# Beyond the Nine-Point Conic

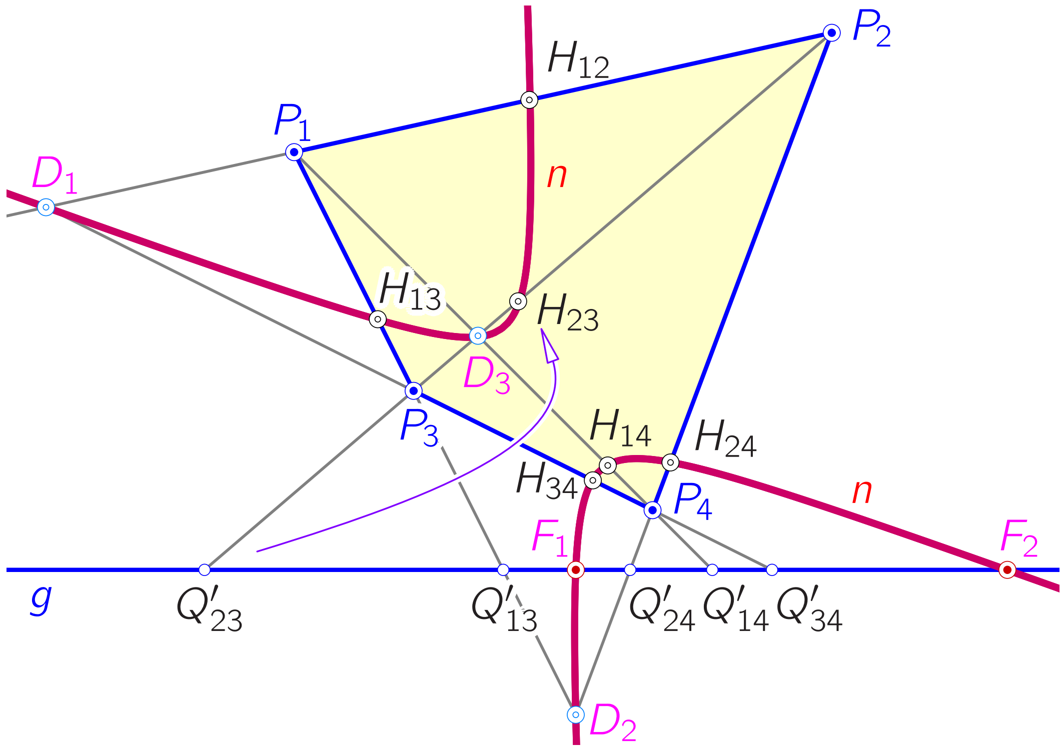
Boris Odehnal

University of Applied Arts Vienna

## rough sketch of the talk

| | |
|---|---|
| 9 points on a single conic | sometimes 11 |
| 10 points on a single conic | in any case |
| proof | analytical |
| superordinate standpoint | relations between 9-point & 10-point conic |
| cyclic quadrilaterals | equilateral hyperbolas & strophoids |
| in the hyperbolic plane | only an 8-point conic |

## Nine points on a single conic




Here (↑), we have an 11-point conic.

complete quadrangle $\mathcal{Q} = P_1 P_2 P_3 P_4$

a "generic" straight line $g$

harmonic conjugates

$H_{ij}$ of $Q'_{ij} = [P_i, P_j] \cap g$

$(i, j) \in \{(1, 2), (1, 3), \ldots, (3, 4)\}$

diagonal points $D_1$, $D_2$, $D_3$ of $\mathcal{Q}$

$F_1, F_2$ fixed points of the Desargues involution induced by $\mathcal{Q}$ on $g$

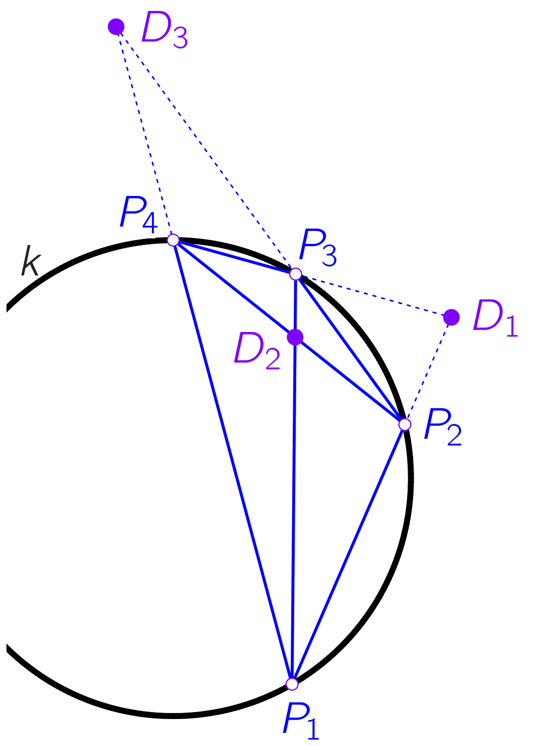
9 conconic points:

$H_{12}, H_{13}, \ldots, D_1, D_2, D_3$

if we are lucky $\Longrightarrow$ 11 conconic points:

