural, simple, familiar, and not so trivial ones
models of geometries	|	creating models and good models
examples	|	where higher dimensional geometries apply to
a view on more	|	some hints to other models



basics and definitions

What is a dimension? What means higher dimensional?

- number of independent coordinates or variables

  (x, y), (x, y, z), $\dots$, $(x_1, x_2, \dots, x_n)$

  coordinates of points in the plane, in 3-space, $\dots$, n-space

  homogeneous coordinates:

  $x_0 : x_1 : \dots : x_n \quad \sim \quad \lambda x_0 : \lambda x_1 : \dots : \lambda x_n \;\; (\lambda \neq 0)$ (Only the ratio matters.)
- number of basis vectors or basis polynomials

  $\mathbf{b}_1 = (1, 0, 0, 0)$, $\mathbf{b}_2 = (0, 1, 0, 0)$, $\dots$,

  $\{1, x, \dots, x^n\}$ ... basis in the space of univariate polynomials of degree n

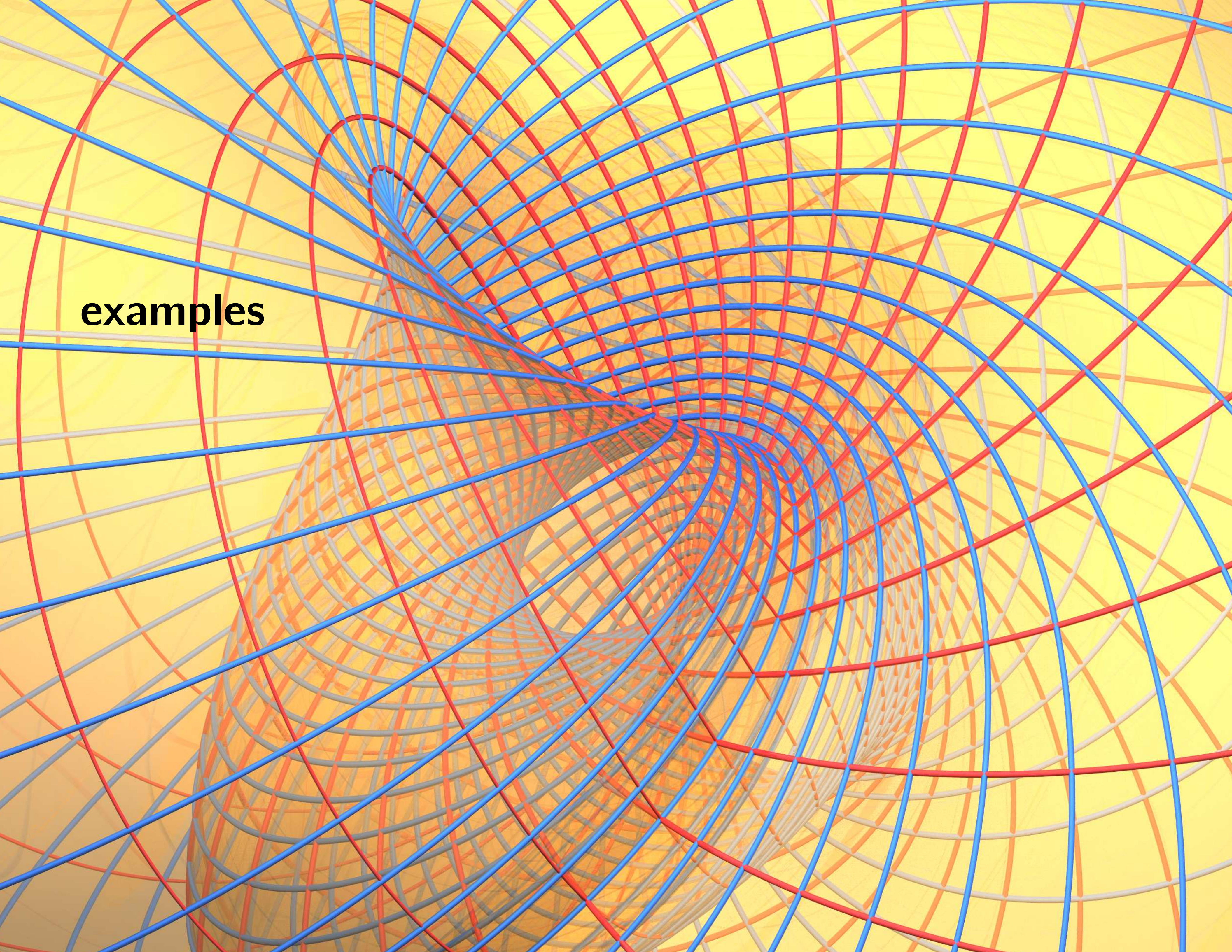
  becomes more complicated if each component can be a collection of polynomials $\longrightarrow$ module over a polynomial ring

  There is always a vector space around.
- number of degrees of freedom

  in kinematic chains, mechanisms, algebraic curves/surfaces ...
- Higher dimensional means more than three,

  more than in the space of our perception!

examples



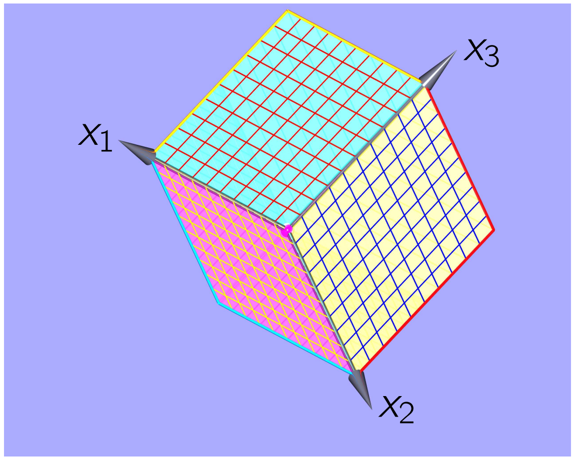
basic objects and their dimensions

object	point	line	curve	plane	surface	...	hyperplane
dimension	0	1	1	2	2	...	$n-1$

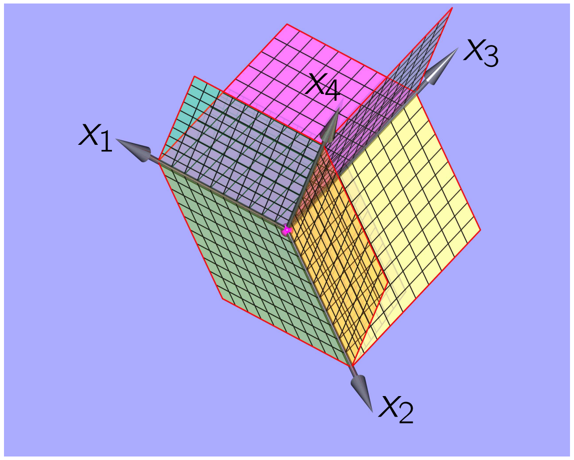
These numbers refer to the dimensions of a line, ... considered as a point space.

Here the dimension equals the number of coordinates to determine a point in space.

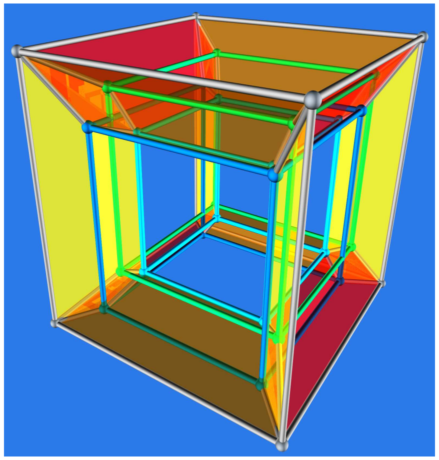
The table above looks different if we do not consider points to be the basic objects.



3-dim. space



4-dim. space

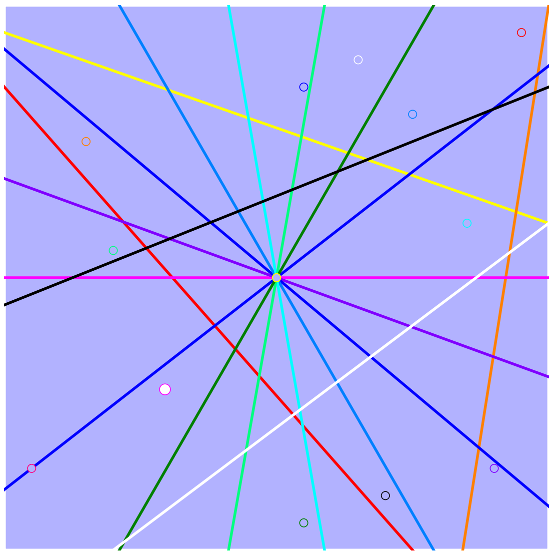


5-dim. space

These are just images showing some *projections* of cuboid corners and a cuboid.

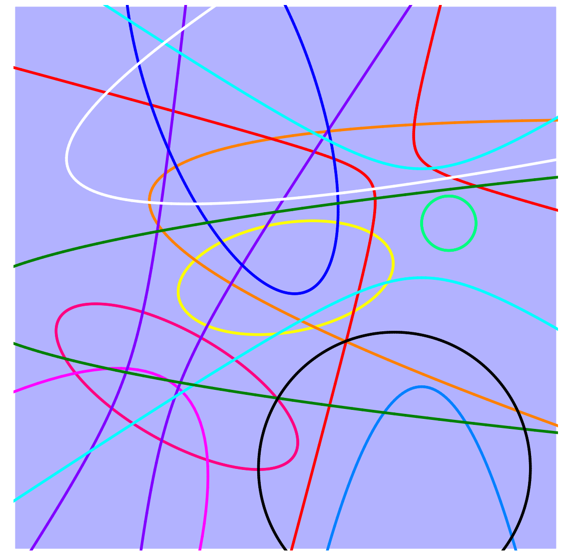
Be careful: Sometimes images of higher dimensional objects are misleading!

simple objects and their dimensions [3,10]



points / lines in a plane:
2-dimensional geometry,
$P = (x_P, y_P)$, $l : \frac{x}{a} + \frac{y}{b} = 1$

circles (incl. lines, points):
3-dimensional geometry,
center + radius = 3 dof



conics (in a plane):
5-dimensional geometry,
$\boldsymbol{a}_0 + 2\boldsymbol{a}_1\boldsymbol{x} + 2\boldsymbol{a}_2\boldsymbol{y} + \boldsymbol{a}_3\boldsymbol{x}^2 + 2\boldsymbol{a}_4\boldsymbol{x}\boldsymbol{y} + \boldsymbol{a}_5\boldsymbol{y}^2 = 0$

space of	dimension	
planar algebraic curves of degree n	$\frac{1}{2}n(n+3)$	V_2^n
k-dimensional subspaces of a projective n-space	$(n-k)(k+1)$	$\mathrm{Gr}_{n,k}$
d-dim. alg. varieties of deg. D in projective space	$\frac{1}{(1+d)!}\prod_{k=1}^{d+1}(D+k)$	

Within a geometry of a certain dimension,
we can find geometries of even higher dimensions!

familiar objects and their dimensions

The weather is at least an 8-dimensional phenomenon:

place (x_1, x_2, x_3), time t, temperature T, air pressure p, ...

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$

$$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = \rho F - \nabla p + \eta \Delta v$$

$$\rho c_P \frac{\mathrm{d}T}{\mathrm{d}t} = \lambda \Delta T$$

$$\frac{\partial}{\partial t} \nabla^2 \Psi = -\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x_1, x_3)} + \nu \nabla^4 \Psi + \rho \alpha \frac{\partial \Theta}{\partial x_1}$$

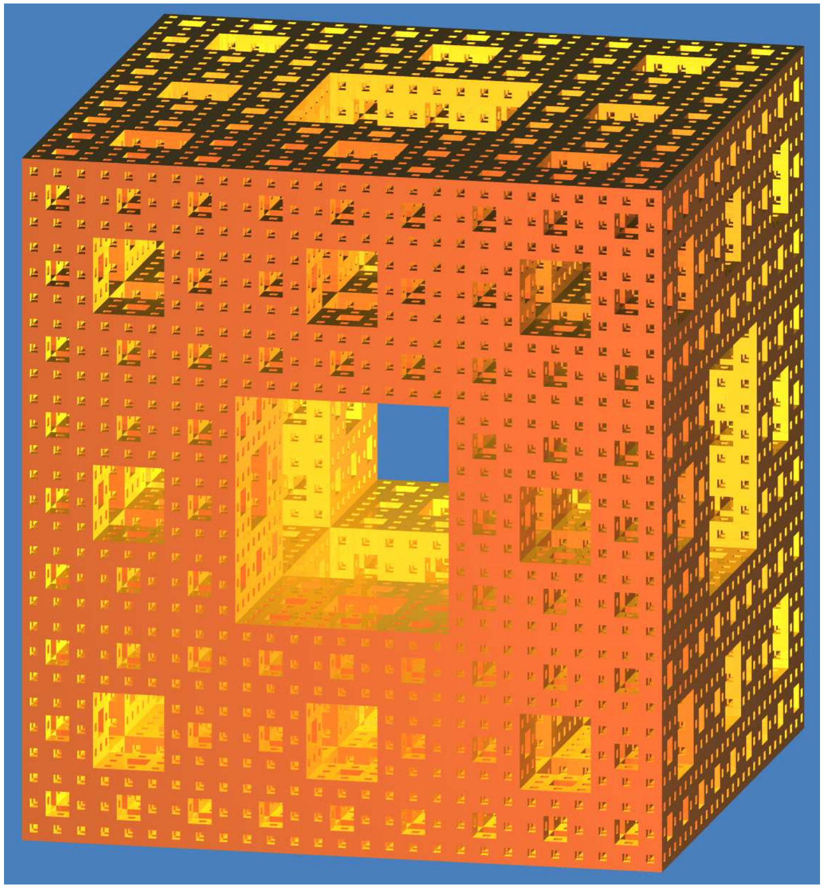
$$\frac{\partial \Theta}{\partial t} = -\frac{\partial(\Psi, \Theta)}{\partial(x_1, x_3)} + \frac{\Delta T}{h} \frac{\partial \Psi}{\partial x_1} + \chi \nabla^2 \Theta$$

M. Roskar

Wheather forecast is **just solving these equations.**

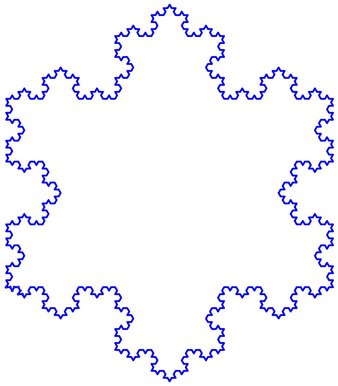
less classical, but more popular objects - fractals

Dimensions are rather computed than counted and are no longer integers!



