Circumparabolas in Chapple's Porism

Boris Odehnal

University of Applied Arts,
Vienna, Austria

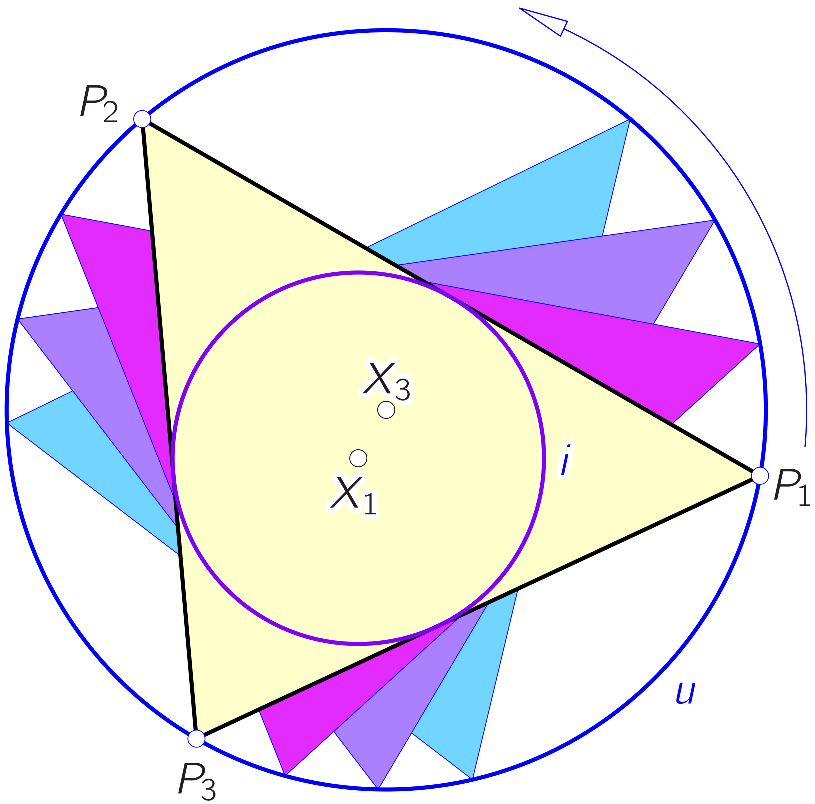
Dan Reznik

Data Science Consulting Ltd.,
Rio de Janeiro, Brazil

rough sketch of the talk

Chapple's porisms	conditions
parametrizations	rational, . . .
technical details	computation
circumparabolas	isogonal transformation
vertices, focal points	a septic and a quintic
envelopes of axes	Steiner cycloids
. . . and their envelopes	porisms between ellipses
traces of point	foci, vertices

Poristic triangle family - Chapple's porism



Each triangle $\Delta = P_1P_2P_3$ has an incircle i and a circumcircle u.