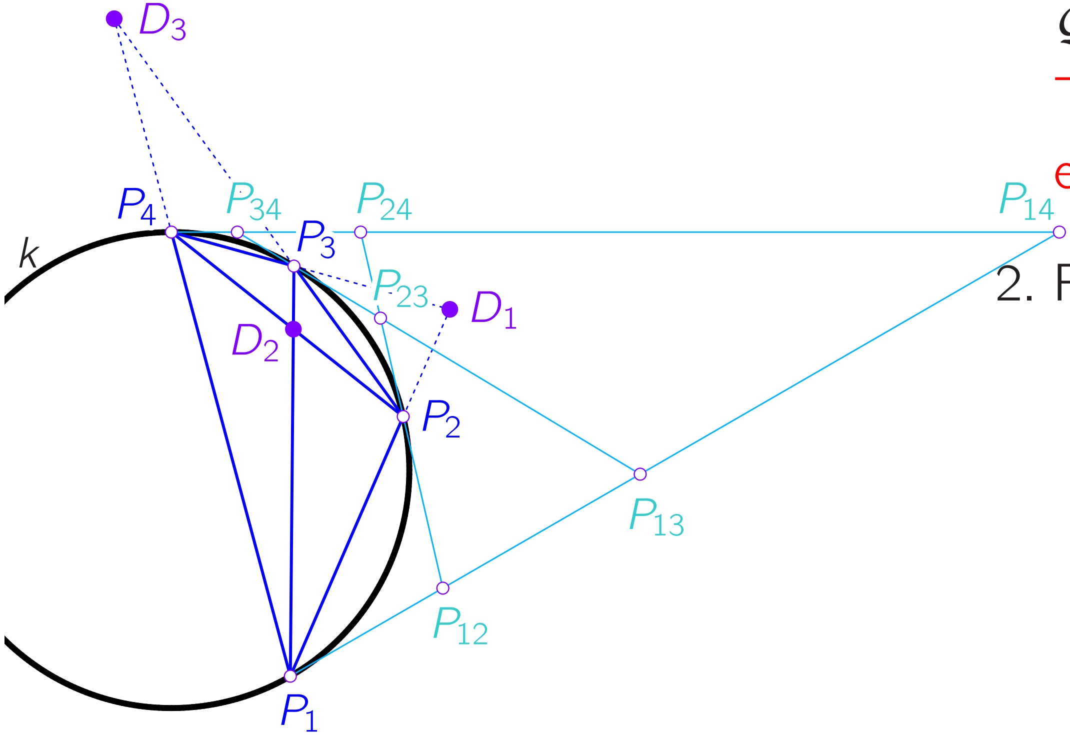
$F_1$, $F_2$ do exist (not guaranteed over the reals)! [1,2,3,4,7,8,9,12,13,15]

## Ten-Point conic I




1. On a conic $k$ choose a quadrilateral $\mathcal{Q} = P_1 P_2 P_3 P_4$ on a conic $k$.
   Through any quadrilateral there exists a pencil of conics. [6]

## Ten-Point conic II

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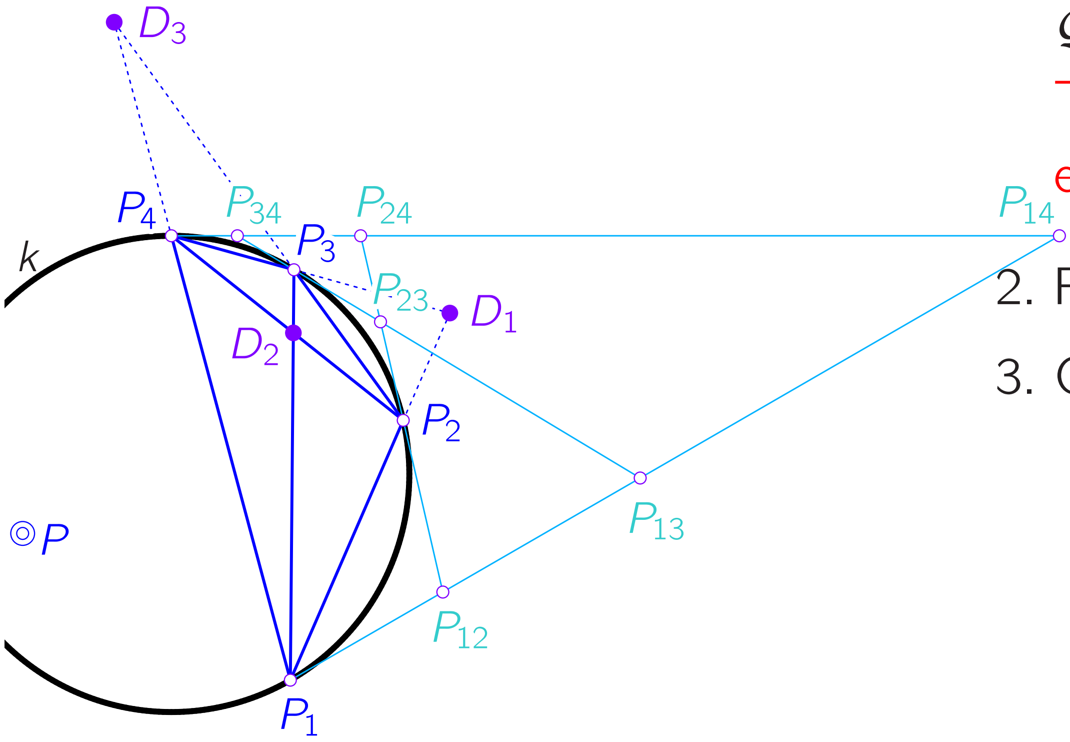
2. Find the poles $P_{ij}$ of $[P_i, P_j]$ w.r.t. $k$.