Hausdorff dimension of the Menger sponge

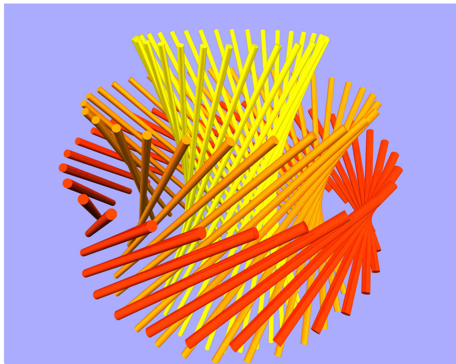
$$d = \frac{\log 20}{\log 3} = 2.72683\ldots$$

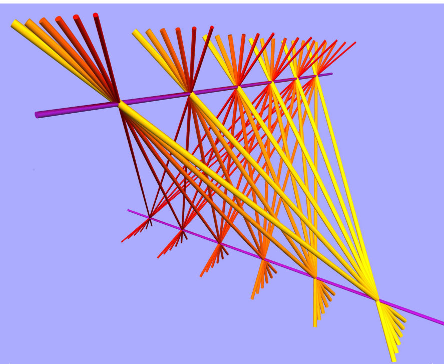


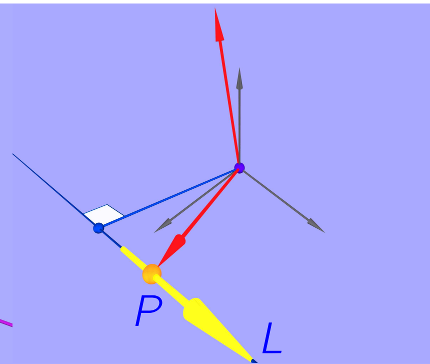
Hausdorff dimension of Koch's snowflake

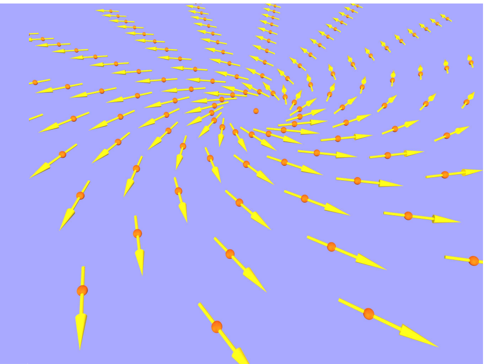
$$d = \frac{\log 4}{\log 3} = 1.2619\ldots$$

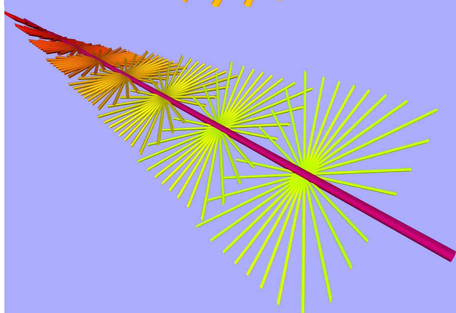
less simple, but more useful



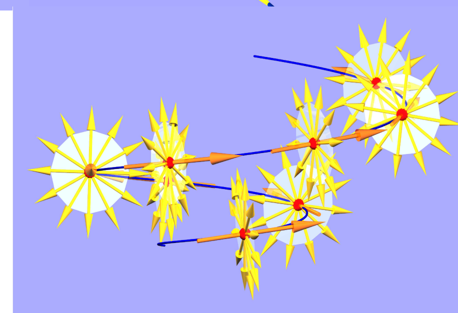




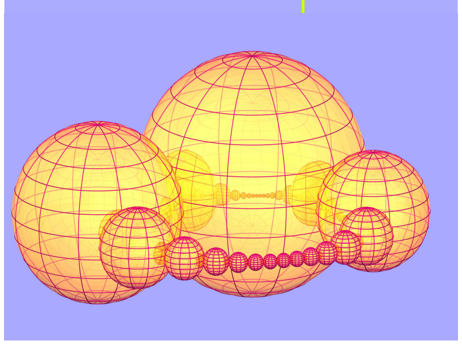




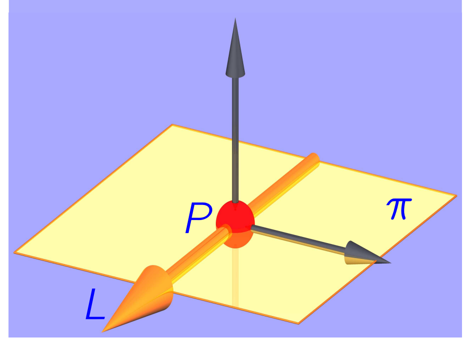
oriented
lines in 3-space
4-dim. geometry



oriented
line elements (L, P)
5-dim. geometry



oriented
spheres in 3-space
4-dim. geometry



oriented flags
(P, L, π) in 3-space
6-dim. geometry

Dimensions need not be finite!

- spaces of functions - Hilbert space

  Fourier series, space of polynomial series, ...

- The solutions of delay differential equations constitute an

  infinite dimensional vector space!

  E.g. $f(t+\tau) - \frac{\mathrm{d}}{\mathrm{d}t} f(t) - f(t) = 0$ ($\tau \in \mathbb{R}^\star$) is solved by

  $\sum_{k=-\infty}^{\infty} \lambda_k e^{p_k \cdot t}$ with $p_k = 1 - \frac{1}{\tau} W(k, \tau e^{2\tau})$ and $\lambda_k \in \mathbb{C}$

- ...

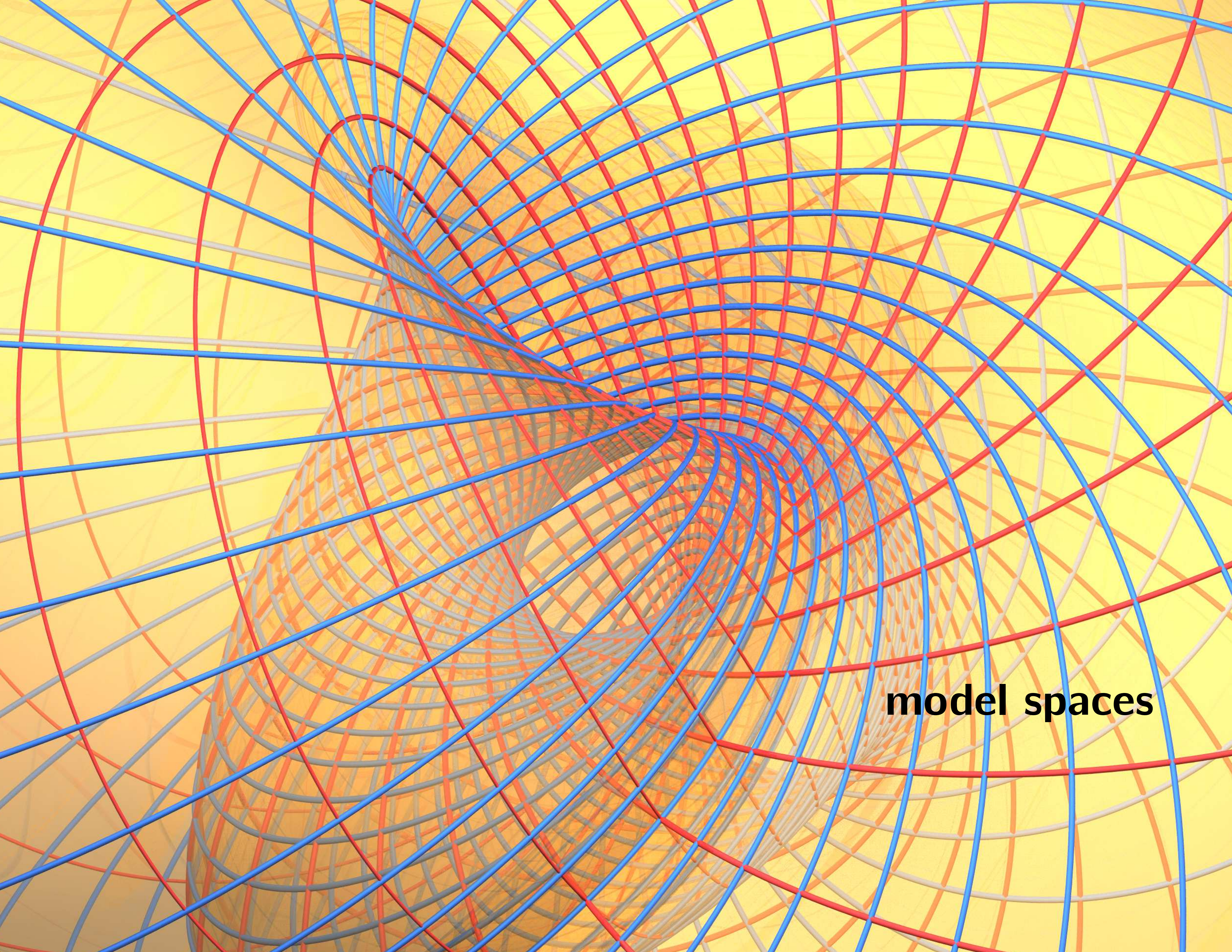
A more geometric curiosity: The Hilbert cube (rather cuboid) is a

cuboid with side lengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

on consecutive orthogonal edges

and has a diagonal with the finite lenght

$$\sqrt{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots} = \sqrt{\zeta(2)} = \frac{\pi}{\sqrt{6}}.$$



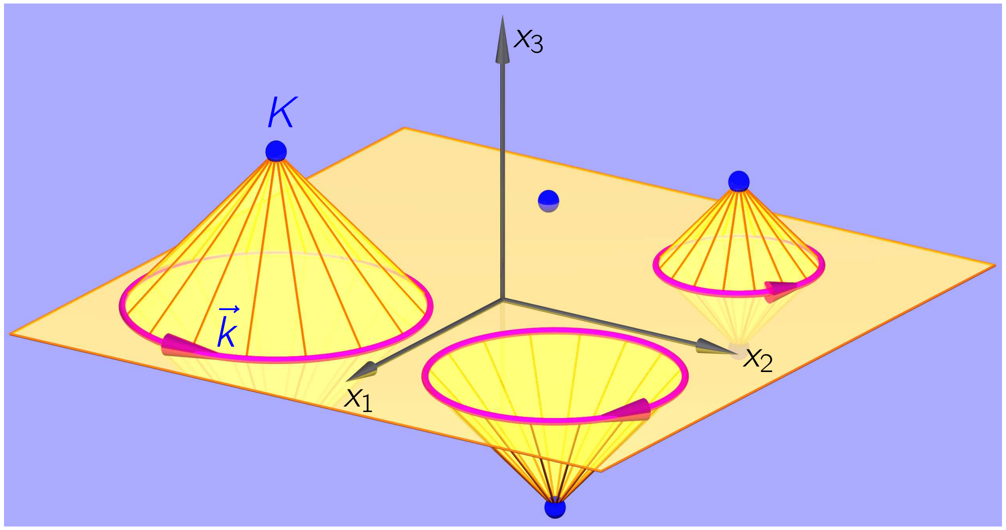
model spaces

we have seen . . .

Some geometric objects depend on a certain/fixed number of constants.

