Permutation Cubics

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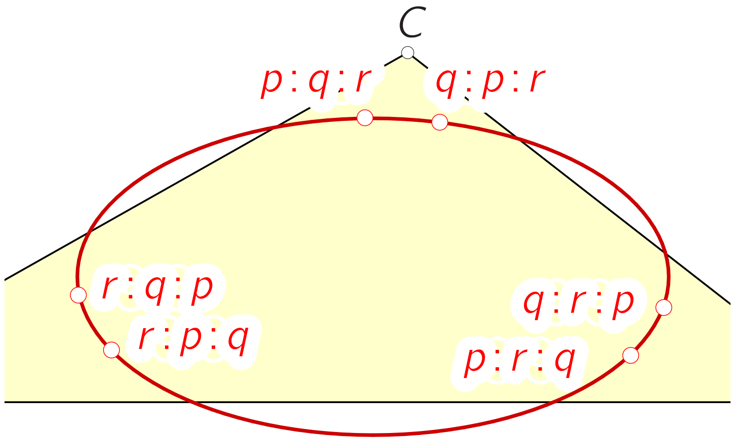
rough sketch of the talk

permutation points	6 conconic points, 12 "concubic" points
a rational cubic	$\mathcal{K}_{228}$ with 8 triangle centers on it
automorphic collineations	discrete group
singular cubics	only degenerate ones
inflection points	three real, common to all cubics
triangle centers as pivot points	groups of centers on cubics
isotomic instead of isogonal conjugation	

Permutation points - conics - cubics

We use homogeneous trilinear coordinates in the plane of the triangle $\Delta = ABC$: $A = 1:0:0$, $B = 0:1:0$, $C = 0:0:1$, unit point $X_1 = 1:1:1$ (X_1 = incenter [3,4]).

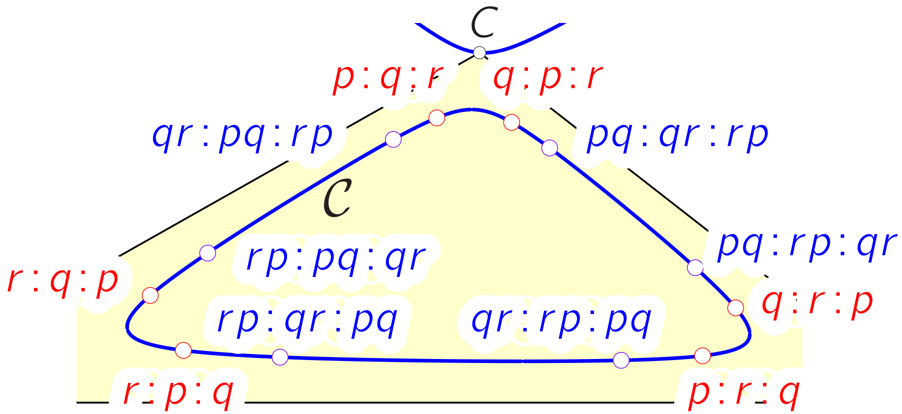
Permuting the trilinears p:q:r of a point P yields six permutation points. [5]



The permutation point lie in the permutation conic

$$\sum^{\circlearrowleft} pq \sum^{\circlearrowleft} \xi^2 = \sum^{\circlearrowleft} p^2 \sum^{\circlearrowleft} \xi\eta.$$

Permutation points of P and its isogonal conjugate $\iota(P)$ are 12 points.