## Ten-Point conic III

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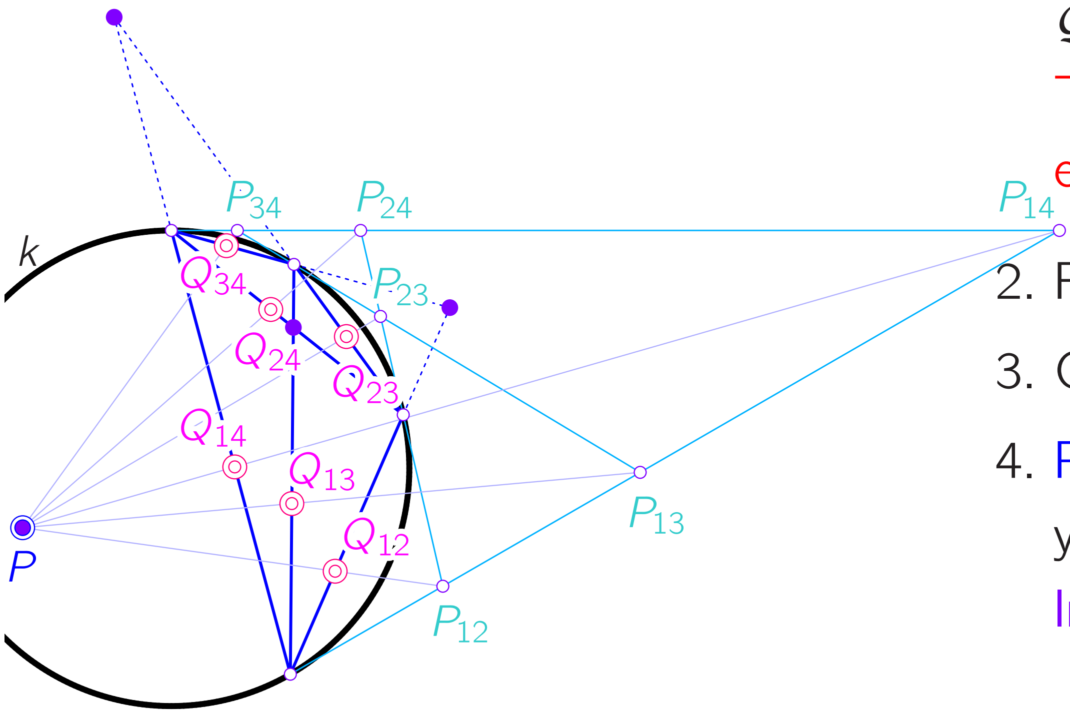
3. Choose an arbitrary point $P$.

## Ten-Point conic IV

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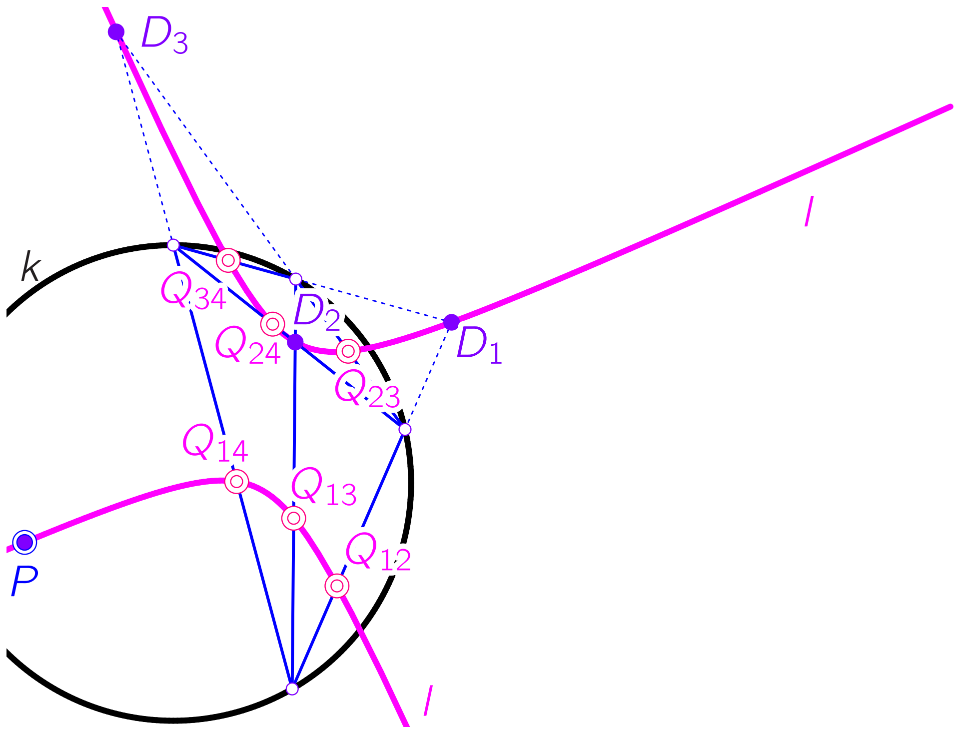
3. Choose an arbitrary point $P$.
4. Project $P_{ij}$ from $P$ onto $[P_i, P_j]$ yields points $Q_{ij}$. $\Longleftrightarrow$
   Inversion of $P_{ij}$ in $k$ with center $P$.