Why not use them as coordinates?

circle in $\mathbb{R}^2$	$\mapsto$	point in $\mathbb{R}^{2,1}$
sphere in $\mathbb{R}^3$	$\mapsto$	point in $\mathbb{R}^{3,1}$
$\vdots$		$\vdots$
sphere in $\mathbb{R}^n$	$\mapsto$	point in $\mathbb{R}^{n,1}$



$\mathbb{R}^{n,1}$ . . . cyclographic image space, model space for the geometry of oriented spheres in Euclidean n-space.

or. circle $\vec{k}$: center $M = (x_M, y_M)$, radius r

$\mapsto$ point $K = (x_M, y_M, r) \in \mathbb{R}^{2,1}$,

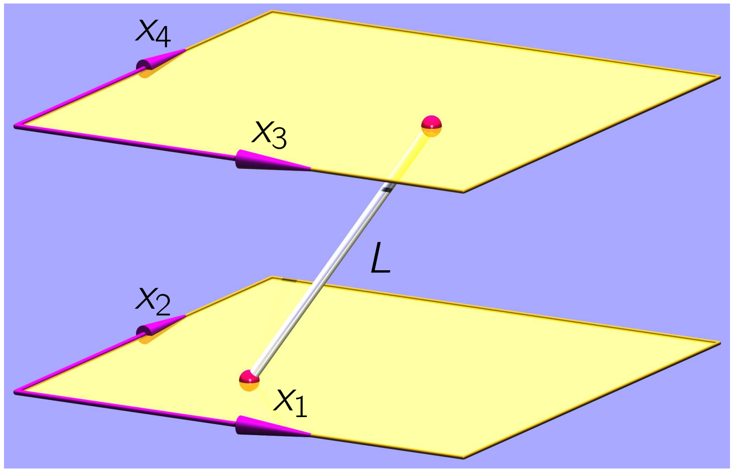
pos. or. $k \longrightarrow \operatorname{sgn}(r) = +1$

$\Longrightarrow$ Geometric objects that are usually described by an equation (or a set of equations) can be represented by just one point in some model space!

What makes a good model?

★ The transition from the original object to its image in the model space, and *vice versa*, should be as simple as possible:

- *e.g.*, cyclographic mapping $\vec{k} \mapsto (x_M, y_M, r)$.



★ lowest possible dimension of the model space:

- *e.g.*, a slice model of line space with, $L \mapsto (x_1, x_2, x_3, x_4)$ is possible, but limited in its applications

★ Almost all relations between objects should also be displayed in the model space:

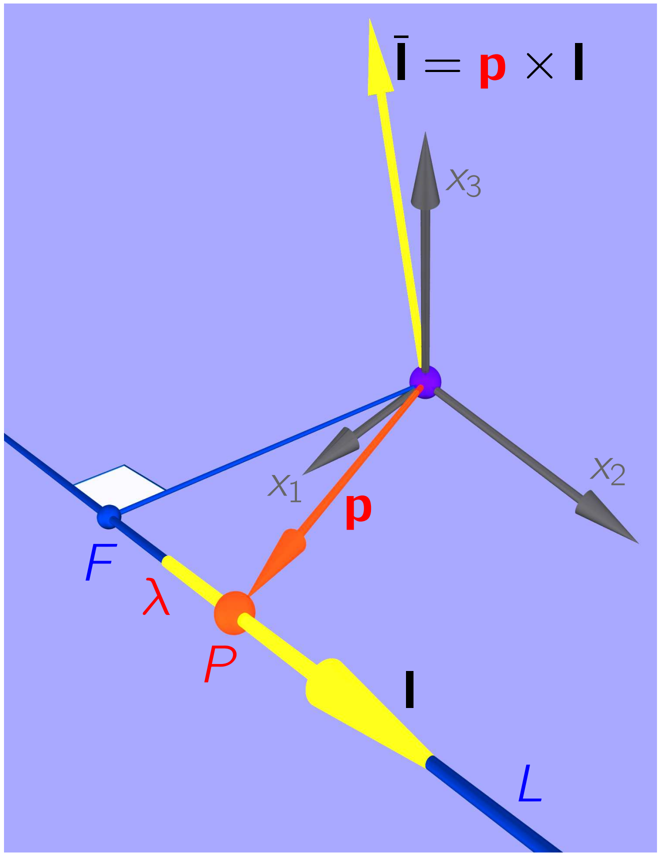
- *e.g.*, intersection of lines, oriented contact of spheres, . . .
- . . . with help of a polar form or a metric

★ Transformations of the objects should induce simple transformations (preferably linear ones) in the model space:

- *e.g.*, Minkowski transformations in $\mathbb{R}^{n,1}$, collinear transformations in models of various other geometries

★ line geometry

- oriented line $L \longleftrightarrow$ point $(\mathbf{l}, \bar{\mathbf{l}}) \in \mathbb{R}^6 \setminus \{\mathbf{o}\} \cong \mathbb{P}^5$
  $(\mathbf{l}, \bar{\mathbf{l}})$ ... Plücker coordinates of L
  $M_2^4 : \ \langle \mathbf{l}, \bar{\mathbf{l}} \rangle = 0$ ... Plücker's quadric $\subset \mathbb{P}^5$
  intersecting lines $\leftrightarrow$ conjugate points w.r.t. M_2^4
  pencils, ruled planes, stars $\longleftrightarrow$
  lines, planes (1. & 2. kind) $\subset M_2^4$
  collineations in $\mathbb{P}^3 \longleftrightarrow$ auto-collineations of M_2^4
- with $\|\mathbf{l}\| = 1 \Longrightarrow$ Euclidean model
- allowing $\mathbf{l} = \mathbf{o} \Longrightarrow$ projective model



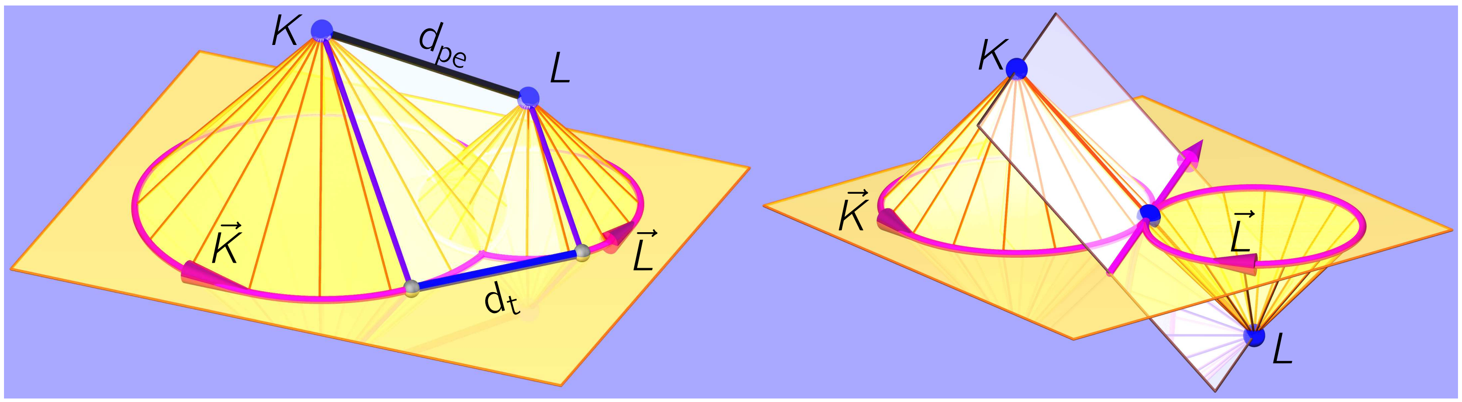
★ line element geometry (partial flags)

- or. line element (L, P) = or. line L + point $P \in L$
  $(\mathbf{l}, \bar{\mathbf{l}}, \lambda) \in \mathbb{R}^7 \setminus \{\mathbf{o}\} \cong \mathbb{P}^6$ ... Plücker coordinates of (L, P)
  with $\|\mathbf{l}\| = 1$ and $\lambda := \langle \mathbf{p}, \mathbf{l} \rangle = \overline{FP}$
  $M_2^5 : \ \langle \mathbf{l}, \bar{\mathbf{l}} \rangle = 0$ ... quadratic cone $\subset \mathbb{P}^6$ as model surface
  Equiform motions induce auto-collineations of M_2^5.

★ **sphere geometry**

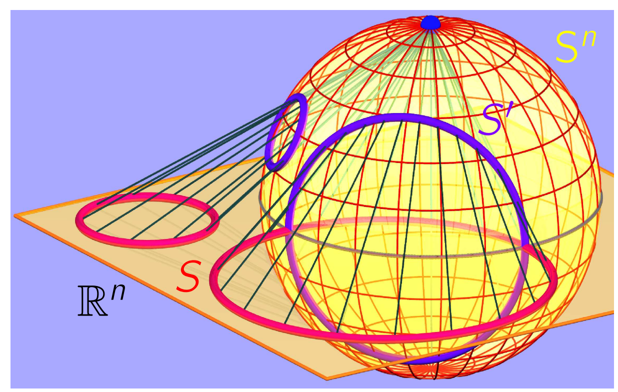
- cyclographic model: oriented sphere $\vec{S} \subset \mathbb{R}^n \mapsto$ point S
  $\mathbb{R}^{n,1}$ . . . $(n+1)$-dimensional Minkowski space, cyclographic model
  affine space with pseudo-Euclidean metric

$$\mathrm{d}_{\mathrm{pe}}(K, L) = \mathrm{d}_{\mathrm{t}}(\vec{K}, \vec{L}) \qquad \mathrm{d}_{\mathrm{pe}} = 0 \iff \text{or. contact}$$



- spherical model: stereographic projection onto S^n: $x_1^2 + \ldots + x_n^2 = 1$

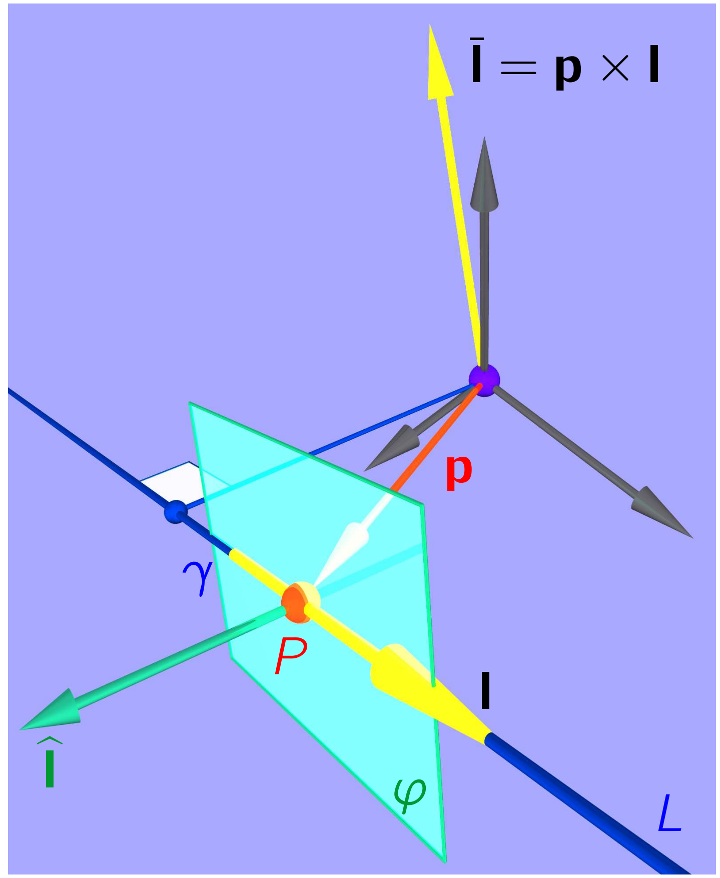
sphere $S \subset \mathbb{R}^n \mapsto$ sphere $S' \subset \mathrm{S}^n$
subsequent polarity w.r.t. S^n:
sphere $S' \subset \mathrm{S}^n \mapsto$ point $S'' \subset \mathbb{R}^{n+1}$
tetra-cyclic, penta-spherical, . . . space