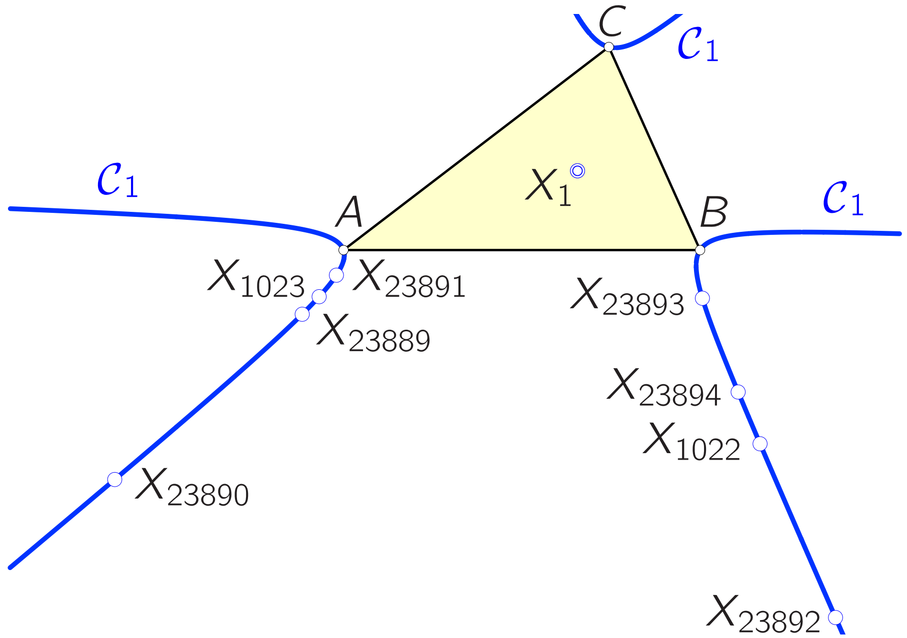


These 12 points lie on a triangle cubic, the permutation cubic

$$\mathcal{C}_P : \sum^{\circlearrowleft} pq(p+q)\xi\eta\zeta = pqr \sum^{\circlearrowleft} \xi\eta(\xi+\eta).$$

Only one is already known!

Choose $p : q : r = 1 : 1 : 1$, *i.e.*, X_1 is the pivot point ...



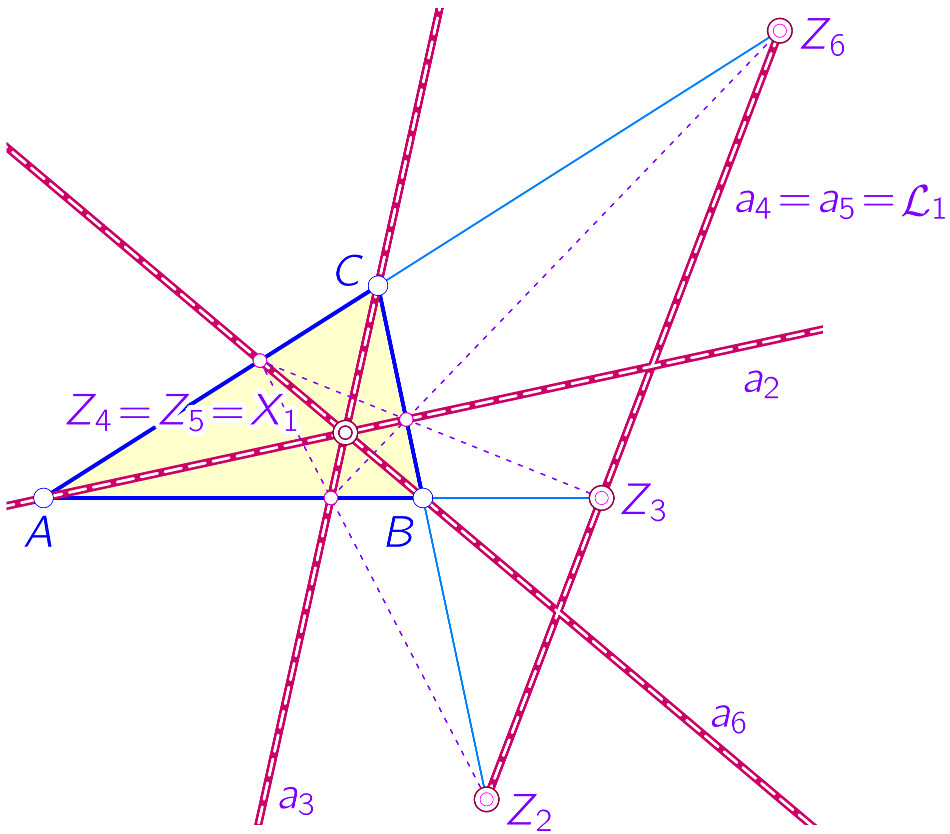
... yields the only rational cubic $\mathcal{C}_1$ with the equation:

$$\mathcal{C}_1 : \; 6\xi\eta\zeta - \sum_{\circlearrowleft} \xi\eta(\xi + \eta) = 0.$$