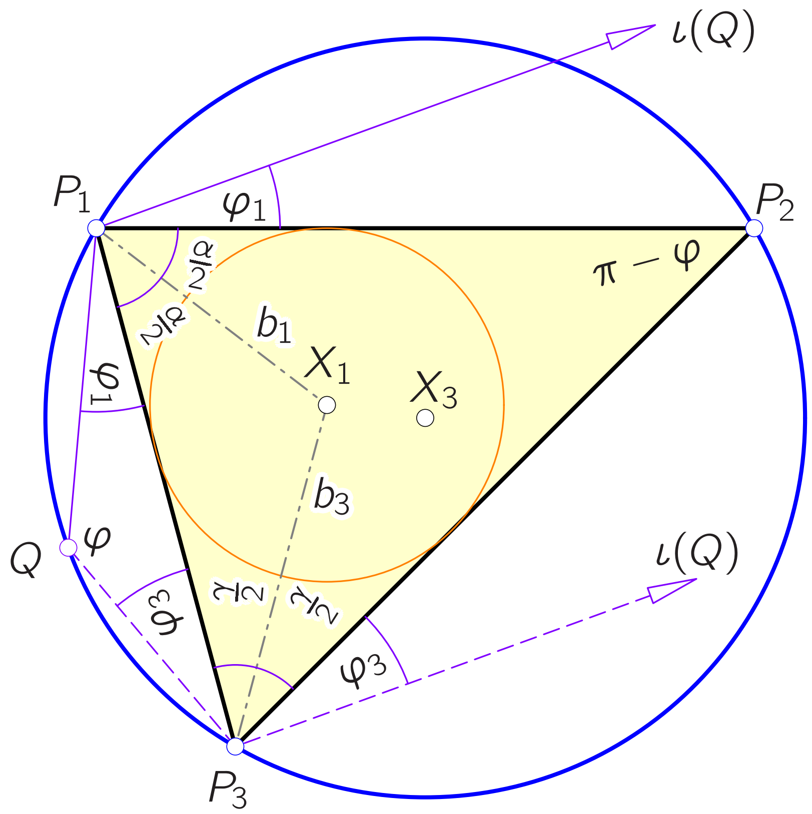
X_1, X_3 ... incenter, circumcenter
$d = \overline{X_1X_3}$ ... central distance [19,20]
r, R ... inradius, circumradius
Euler triangle equation $d^2 = R^2 - 2rR$

poristic triangle family:
$\forall P_1 \in u \exists P_{2,3} \in u : [P_i, P_j]$ tangent to i
Two such circles always allow for the construction of a smooth one-parameter family of interscribed triangles. [1,4,5,18,19,21]

Traversing the poristic triangle family is not a rigid body motion.
Traces of centers, numerical invariants, ... derived in: [8–11,15,16,17,22–24].

Isogonal conjugation and circumparabolas

[14,19,20]



isogonal conjugation:

a quadratic Cremona transformation

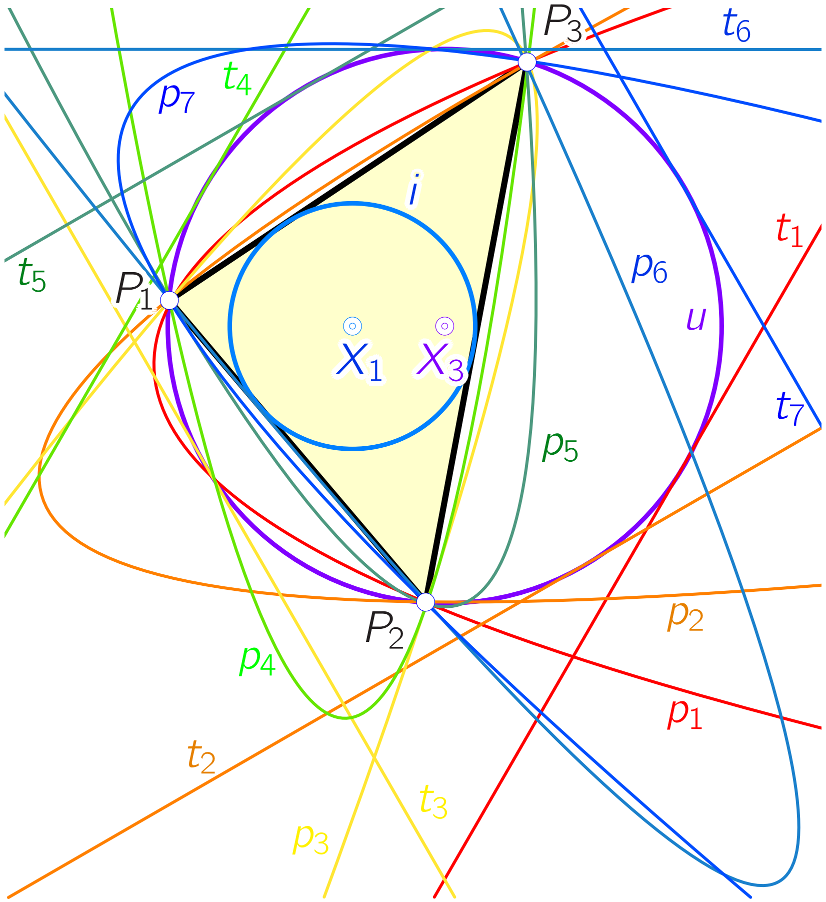
Reflections of the Cevians of some point P in the (interior) angle bisectors concur in the isogonal conjugate $\iota(P)$ of P.