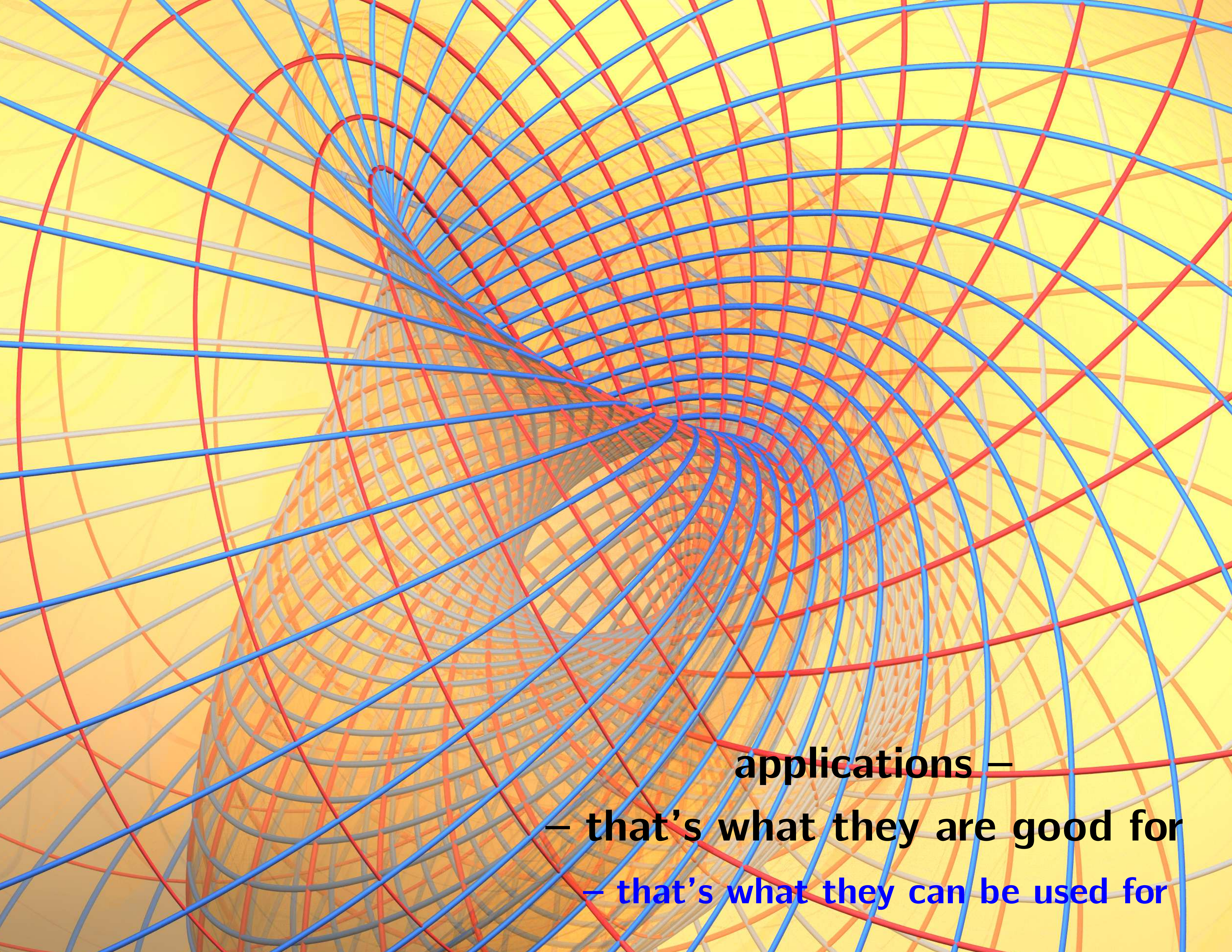


- Lie's quadric – quadric model
  stereographic projection $\mathbb{R}^{n+1} \to L_2^{n+1}$
  $L_2^{n+1}: \; x_1^2 + \ldots + x_n^2 - x_{n+1}^2 - x_{n+2}^2 = 0$
  L_2^{n+1} … Lie's quadric
- or. contact $\longleftrightarrow$ conjugacy w.r.t. L_2^{n+1}
- sphere-preserving contact transformations
  $\longleftrightarrow$ auto-collineations of L_2^{n+1}
- Lie's line-sphere-mapping relates M_2^4 and L_2^4.
  It is just a projective collineation!

★ **geometry of flags** (complete flags)

- Or. flag $\mathcal{F} = (P, L, \varphi)$ in Eucl. 3-space with $P \in L \subset \varphi$ determines a Cartesian frame.
- coordinates of an or. flag $\mathcal{F} = (\mathbf{l}, \bar{\mathbf{l}}, \hat{\mathbf{l}}, \lambda) \in \mathbb{R}^{10}$ with $M_5^6 : \langle \mathbf{l}, \bar{\mathbf{l}} \rangle = \langle \mathbf{l}, \hat{\mathbf{l}} \rangle = 0 \quad \& \quad \|\mathbf{l}\| = \|\hat{\mathbf{l}}\| = 1$
- M_5^6 admits a rational parametrization.
- $\mathcal{F}_0 = (1, 0, 0; 0, 0, 0; 0, 0, 1, 0,) \mapsto \mathcal{F}$ determines a Euclidean motion.
- Euclidean motions induce automorphic collineations of M_5^6.





applications –

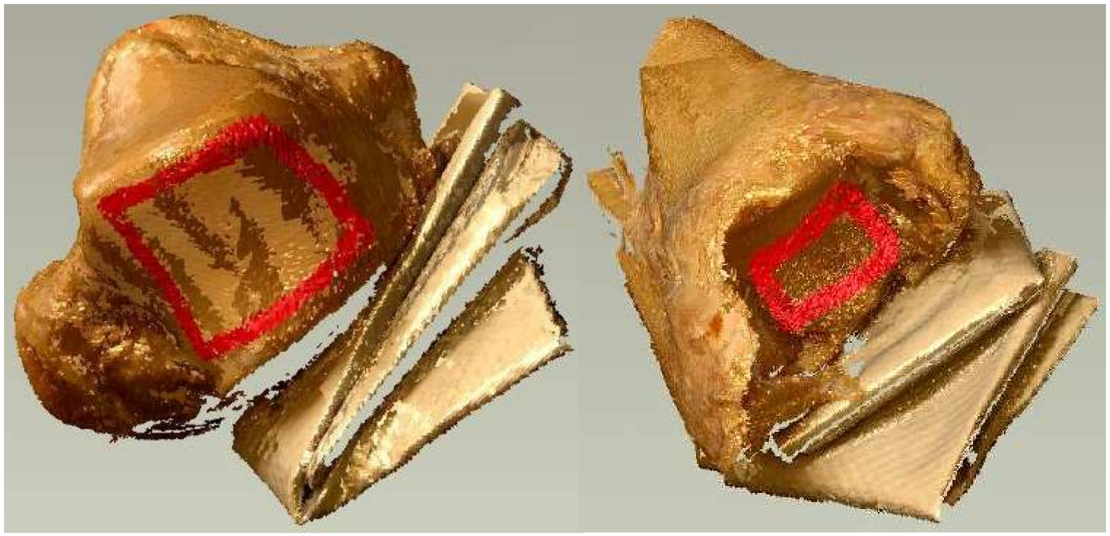
– that's what they are good for

– that's what they can be used for

surface recognition and reconstruction - line geometry [27,33]

The path normals $(\mathbf{n}, \overline{\mathbf{n}})$ of a one-parameter subgroup of the Euclidean group form a linear complex of lines $\mathcal{C} = (\mathbf{c}, \overline{\mathbf{c}})$, *i.e.*, $\langle \mathbf{c}, \overline{\mathbf{n}} \rangle + \langle \overline{\mathbf{c}}, \mathbf{n} \rangle = 0$.

The complex $\mathcal{C} = (\mathbf{c}, \overline{\mathbf{c}})$ determines a helical motion with pitch $p = \langle \mathbf{c}, \overline{\mathbf{c}} \rangle \langle \mathbf{c}, \mathbf{c} \rangle^{-1} \neq 0$.



3D scans of the articulate surfaces of the human ankle joint

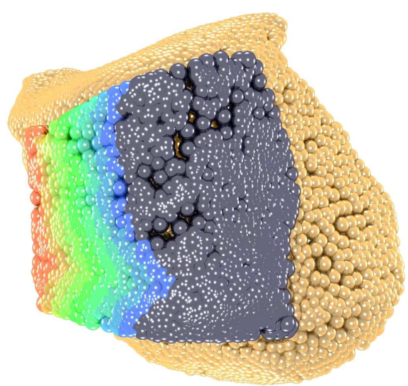
estimation of surface normals $(\mathbf{n}_i, \overline{\mathbf{n}}_i)$ from the point cloud

$\mathcal{C} = (\mathbf{c}, \overline{\mathbf{c}}) \to$ eigenvector of

$$\sum_i \begin{pmatrix} \mathbf{n}_i^T \\ \overline{\mathbf{n}}_i^T \end{pmatrix} (\mathbf{n}_i, \overline{\mathbf{n}}_i) \in \mathbb{R}^{6\times 6}$$

with the largest eigenvalue.

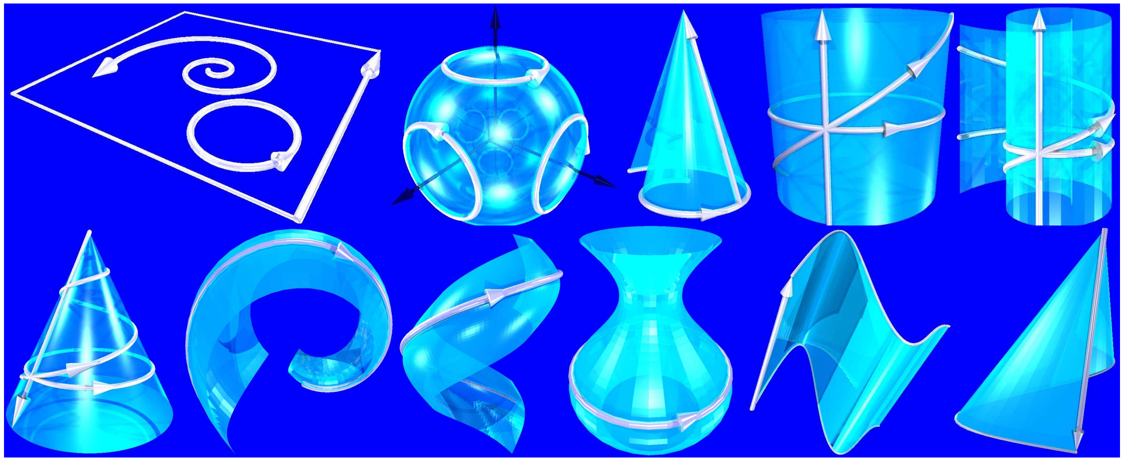
$\Longrightarrow$ Gliding of articulate surfaces along each other forces them to perform a helical motion.



According to Gray's anatomy, it would be a pure rotation.

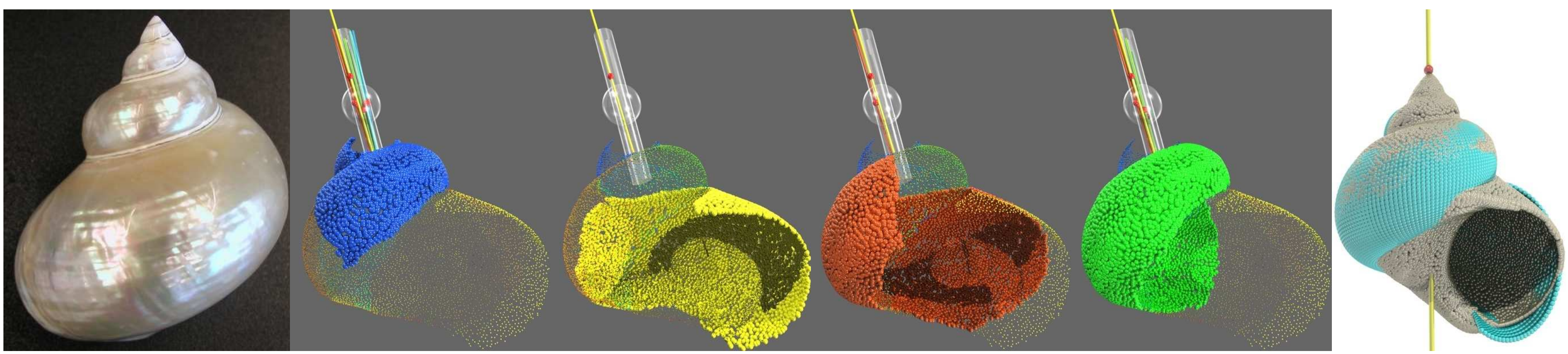
The path normal elements $(\mathbf{n}, \overline{\mathbf{n}}, \nu)$ of a uniform equiform motion form a linear complex $(\mathbf{c}, \overline{\mathbf{c}}, \gamma)$ of line elements, *i.e.*, $\langle \mathbf{c}, \overline{\mathbf{n}} \rangle + \langle \overline{\mathbf{c}}, \mathbf{n} \rangle + \gamma\nu = 0.$

Eigenanalysis of a 7×7-matrix makes 11 classes of surfaces cognoscible:



planes, spheres, spiral cones, cylinders of rev., spiral cylinders, cones of rev., **spiral surfaces**, helical surfaces, surfaces of rev., generic cylinders and cones.