## Ten-Point conic V

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5. Observation:
   **Ten conconic points**: $D_i$, $Q_{ij}$, $P$!

## Ten-point conic

**Thm. 1.** The 10 points $Q_{ij}$, $D_i$, and $P$ lie on a single conic

$$l:(x_0,x_1,x_2)\begin{pmatrix} 2tp_2 & 2(tp_1-tp_2-p_2) & p_2-tp_0 \\ 2(tp_1-tp_2-p_2) & 4(p_2-tp_0) & 2(tp_0+p_0-p_1) \\ p_2-tp_0 & 2(tp_0+p_0-p_1) & -2p_0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}=0.$$

Proof (unfortunately only analytic):

Nearly canonical projective frame in homogeneous coordinates:

$P_1 = 1:0:0$, $P_2 = 0:0:1$, $P_3 = 1:1:1$, $P_4 = 1:t:t^2$ with $t \in \mathbb{R} \setminus \{0,1\}$

$\Longrightarrow$ $k$ has simplest equation: $x_0x_2 - x_1^2 = 0$ [6]

$P = p_0 : p_1 : p_2$ is **not** allowed to lie on (i) a line $[P_i, P_j] \Longleftrightarrow$

$$p_1(p_1-p_2)(p_2-tp_1)(p_1-p_0)(p_1-tp_0)(tp_0-(1+t)p_1+p_2) \neq 0$$

(ii) a tangent of $k$ at $P_i \Longleftrightarrow$

$$p_0p_2(p_0-2p_1+p_2)(t^2p_0-2tp_1+p_2) \neq 0$$

(iii) a side of the diagonal triangle $\Longleftrightarrow$

$$(tp_0-2p_1+p_2)(tp_0-2tp_1+p_2)(tp_0-p_2) \neq 0$$

Compute $Q_{ij}$ and $D_{ij}$ and substitute into the given equation of $l$. □

## Ten-point conic

**Thm. 2.** The ten-point conic $l$ from Thm. 1 is the nine-point conic of $\mathcal{Q}$ w.r.t. the polar line $p$ of $P$ w.r.t. the conic $k$.

Proof:

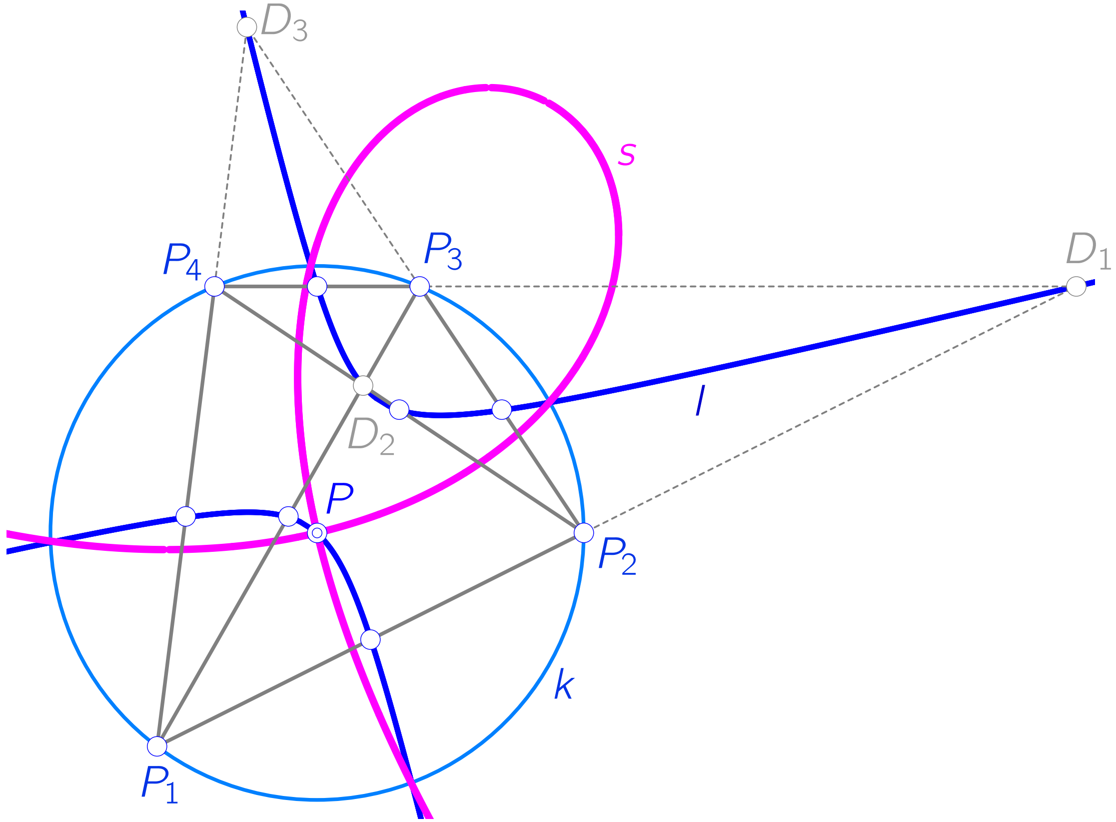
The arbitrarily chosen point $P$ has a polar line $p$ w.r.t. $k$. The points $Q_{ij}$ are the harmonic conjugates of $Q'_{ij} = [P_i, P_j] \cap p$ w.r.t to $k$. □

**Thm. 3.** The nine-point conic of a quadrilateral $\mathcal{Q}$ on a conic $k$ equals the ten-point conic $l$ if $P$ is chosen as the center of $k$.