It has an acnode at X_1 and
shows up as $\mathcal{K}_{228}$ in B. Gibert's list [1],
isogonal circum-conico-pivotal cubic,
contains triangle centers X_i with
$i = 1, 1022, 1023, 23889 - 23894$.

isogonal pairs: (1022,1023), (23889,23894), (23890,23893), (23891,23892)

Discrete group of automorphic collineations

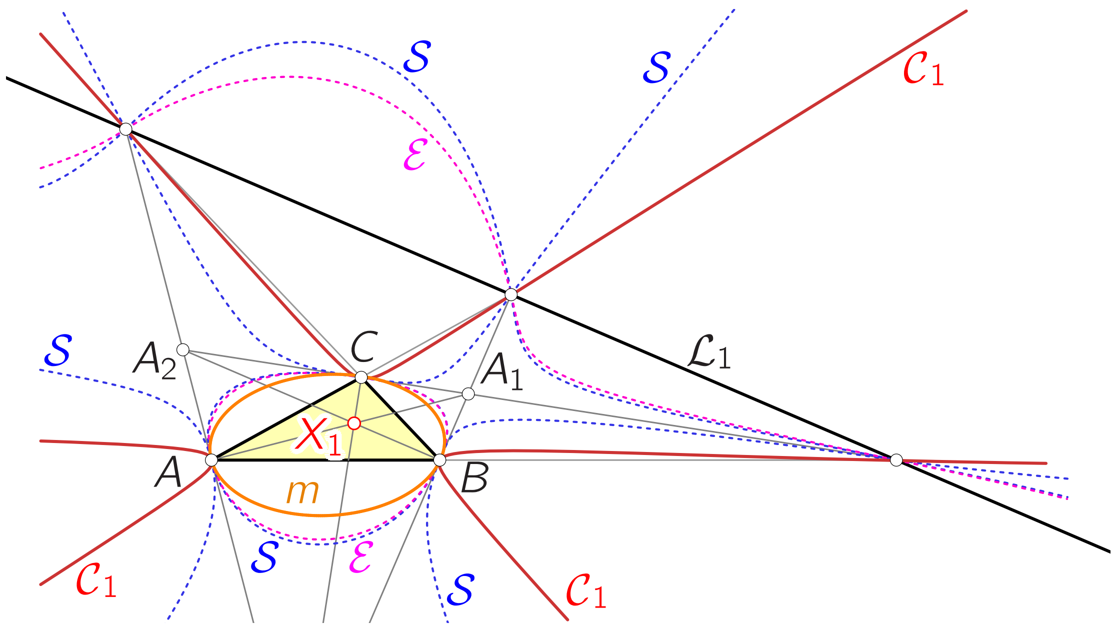


The 6 permutations of the coordinates define 6 automorphic collineations of the permutation (conics and) cubics.

The six collineations constitute a group.

Fixed points and fixed lines form a self-dual figure which is not a configuration.

Singular permutation cubics



Singular permutation cubics occur if the pivot point is chosen on:

(i) $\mathcal{C}_{44} = m \cup \mathcal{L}_1$ and $\mathcal{C}_1$

m: $\circlearrowleft\!\!\sum \xi\eta = 0$ Mandart ellipse

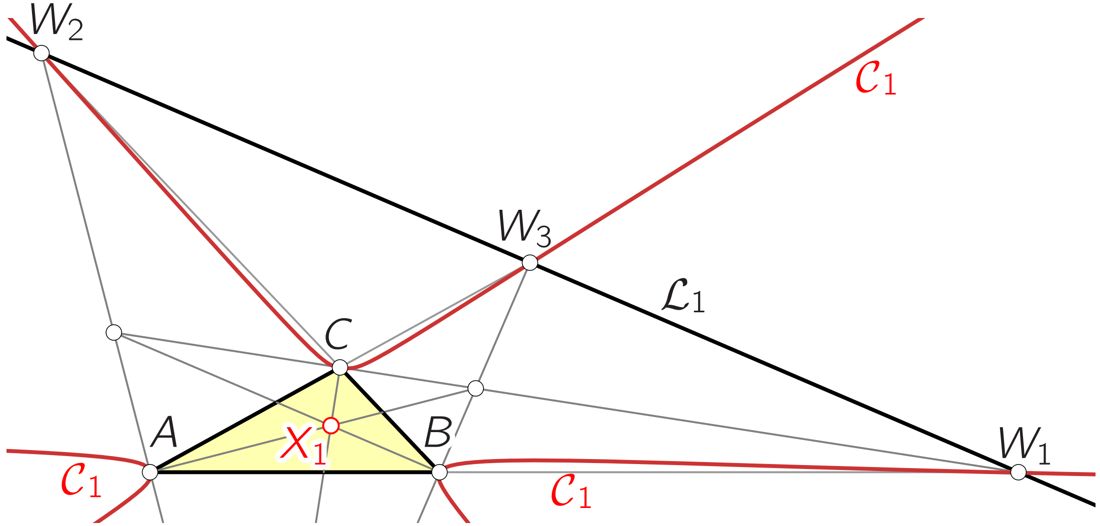
$\mathcal{L}_1$: $\circlearrowleft\!\!\sum \xi = 0$ antiorthic axis