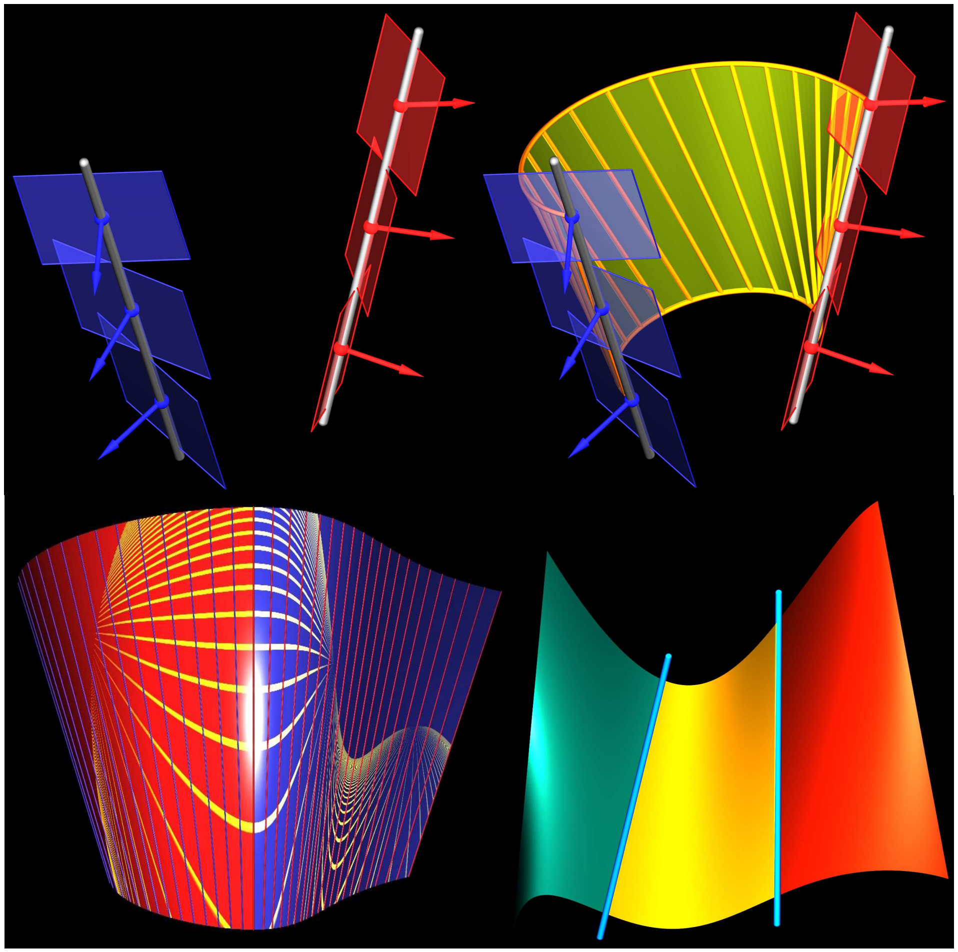
Reconstruction with the best fitting uniform equiform motion.



Hermite interpolation of/with ruled surfaces [17,24,25]

In any model of line geometry, a ruled surface appears as a curve.

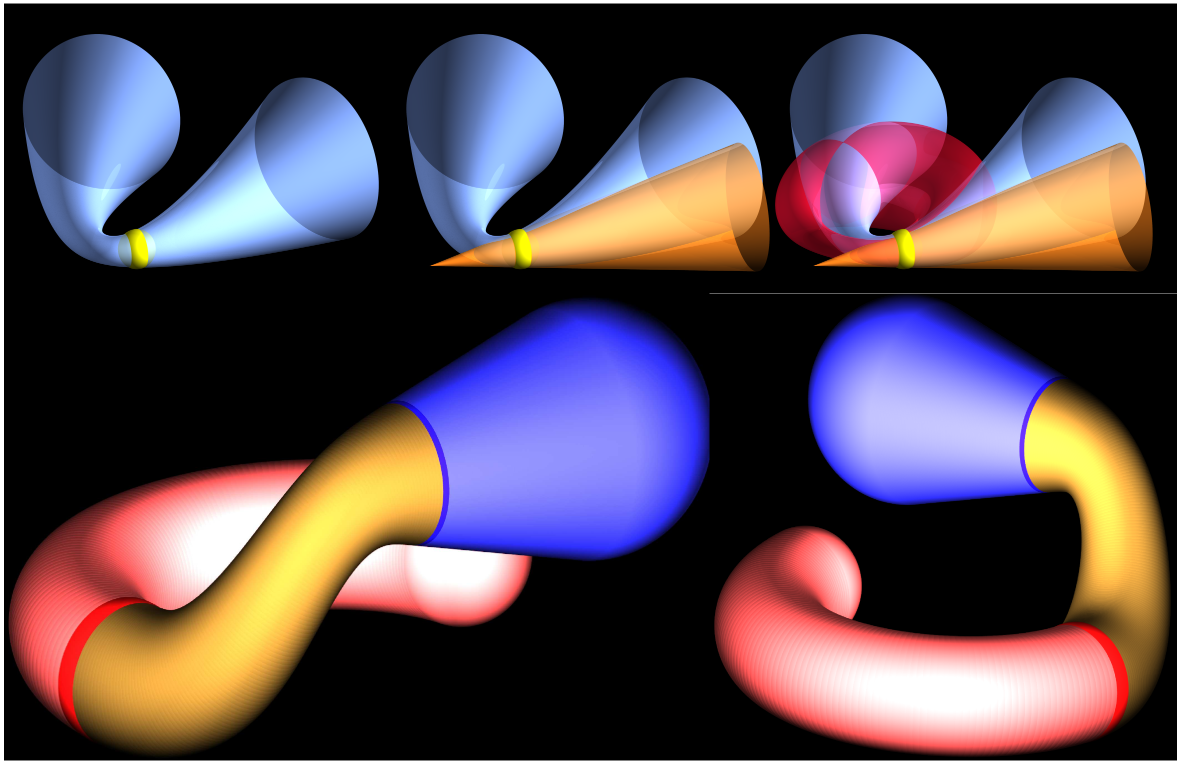
$\Longrightarrow$ Ruled surface interpolation is simplified to curve interpolation.



line geometric Hermite data:
rulings,
contact projectivities,
osculating quadrics, ...

G^2- and G^3-interpolation
of ruled surface data
could hardly be achieved in a different way, if at all.

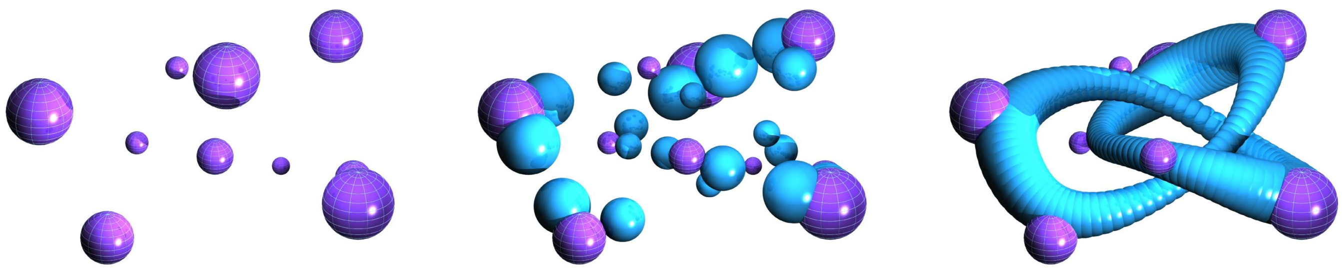
In any model of sphere geometry, a channel surface appears as a curve.
$\Longrightarrow$ Channel surface interpolation is simplified to curve interpolation.



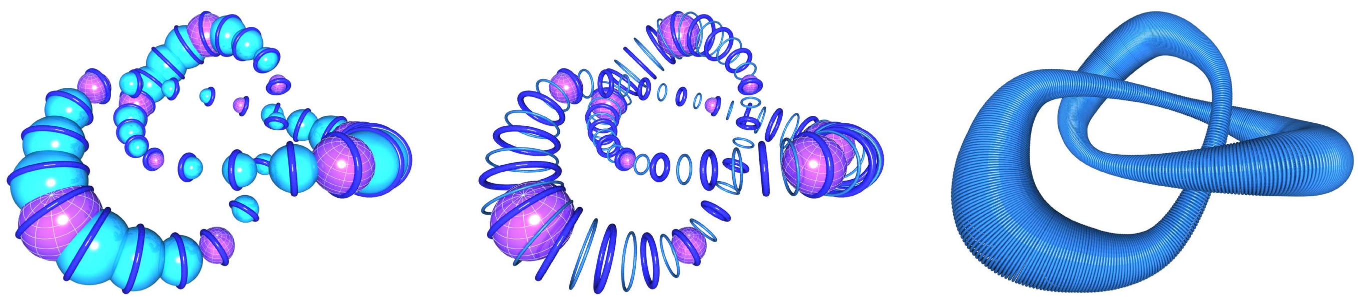
sphere geometric Hermite data:
spheres,
tangent cones,
osculating Dupin cyclides, ...

G^2- & G^3-interpolation
of channel surface data
could also be achieved within
the cyclographic model.

The standard subdivision schemes are defined for data in affine spaces.
One round of a combined scheme consists of subdivision in the (ambient) model space and a subsequent projection onto the model manifold (L^4_2, M^4_2, M^6_5, SO(3), . . .).



This works even on the manifold of circles in Euclidean three-space . . .

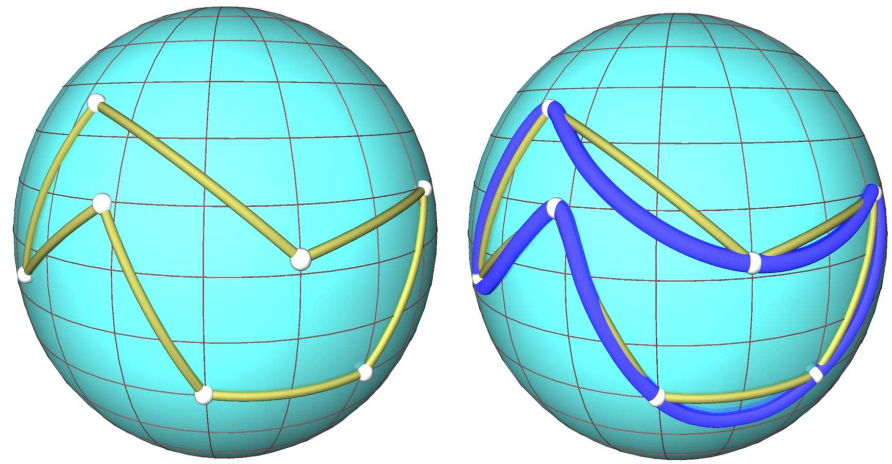


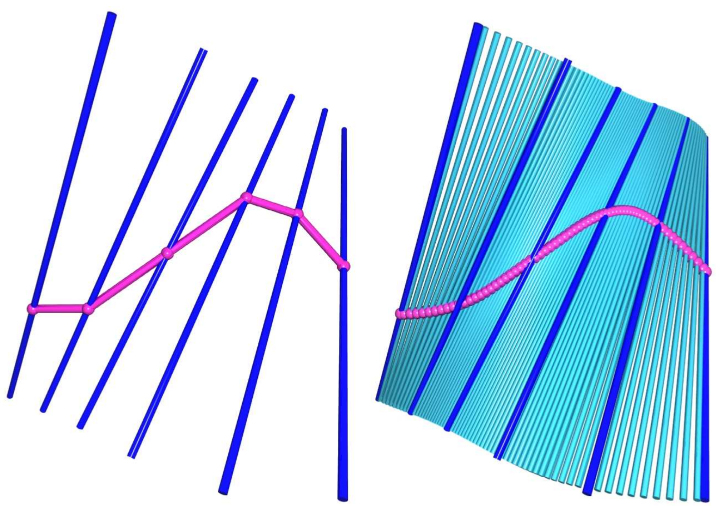
. . . which is a six-dimensional cone with a one-dimensional vertex.
After an initial approximation of the characteristic circles on a discrete channel surface, this family of circles can be refined by a combined subdivision scheme.