The isogonal images of points Q on u are the ideal points (points at infinity).

The isogonal image of a straight line is a conic (in general).

$\Longrightarrow$ The isogonal images of u's tangents are circumparabolas of any triangle interscribed between i and u.

Circumparabolas



The isogonal images $\iota(Q)$ of points Q on u are the ideal points (points at infinity).

The circumparabolas p_i of a triangle $\Delta = P_1P_2P_3$ are the isogonal images of the tangents t_i of Δ's circumcircle u.

For each triangle pose there are three degenerate parabolas (ideal points in the direction of the sides of the triangle).

Computational approach towards porisms [8,9,14,21]

circumcircle u, incircle i

$$u: \ (x-d)^2+y^2=R^2, \ i: \ x^2+y^2=r^2, \qquad R>r>0, \ d^2=R^2-2rR$$

rational parametrization of u

$$\mathbf{u}=(R\cos\tau+d, R\sin\tau) \ \tau\in\mathbb{R}$$

$$\cos\tau=\frac{1-T^2}{1+T^2}, \quad \sin\tau=\frac{2T}{1+T^2}, \ T\in\mathbb{R}$$

poristic triangle family $\Delta=P_1P_2P_3$

$P_1=\mathbf{u}(T)$ rational, P_2, P_3 not rational

$Q=\mathbf{u}(U)$ pivot for parabola, $\qquad U\in\mathbb{R}$

$A=\iota(Q)$ ideal point of parabola (isogonal image of Q)

yields the ideal points of parabolas

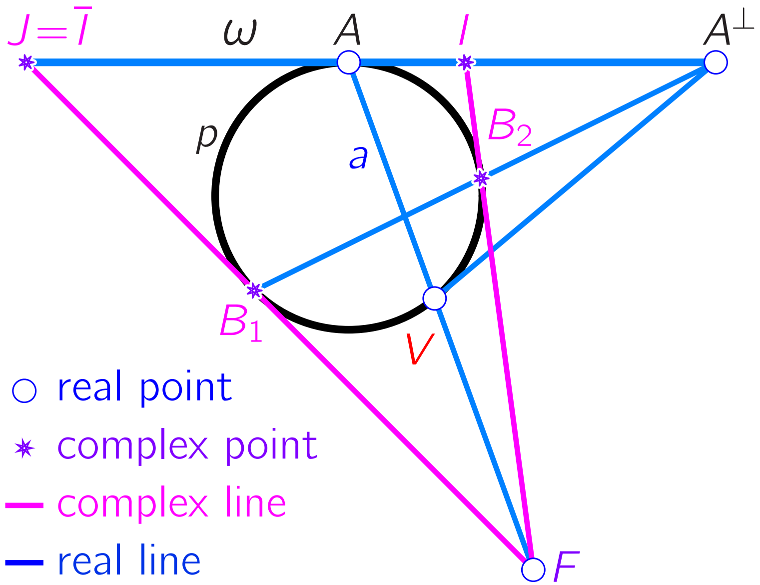
$A=0:T^3\delta^2-\delta(\delta+2\sigma)T^2U-\sigma(2\delta+\sigma)T+\sigma^2U:-T^3U\delta^2-\delta(\delta+2\sigma)T^2+\sigma(2\delta+\sigma)TU+\sigma$

(homogeneous coordinates, $\sigma=R+d$, $\delta=R-d$)

...rational as well as many other things!

What are foci and vertices? [6,7,14]

The construction of foci and vertices shows the simplest way to their computation.