(ii) the sides lines of the excentral triangle $\Delta_e = A_1A_2A_3$

(iii) the sides lines of the base triangle Δ

The elliptic cubic $\mathcal{E}$ and the elliptic sextic $\mathcal{S}$ (with three tacnodes at A, B, C and ordinary double points at $W_1 = [A, B] \cap \mathcal{L}_1 = 1 : -1 : 0, \ldots$) are artefacts of the computation.

Inflection points



The permutation cubics share the three real inflection points

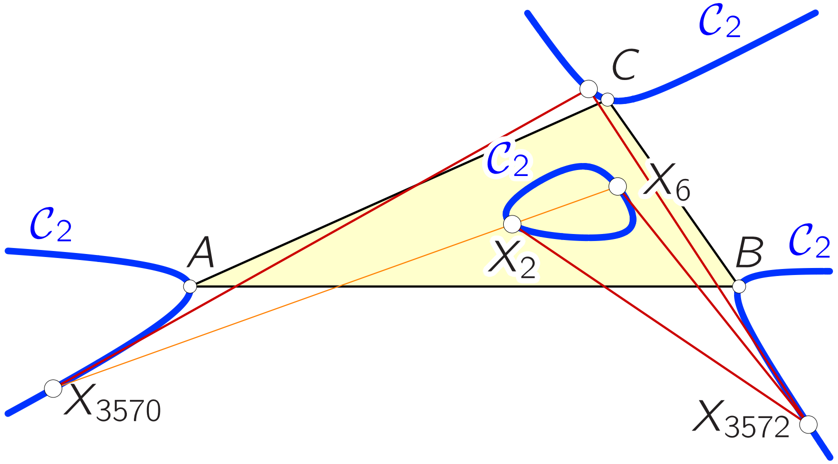
$$W_1, W_2, W_3$$

which are the intercepts of the triangle sides and the antiorthic axis $\mathcal{L}_1$.

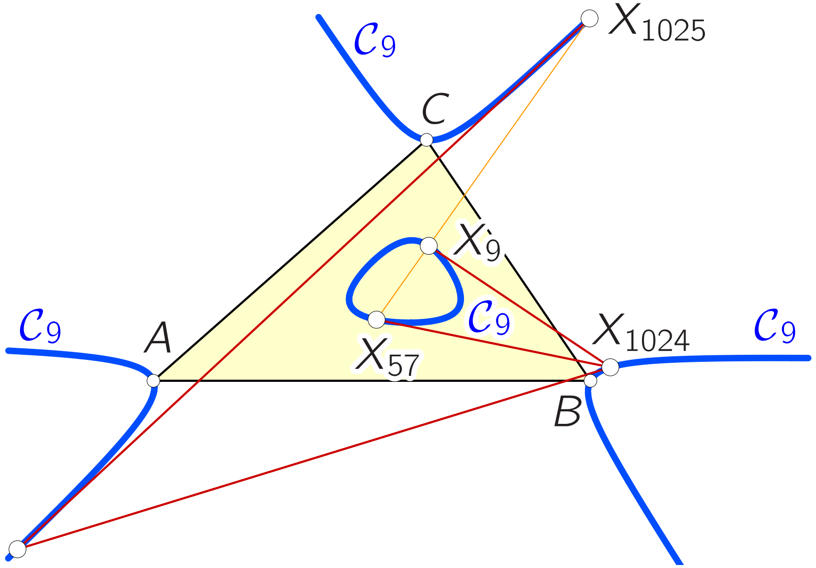
Behave like the distance product cubics. [6]

Triangle centers as pivot points

With each triangle center X_i its isogonal conjugate $\iota(X_1)$ is also contained in $\mathcal{C}_i$. Only a small number of cubics contains more than one pair of isogonal conjugates.