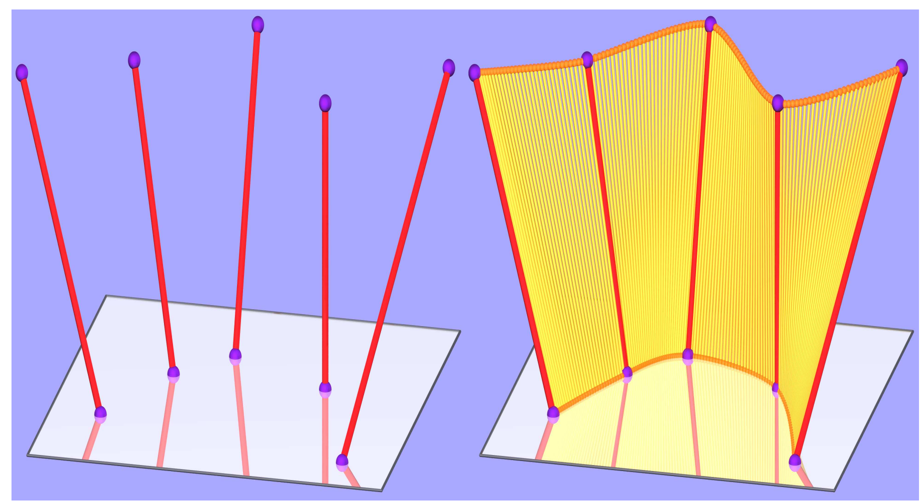
adapted subdivision schemes [5,8,24]

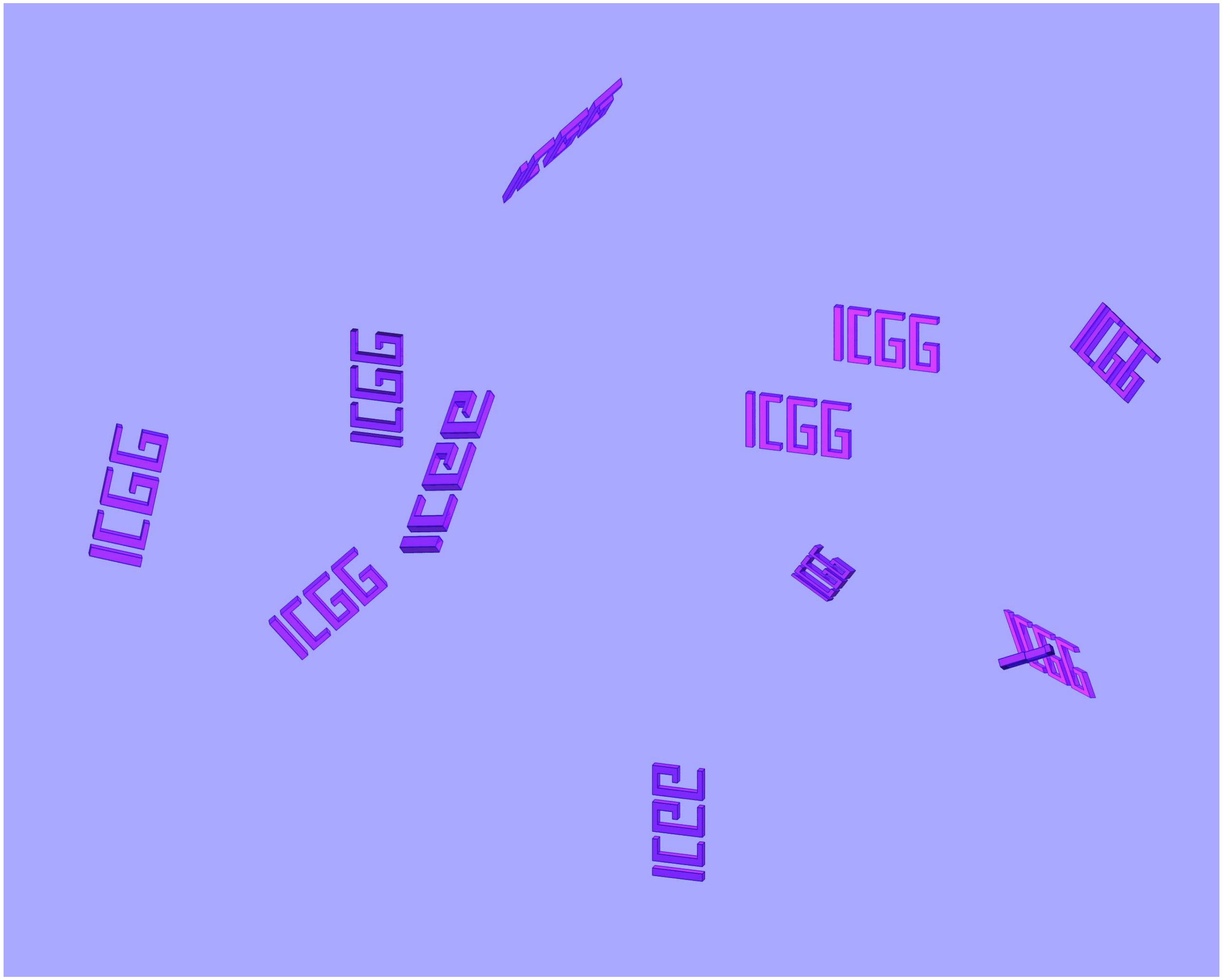
Ruled surfaces can be refined with combined schemes in at least two ways:

- SLERP for the director cone +
  + ordinary subdivision for the striction curve ↓
- subdivision in the model space +
  + projection onto M_2^4 ↘







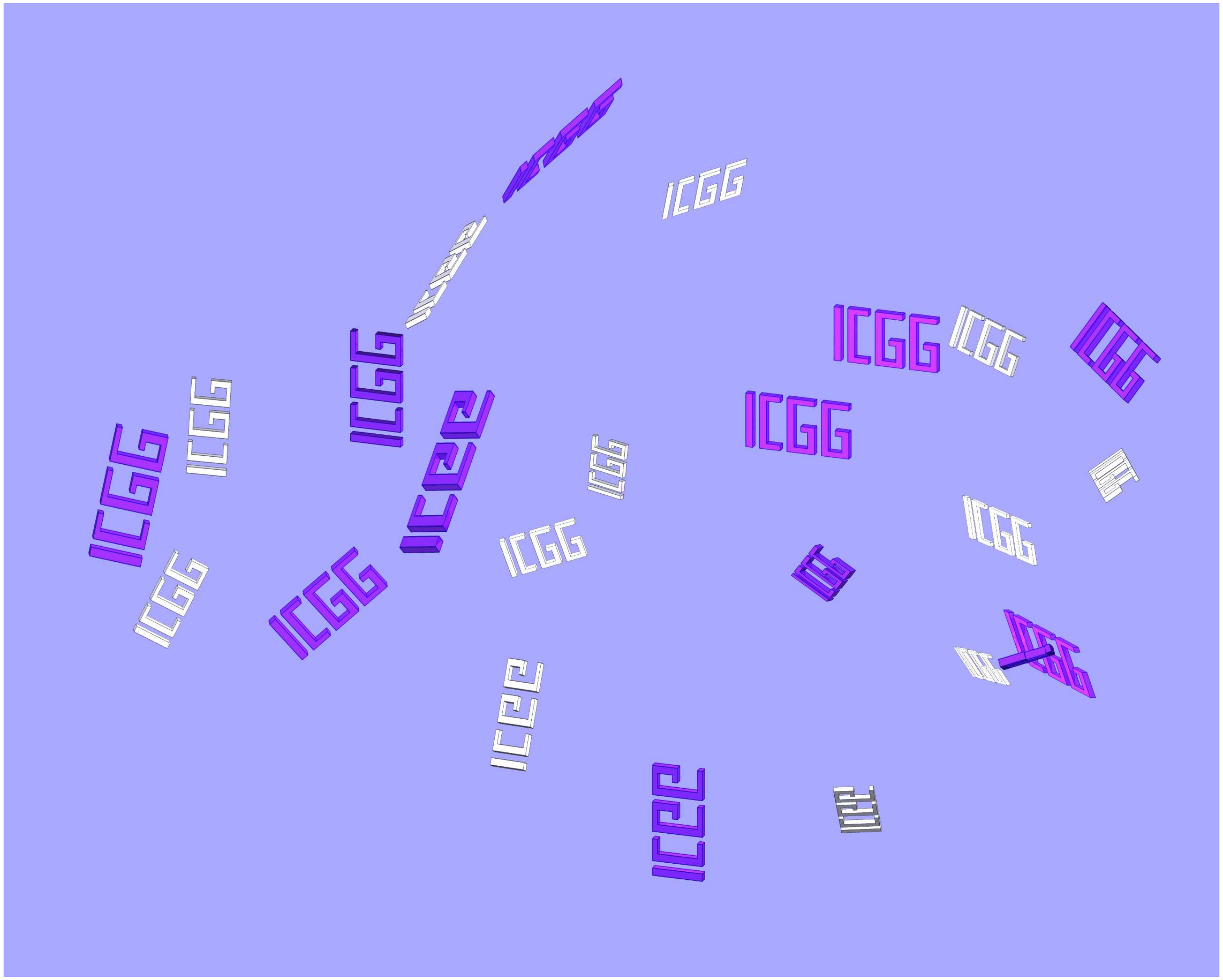


Or. flags determine Euclidean motions.

A combined scheme consists of subdivison and projection to the group (manifold) of Euclidean motions.

interpolating schemes preferred

$\Longrightarrow$ in the limit: a smooth motion interpolating a sequence of given poses



Or. flags determine Euclidean motions.

A combined scheme consists of subdivison and projection to the group (manifold) of Euclidean motions.

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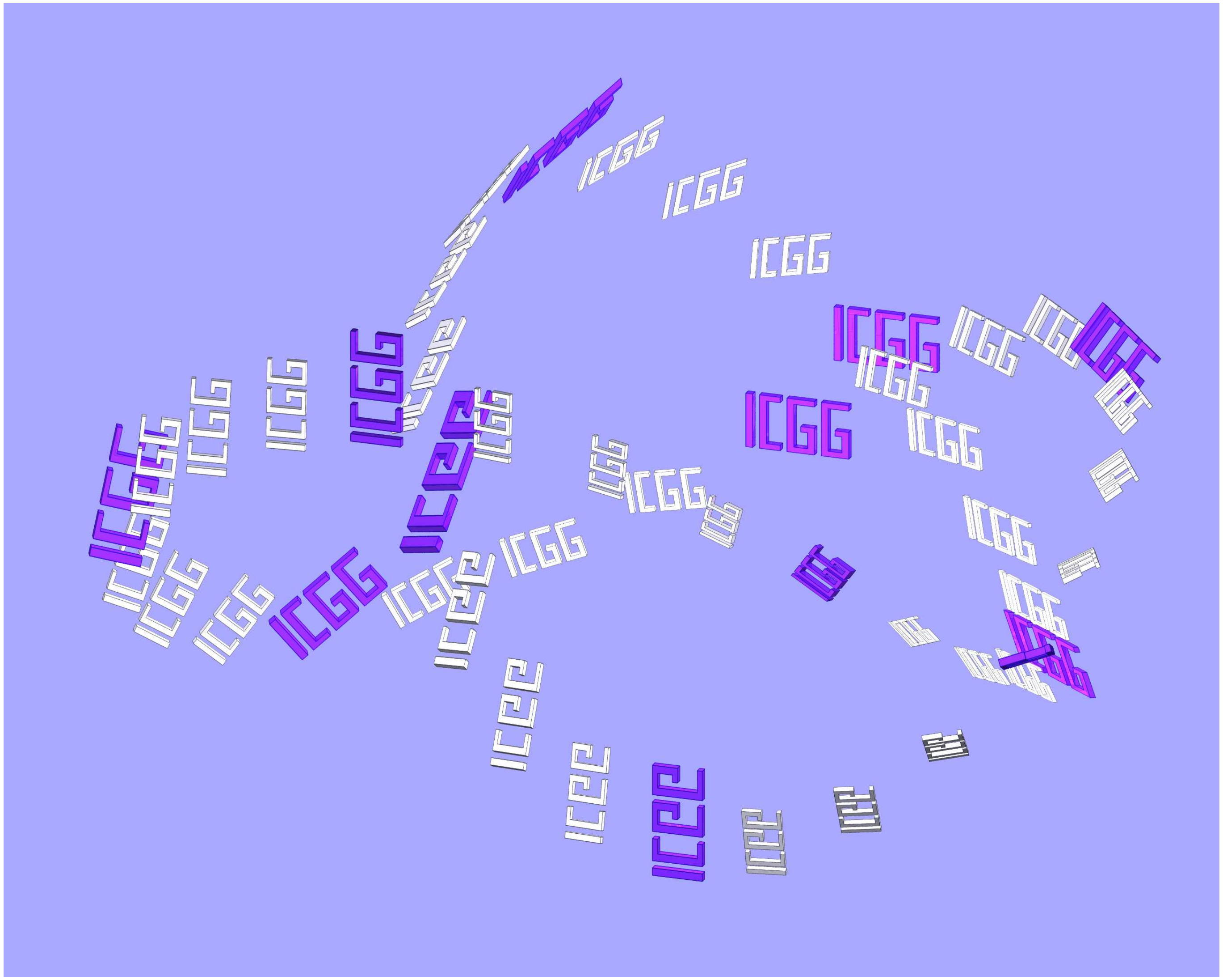