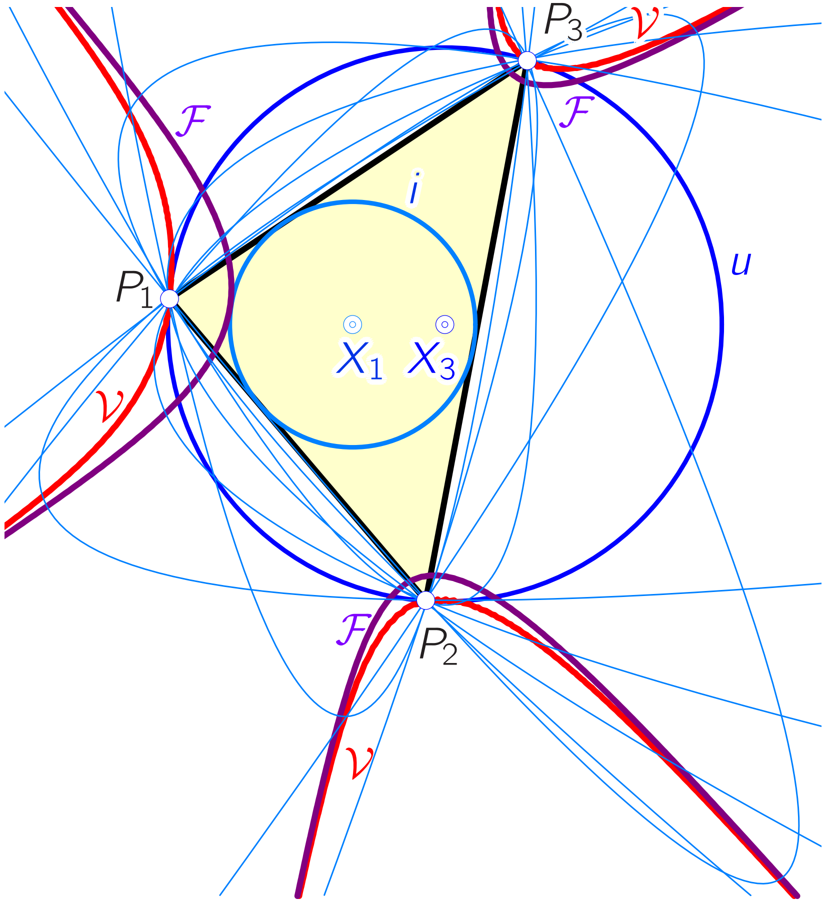


A	parabola's point at infinity
$A^{\perp}$	absolute polue of A = vertex tangent's point at infinity
a	parabola's axis
V	parabola's vertex
I, $J = \bar{I}$	absolute points of Eucl. geom.
$[I, B_1]$, $[J, B_2]$	parabola's isotropic tangents
F	parabola's focus

F is real since

$F = [I, B_1] \cap [J, B_2] = [I, B_1] \cap \overline{[I, B_1]}$.

Loci of vertices and foci of circumparabolas



Independent of porisms:

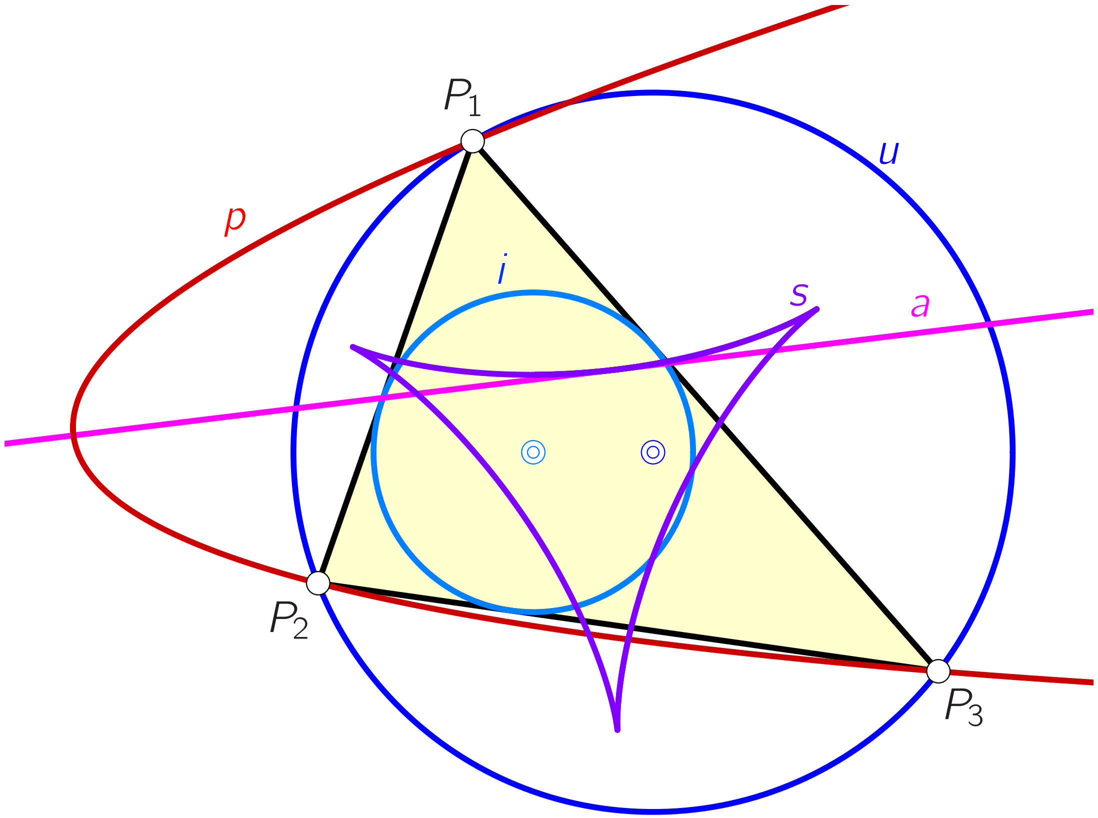
The vertices of all circumparabolas lie on a septic curve $\mathcal{V}$. [13]

The foci of all circumparabolas lie on a quintic curve $\mathcal{F}$. [12]

In connection with porisms: The manifold of all circumparabolas of the triangles in a poristic family is a quadratic cone in the Veronese manifold.