Proof:

If $P$ is the center of $k$, then its polar $p$ w.r.t. $k$ is the line at infinity and the harmonic conjugates $Q_{ij}$ of the ideal points $Q'_{ij}$ of the lines $[P_i, P_j]$ w.r.t. the pairs $(P_i, P_j)$ are the midpoints of the segments $P_iP_j$. □

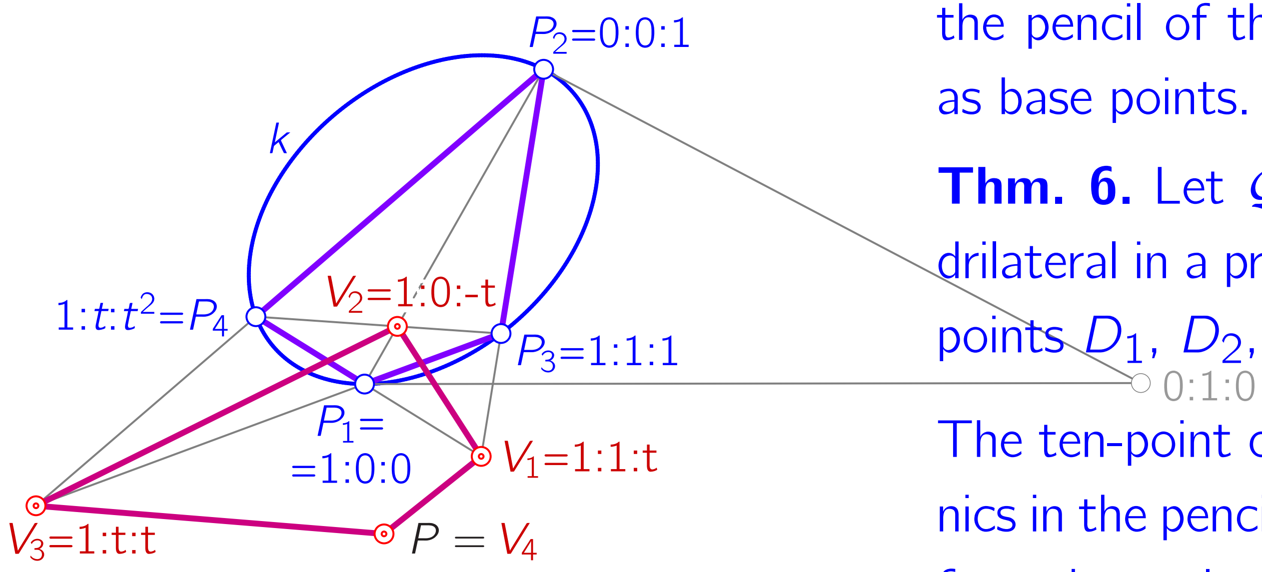
# Cyclic quadrilaterals - hyperbolas - strophoids




**Thm. 4.** The ten-point conic $l$ of a cyclic quadrilateral $P_1P_2P_3P_4$ is an equilateral hyperbola, provided $P$ is the center of the circle $k$.

Its inverse in $k$ (center of inversion $= P$) is a strophoid $s$ with double point at $P$. [14]