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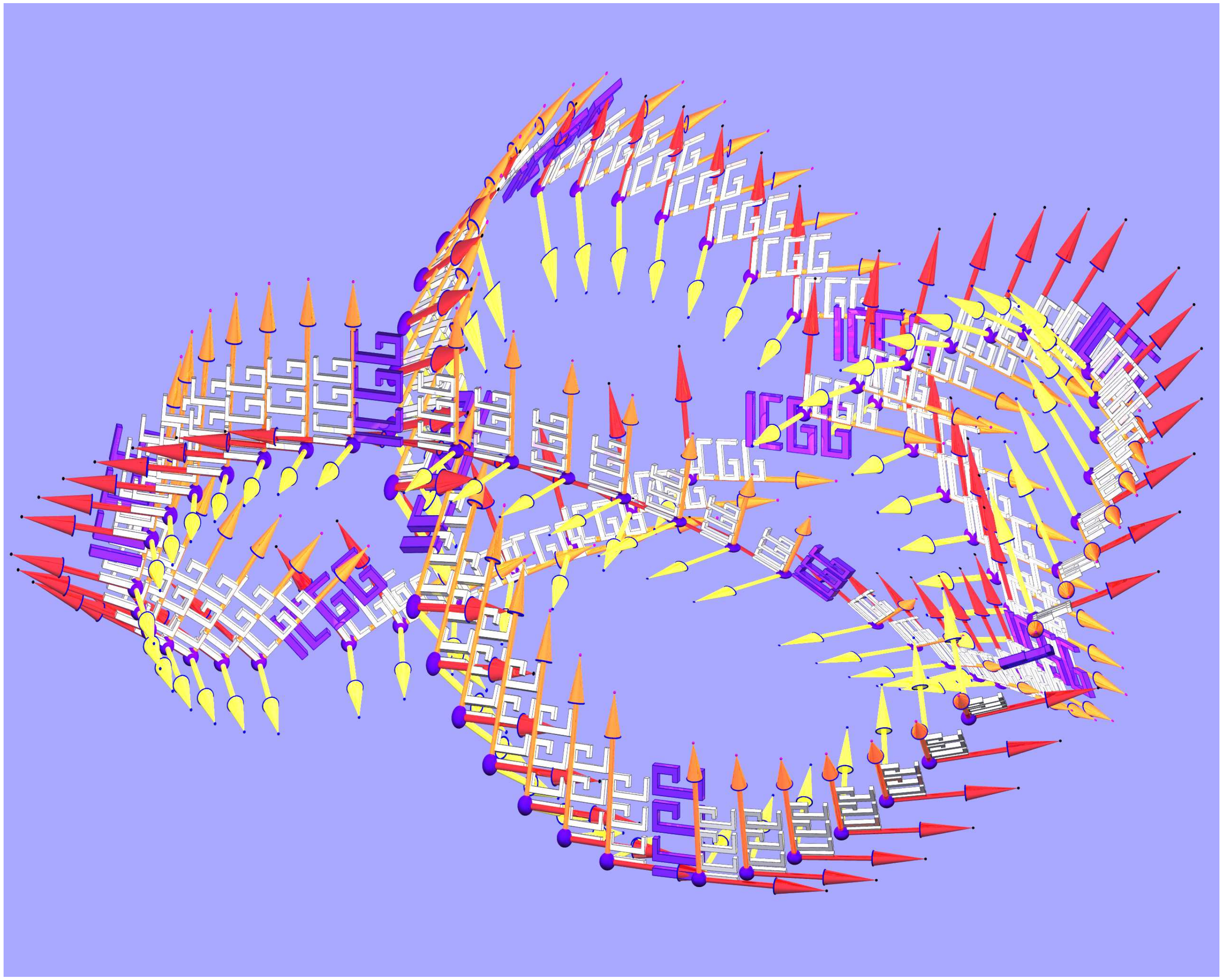
adapted and combined schemes [24,29,31]



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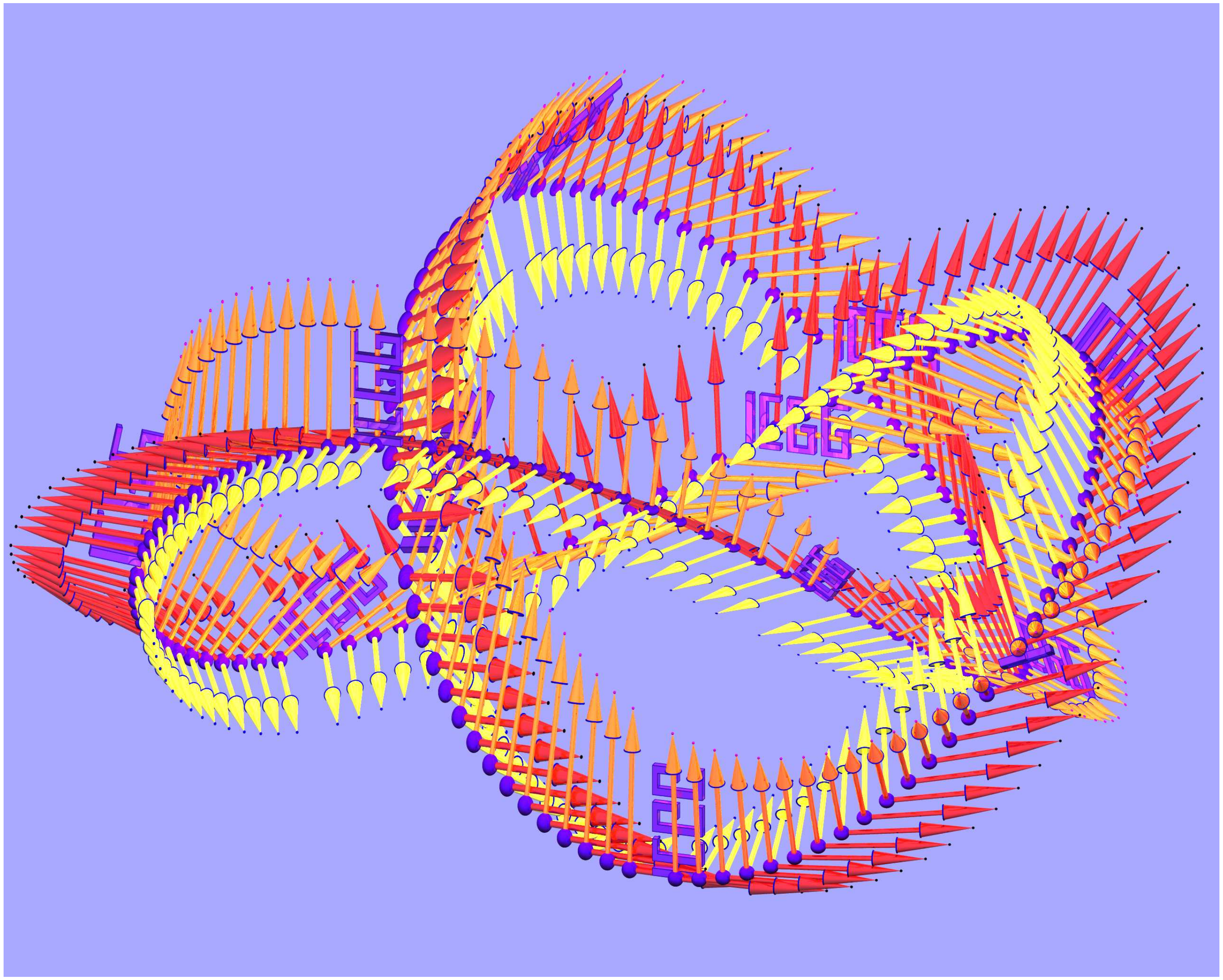

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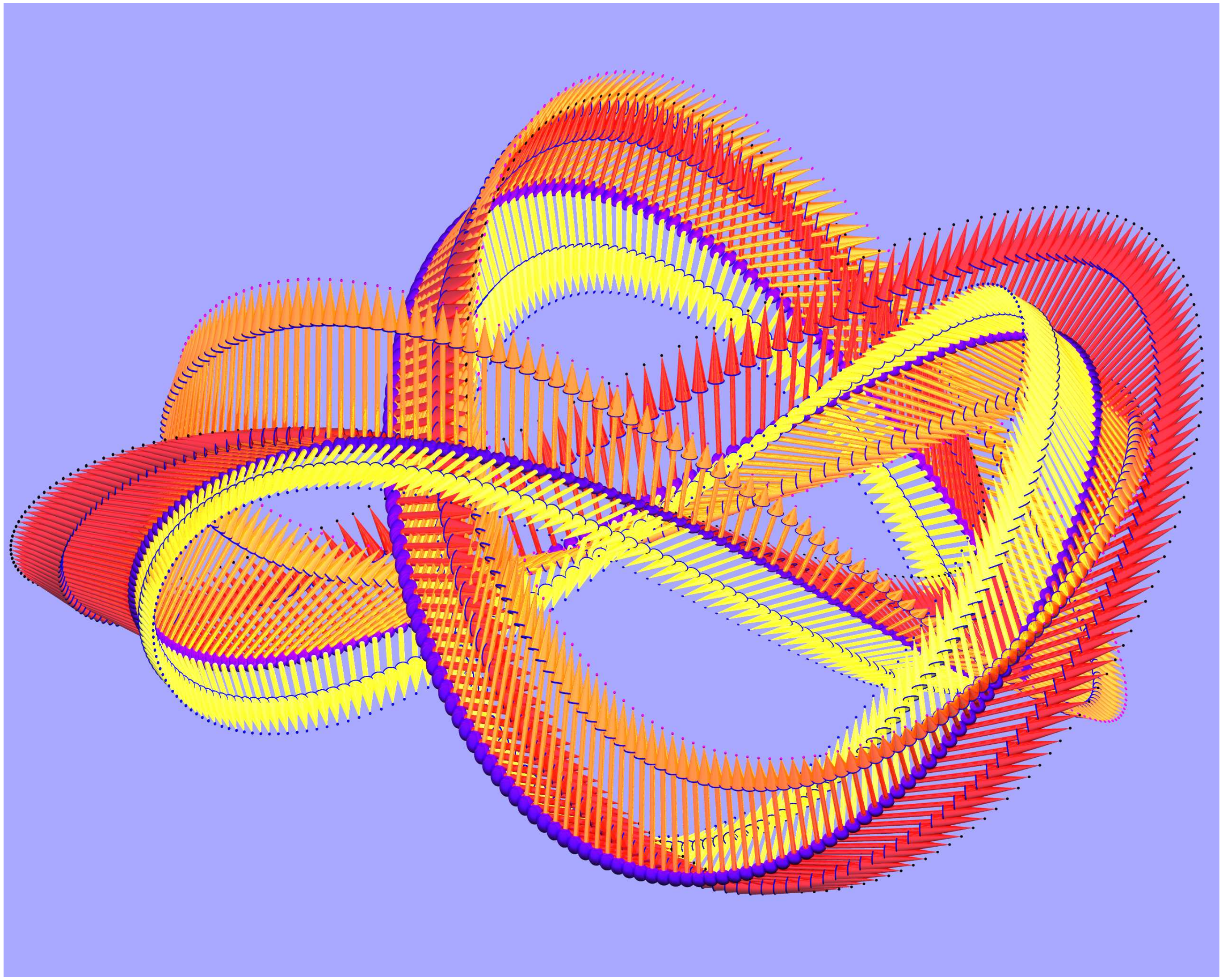

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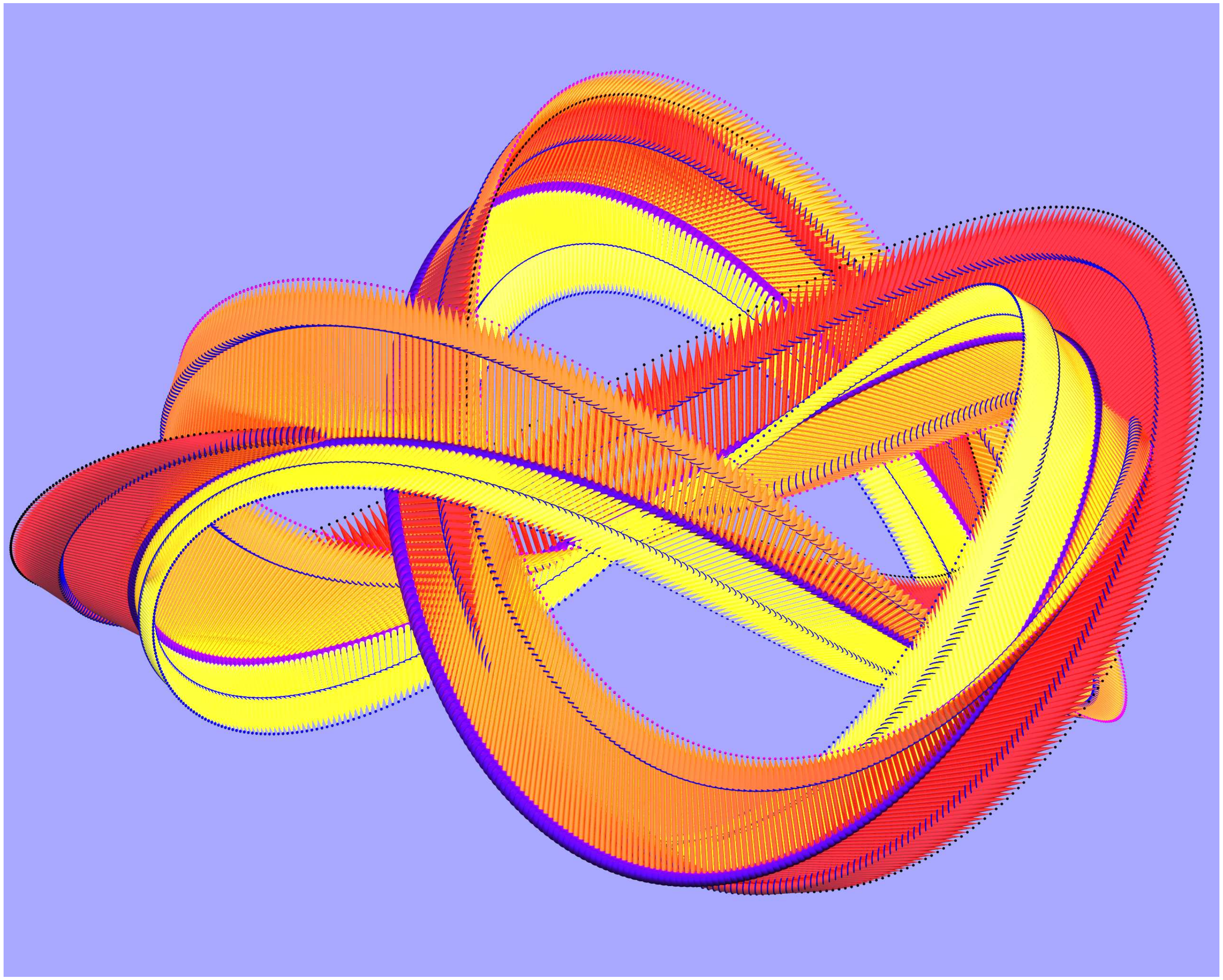