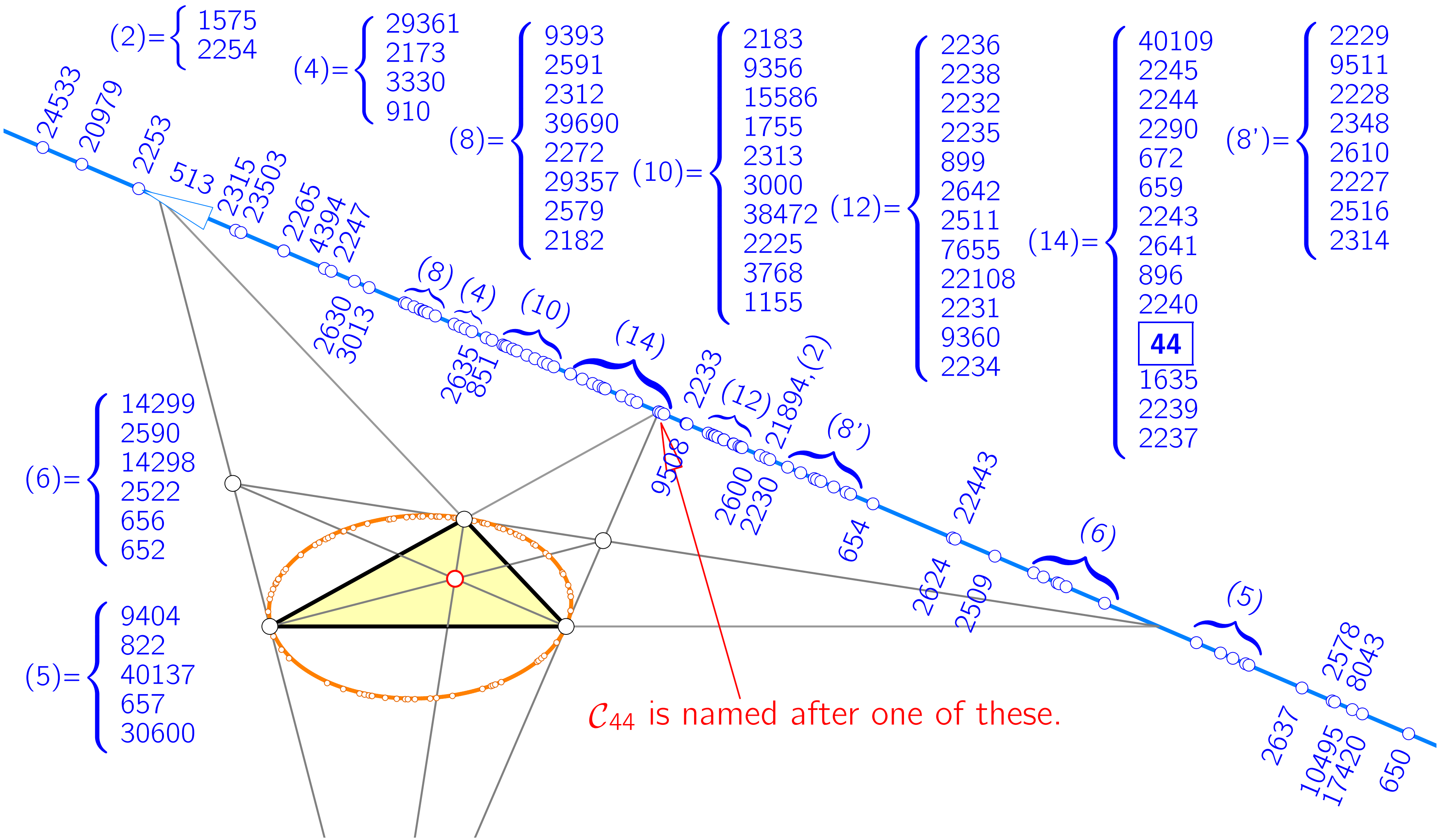


$$\mathcal{C}_2 = \mathcal{C}_6 = \mathcal{C}_{3570} = \mathcal{C}_{3572}$$