## Pencils of ten-point conics



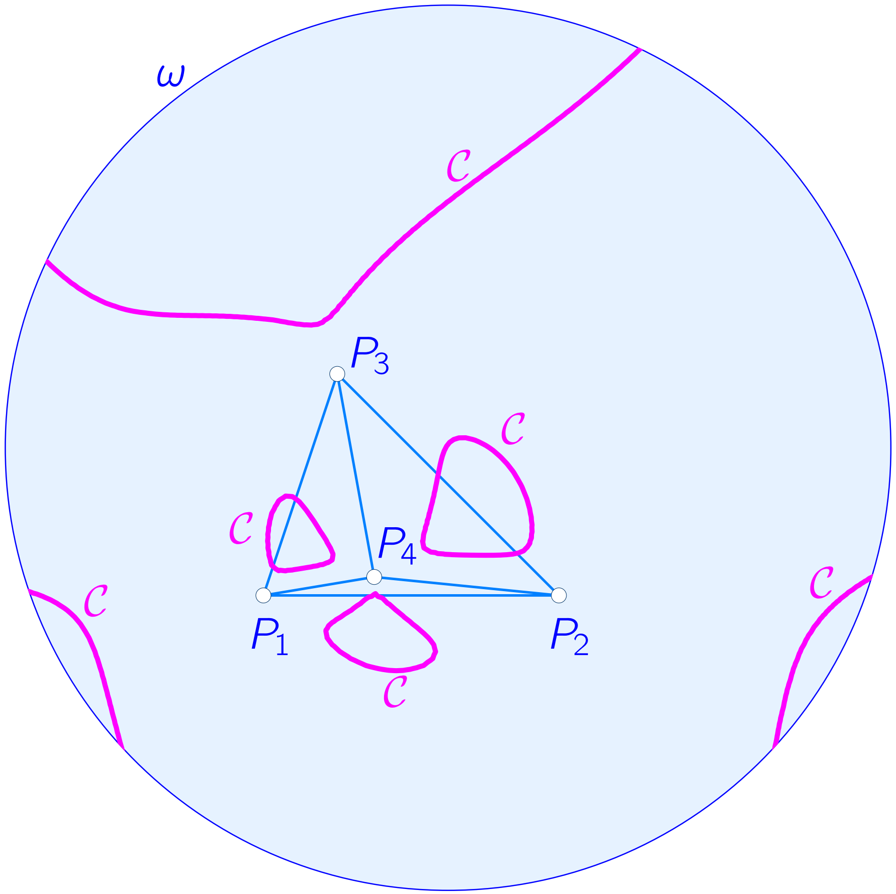

**Thm. 5.** Thm. 1 is true for any conic in the pencil of the 1. kind with $P_1, \ldots, P_4$ as base points.

**Thm. 6.** Let $\mathcal{Q} = P_1, P_2P_3P_4$ be a quadrilateral in a projective plane with diagonal points $D_1$, $D_2$, $D_3$.

The ten-point conics associated to the conics in the pencil of the 1. kind with basis $\mathcal{Q}$ form themselves a pencil of conics of the 1. kind with the pivot point $P$ and the diagonal points $D_1$, $D_2$, $D_3$ for its base points.

$P \notin [P_i, P_j]$, $P \notin [D_i, D_j]$

## How did I come across? [10,11,14]




The set $\mathcal{C}$ of points in $\mathbb{H}^2$ / $\mathbb{E}^2$ with conconic pedal points on the six sides of a complete quadrilateral $\mathcal{Q} = P_1P_2P_3P_4$ is a curve of degree 12 if no vertex $P_i$ is an absolute point.

For each asymptotic vertex, the degree of $\mathcal{C}$ drops by 3.

Consequently, if $\mathcal{Q}$ is an asymptotic quadrilateral, $\mathcal{C}$ is empty (degree 0).

The reason for that: The existence of the ten-point conic.

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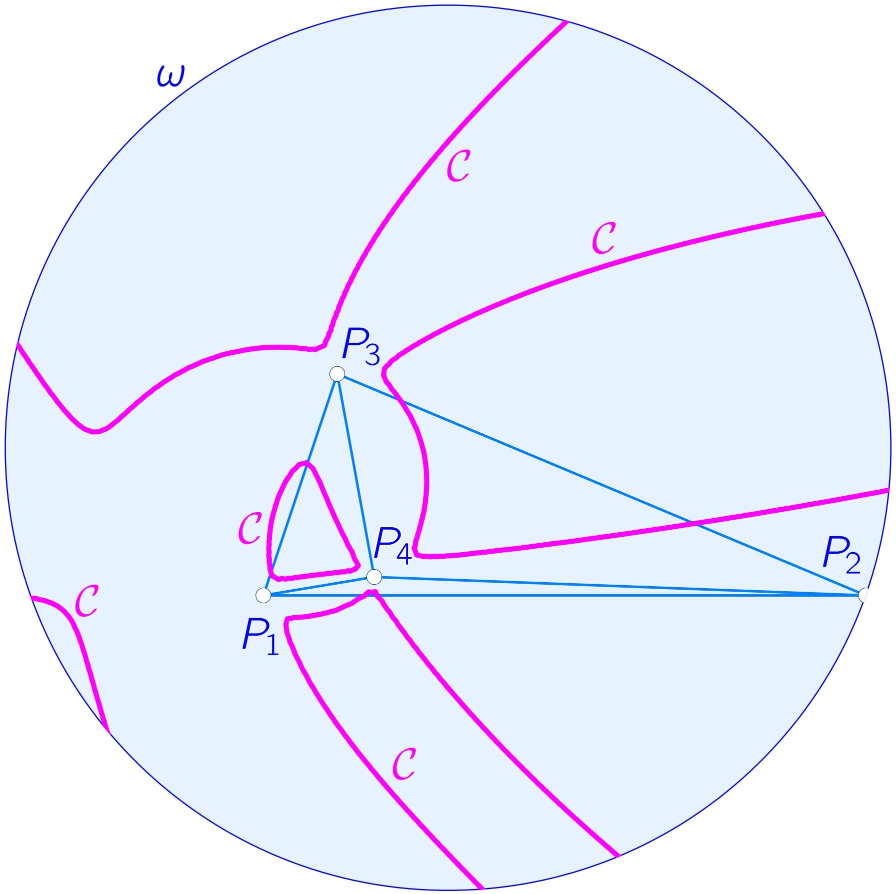