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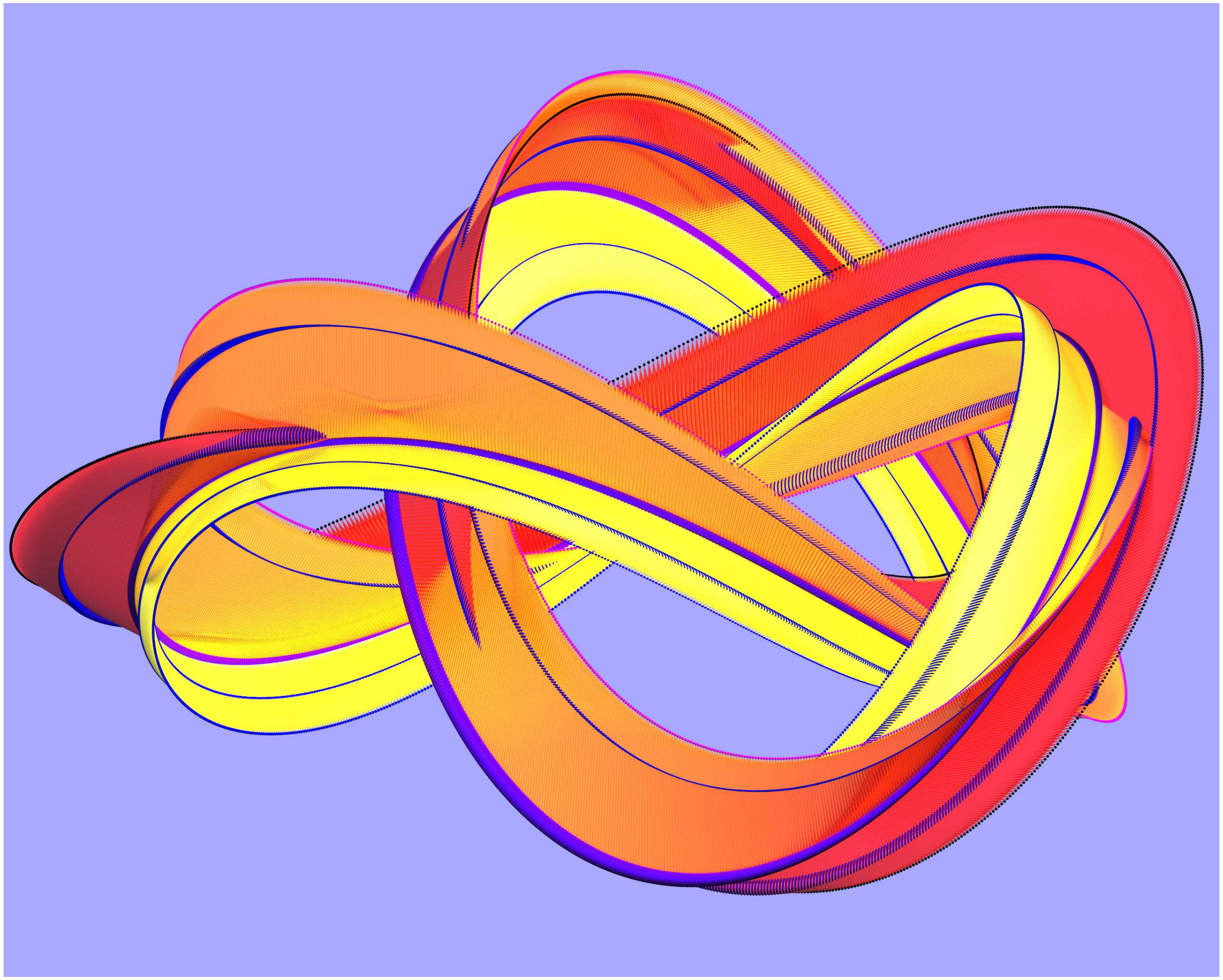

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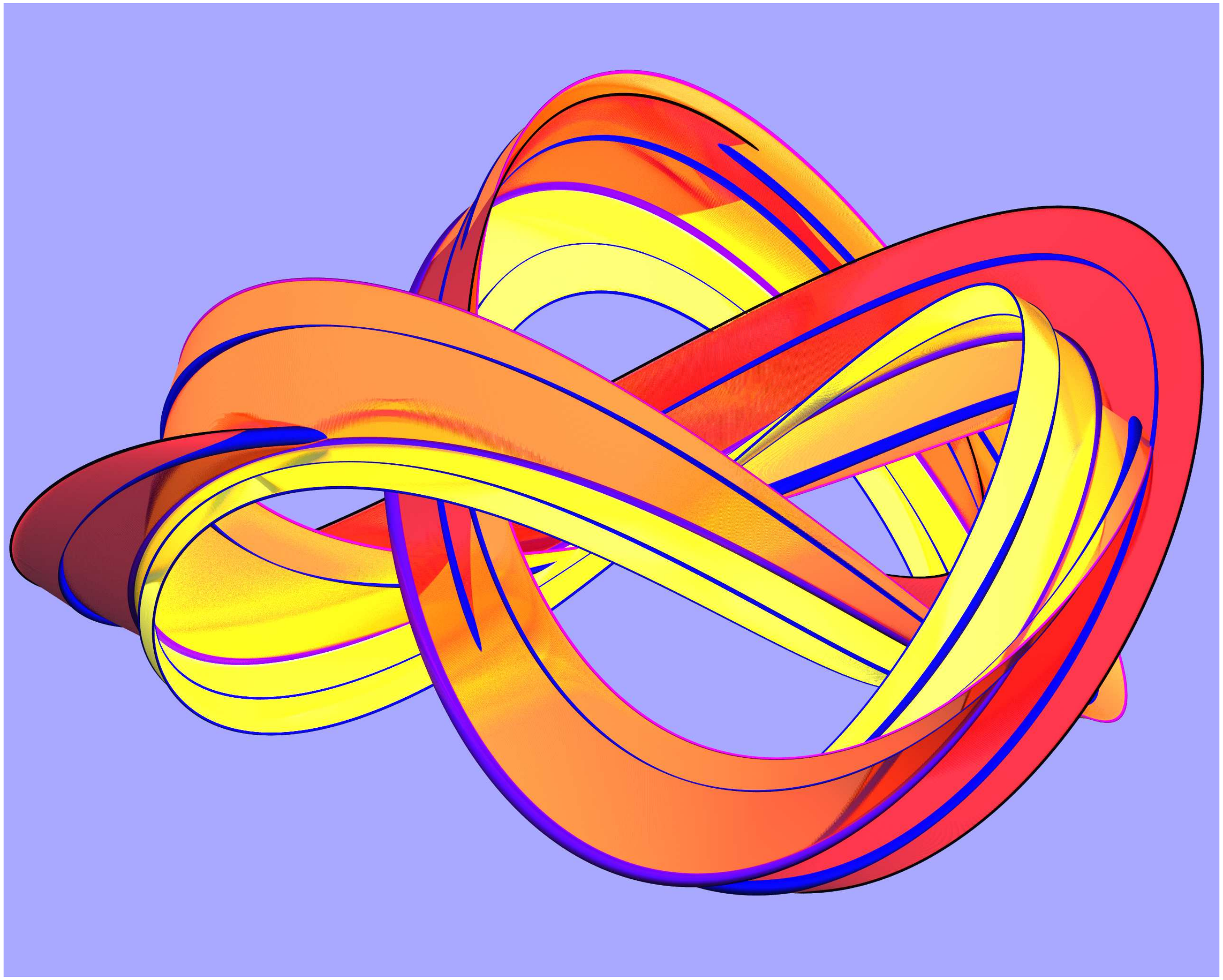

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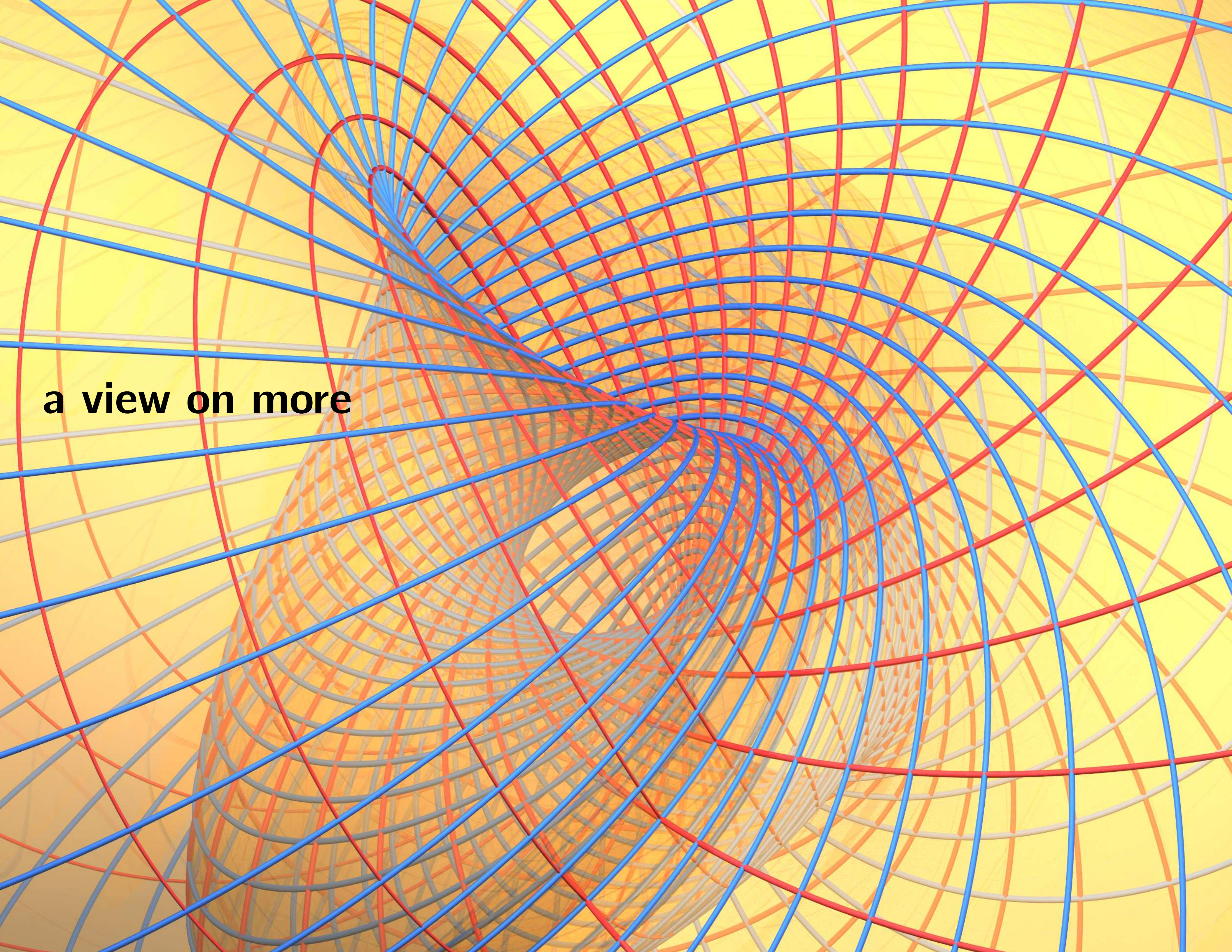
a view on more

- **donut $\mapsto$ coffee cup**

  Shape spaces serve as models for moving and deforming objects.
  A moving and deforming object is represented by a point in some shape space.
  The transformation donut $\mapsto$ coffee cup is a curve in shape space.
  The dimension of the shape space depends on the complexity (resolution) of the object.

- **really high dimensional spaces**

  Grassmannians $\mathrm{Gr}_{n,k}$, Veronese V_k^n, and Segre S_k^n manifolds occupy lots of space and serve as models for the geometries of k-dimensional subspaces in projective n-space, forms of degree n in k variables, products of and mappings between projective spaces.

- **flag manifolds, exterior algebras, . . .**

  . . . have applications in kinematics and physics.
  Geometry in Study's quadric serves curious phenomena: triality, . . .

we have seen and learned

- Complicated geometric objects can be represented by points.
- Relations between objects can be translated into metric properties of points in the model space.
- Computations become simple or even possible in higher dimensional spaces.
- Transformations of the original objects can be transferred to linear transformations in the model space.
- Everything should be linear, a vector space, . . .

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Attention, Interest, Patience, . . . !