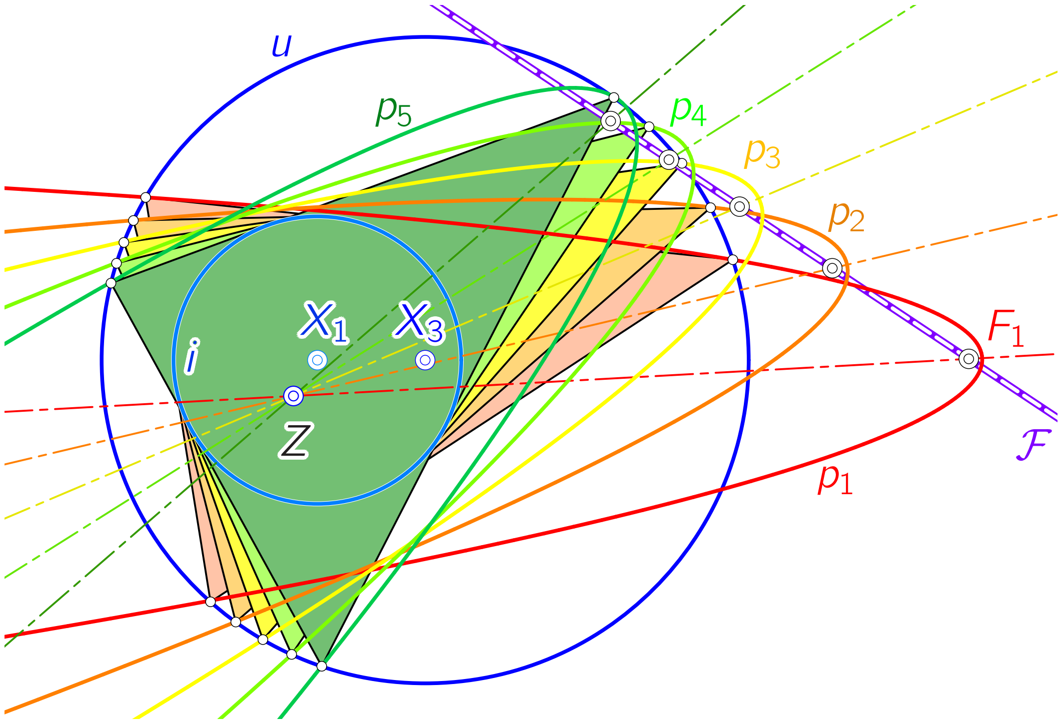
Axes of circumparabolas

[2,3,6,7,25]



While the circumparabola p traverses the family of all circumparabolas (variable U) of a fixed triangle in the poristic family (fixed T), its axes envelop a Steiner cycloid s.

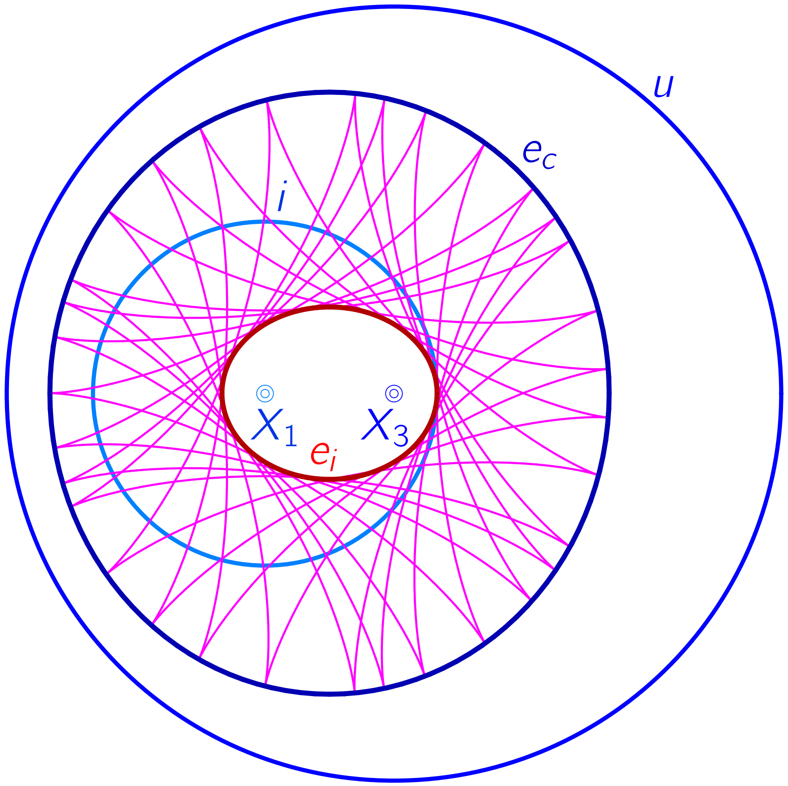
Focal trace & envelope of axes



Over poristic triangles (variable T), if U (and thus, Q) is fixed, the foci of circumparabolas (for fixed U) move on a straight line.

Over poristic triangles (variable T), if U (and thus, Q) is fixed, the axes pass through a fixed point Z.

Envelope of the Steiner cycloids [2,3,6,7,25]