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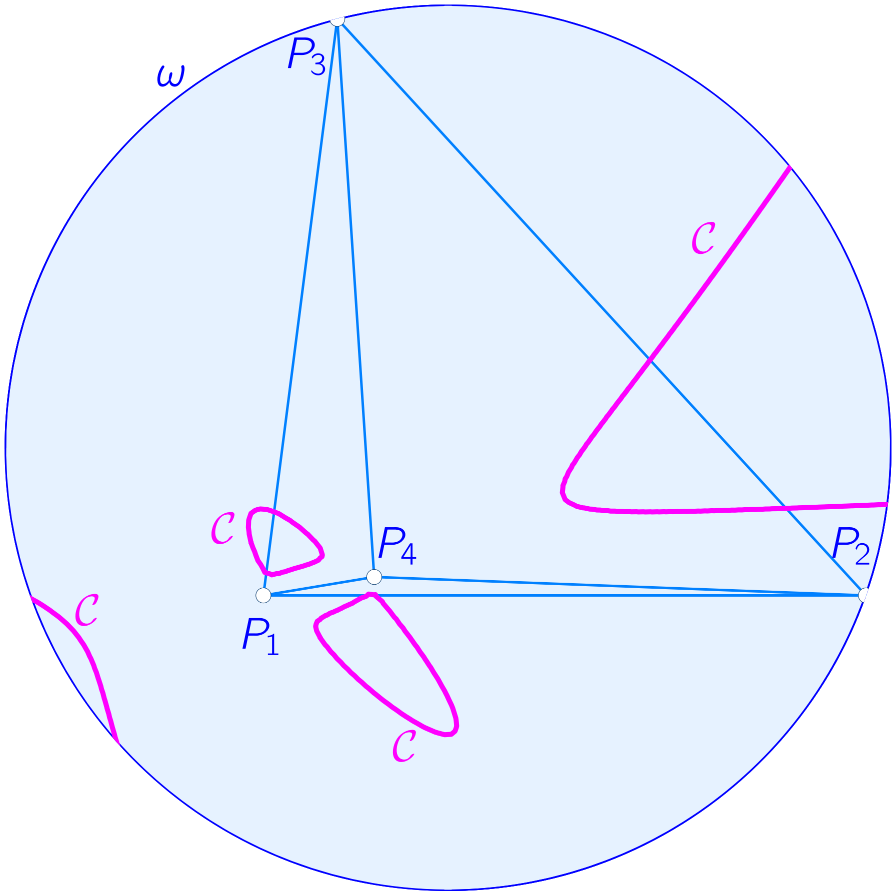


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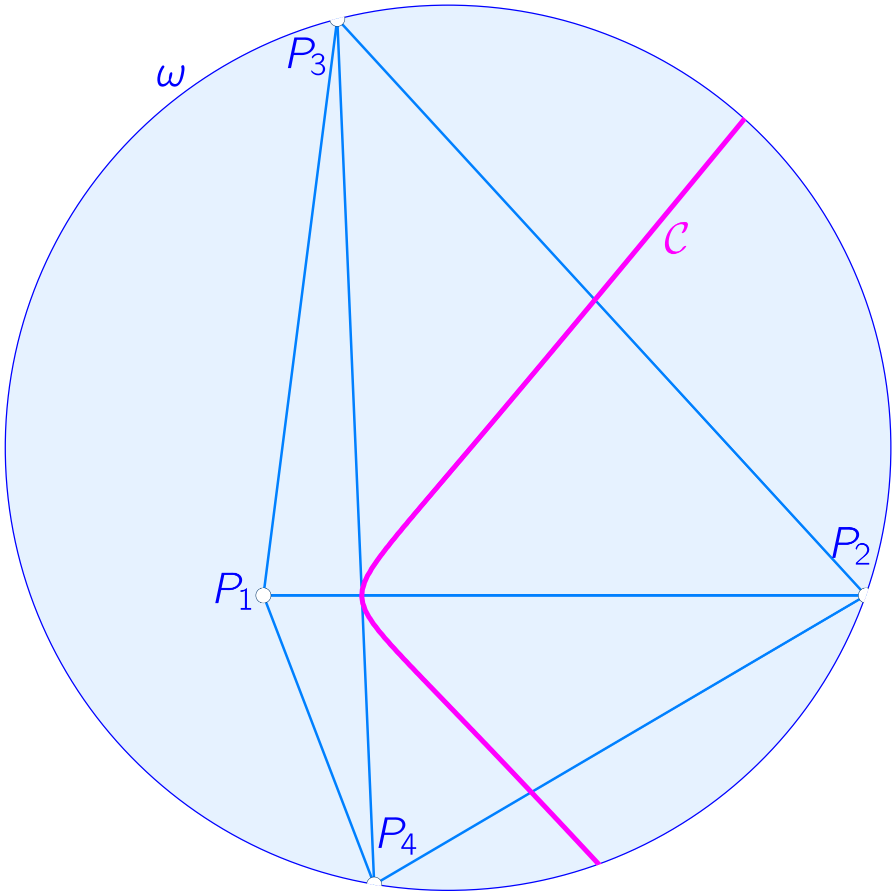