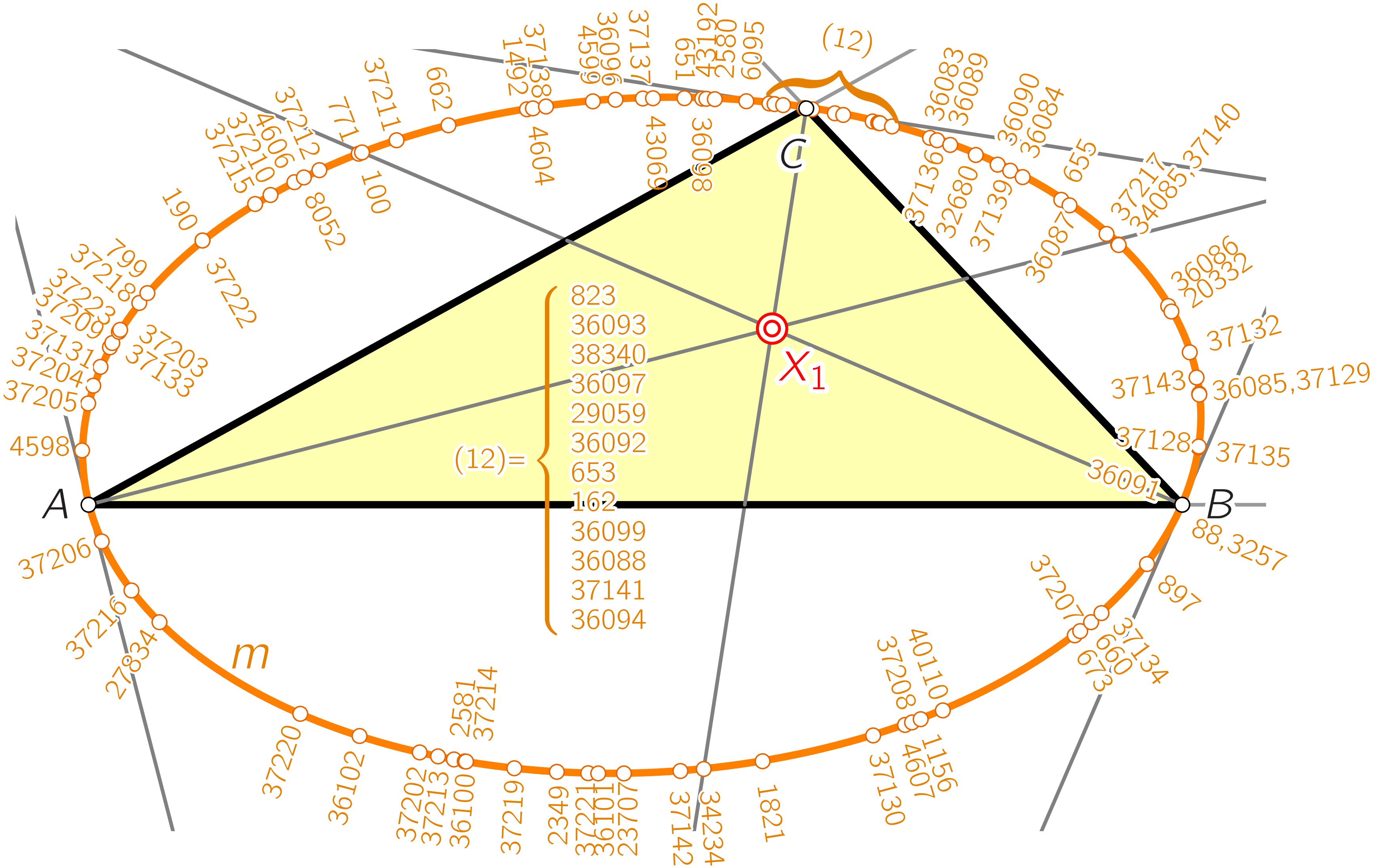
$$\mathcal{C}_9 = \mathcal{C}_{57} = \mathcal{C}_{1024} = \mathcal{C}_{1025}$$