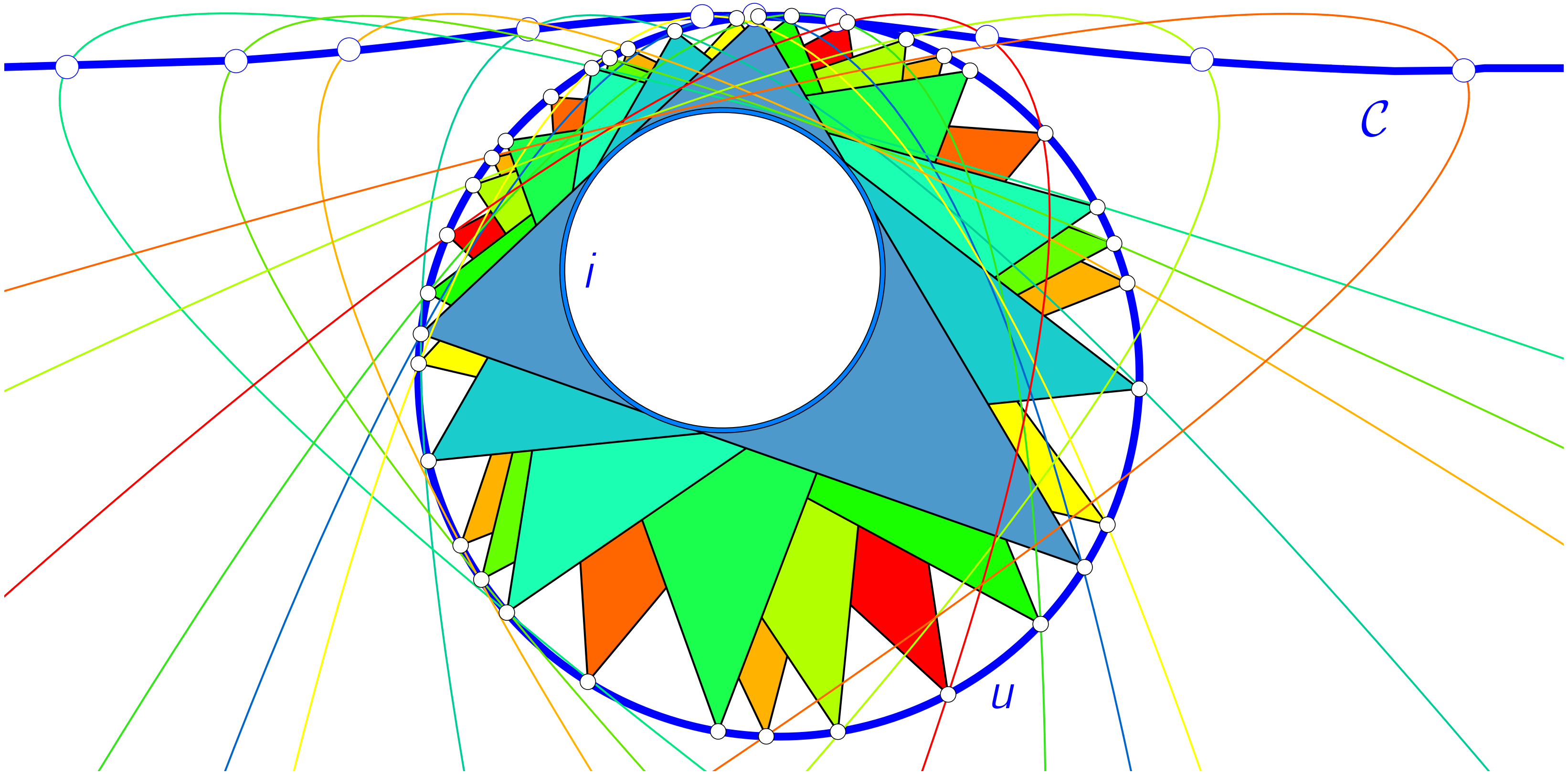


Over poristic triangles, the Steiner cycloids as envelopes of the axes of the circumparabolas envelop two ellipses e_i and e_c. The triangles' incenter X_1 and circumcenter X_3 are the real foci of e_i. X_{1364} is the center of e_i. [19,20]

This yields another porism:

If it is possible to draw a Steiner hypocycloid with its three cusps on e_c and thrice tangent to e_i for one choice of a cusp on e_c, then it is possible for any choice.

Vertices of circumparabolas



The poristic trace of the vertices of the circumparabolas is a rational cubic $\mathcal{C}$.

The cubics' isolated nodes are located on the ellipse e_i. [2,3,6,7,25]

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Thank You For Your Attention!