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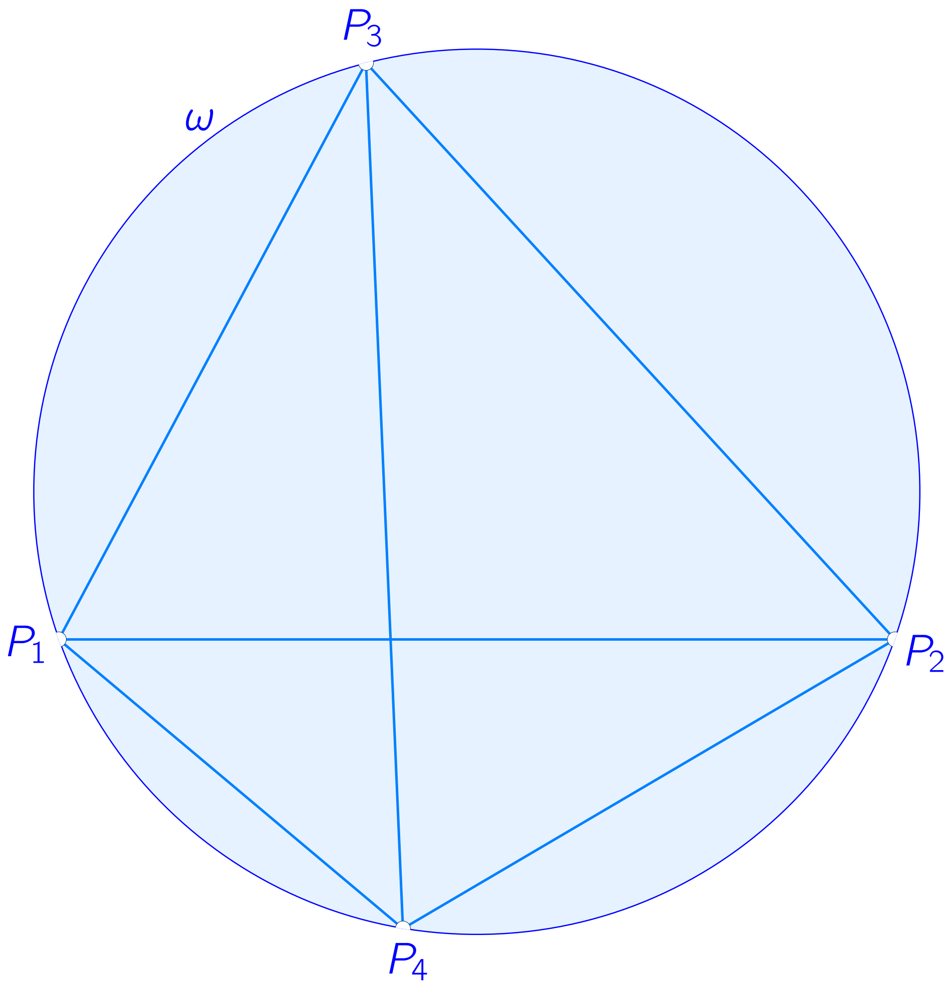


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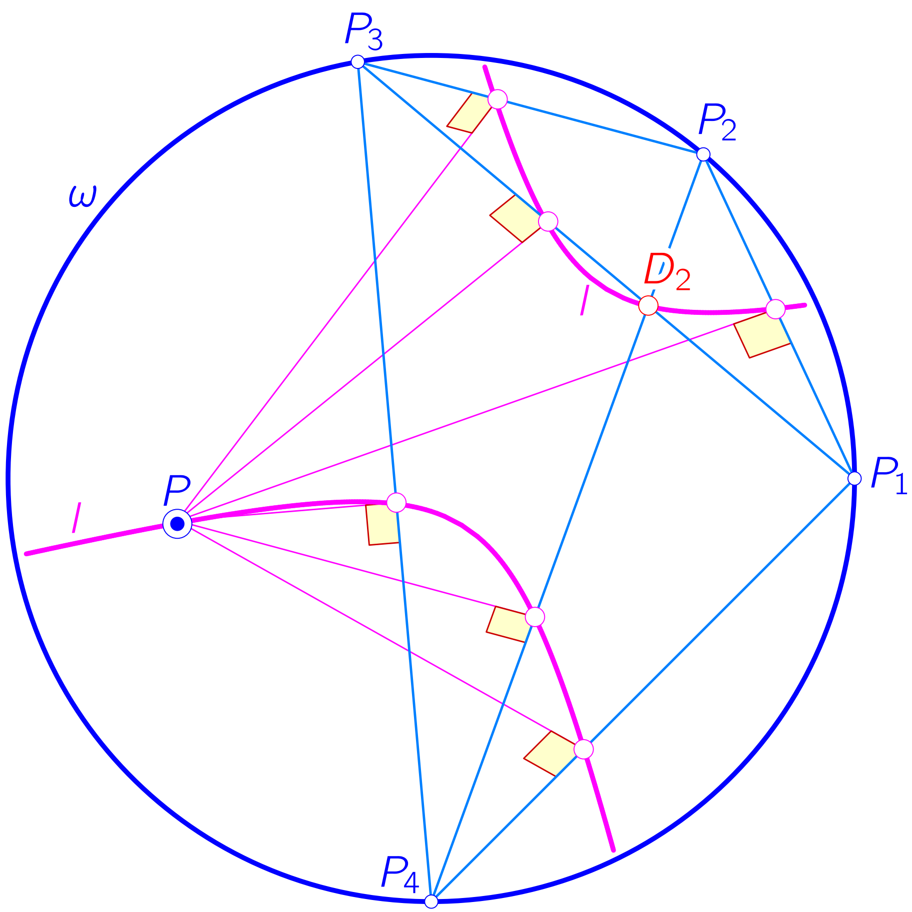
# Hyperbolic plane - eight-point conic




The six hyperbolic pedal points of $P$ on the sides of a completely asymptotic quadrilateral lie on a single conic $l$ **independent of the choice of $P$.**

Further, the conic $l$ passes through $P$ and the only (hyperbolic) diagonal point $D_2$. [5]

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Thank You For Your Attention!