Triangle centers that define singular permutation cubics I



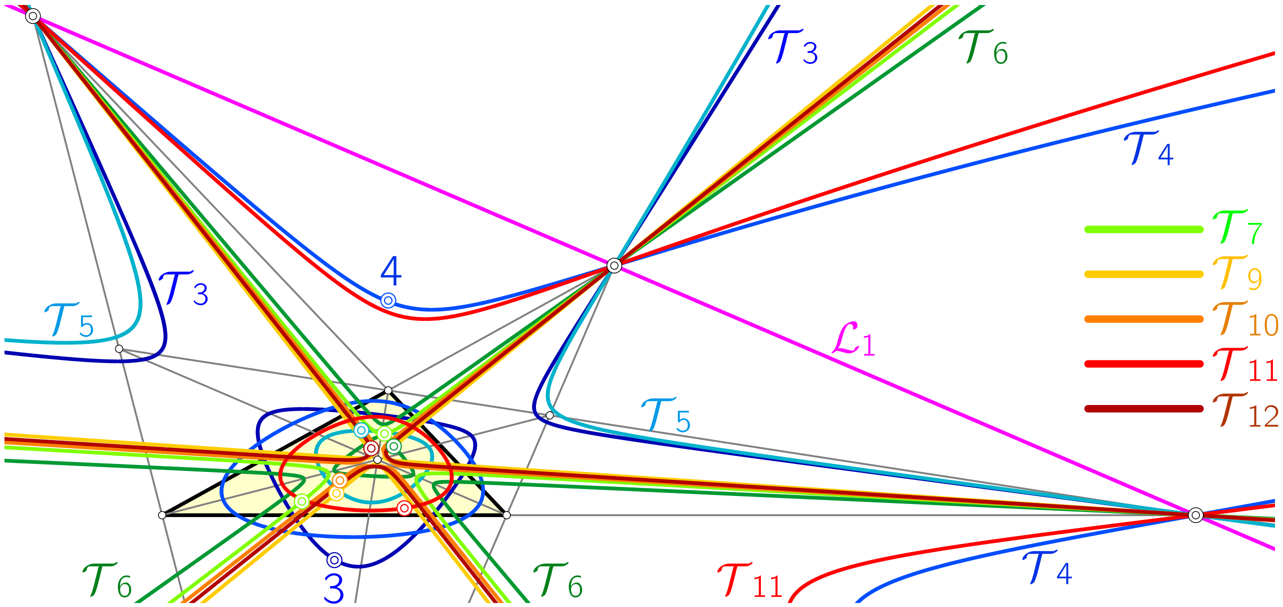
distribution of triangle centers on the antiorthic axis $\mathcal{L}_1 \subset \mathcal{C}_{44}$

Triangle centers that define singular permutation cubics II



distribution of triangle centers on the Mandart ellipse $m \subset \mathcal{C}_{44}$

Isotomic conjugation instead of isogonal conjugation



The isogonal conjugation can be replaced with any quadratic Cremona transformation.

for example:
the isotomic conjugation

$\mathcal{T}_i$ . . . permutation cubic defined by the triangle center X_i

Permutation cubics defined with the isogonal conjugation are self-isogonal.

Permutation cubics defined with the isotomic conjugation are NOT self-isotomic.

Literature

[1] B. Gibert: *Cubics in the Triangle Plane.* Available at: `https://bernard-gibert.pagesperso-orange.fr/index.html`. Last accessed on April 1, 2022.

[2] G. Glaeser, H. Stachel, B. Odehnal: *The Universe of Concis.* From the ancient Greeks to 21st century developments. Springer-Spektrum, Springer-Verlag, Heidelberg, 2016.

[3] C. Kimberling: *Triangle Centers and Central Triangles.* (Congressus Numerantium Vol. 129) Utilitas Mathematica Publishing, Winnipeg, 1998.

[4] C. Kimberling: *Encyclopedia of Triangle Centers*. Available at: `http://faculty.evansville.edu/ck6/encyclopedia`. Last accessed on April 1, 2022.

[5] C. Kimberling: *Permutation Ellipses.* J. Geom. Graphics **24**/2 (2020), 233–247.

[6] B. Odehnal: *Distance Product Cubics.* KoG **24** (2020), 29–40.

Thank You For Your Attention!